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STATISTICAL EXTENSIONS OF SOME CLASSICAL TAUBERIAN THEOREMS IN NONDISCRETE SETTING

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Abstract. Schmidt's classical Tauberian theorem says that if a sequence $(s_k : k = 0, 1, \ldots)$ of real numbers is summable $(C,1)$ to a finite limit and slowly decreasing, then it converges to the same limit. In this paper, we prove a nondiscrete version of Schmidt's theorem in the setting of statistical summability $(C,1)$ of real-valued functions that are slowly decreasing on $\mathbb{R}_+$. We prove another Tauberian theorem in the case of complex-valued functions that are slowly oscillating on $\mathbb{R}_+$. In the proofs we make use of two nondiscrete analogues of the famous Vijayaraghavan lemma, which seem to be new and may be useful in other contexts.

1. Introduction. We consider real- or complex-valued functions that are measurable (in Lebesgue's sense) on some interval (a, ∞), where $a \geq 0$. We recall (see [5]) that a function f has *statistical limit at* ∞ if there exists a number ℓ such that for every $\varepsilon > 0$,

$$\lim_{b\to\infty} \frac{1}{b-a} |\{x \in (a,b) : |f(x) - \ell| > \varepsilon\}| = 0, \tag{1.1}$$

where by $|\{\cdot\}|$ we denote the Lebesgue measure of the set indicated in $\{\cdot\}$. If this is the case, we write

$$\operatorname*{st-lim}_{x\to\infty} f(x) = \ell.$$

Clearly, the statistical limit ℓ in (1.1) is uniquely determined. The existence of the ordinary limit of a function f at ∞ implies the existence of the statistical limit of f at ∞ with the same value. The notion of statistical

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limit also enjoys the property of additivity and homogeneity. (See [5] for further details.)

It is easy to see that the particular choice of the left endpoint a of the definition domain of f is indifferent in (1.1). That is, if (1.1) is satisfied for some $a \geq 0$, then it is satisfied for any $a_1 \geq 0$ in place of a. For the sake of simplicity in writing, in what follows we assume that $a := 0$.

We recall that a real-valued function f is said to be *slowly decreasing* (in the sense of Schmidt; see [7] for the discrete case) if

$$\lim_{\lambda \to 1+} \liminf_{x \to \infty} \inf_{x \leq t \leq \lambda x} [f(t) - f(x)] \geq 0. \tag{1.2}$$

Since the auxiliary function

$$a(\lambda) := \liminf_{x \to \infty} \inf_{x \leq t \leq \lambda x} [f(t) - f(x)]$$

is evidently decreasing in λ on the interval $(1, \infty)$, the right-hand limit in (1.2) exists, and $\lim_{\lambda \to 1+}$ in it can be equivalently replaced by $\sup_{\lambda > 1}$.

It is easy to check that (1.2) is satisfied if and only if for every $\varepsilon > 0$ there exist $x_0 = x_0(\varepsilon) > 0$ and $\lambda_0 = \lambda_0(\varepsilon) > 1$, the latter as close to 1 as we want, such that

$$f(t) - f(x) \geq -\varepsilon \quad \text{whenever } x_0 \leq x \leq t \leq \lambda_0 x. \tag{1.3}$$

We note that the symmetric counterpart of the notion of slow decrease is the following: a real-valued function f is said to be *slowly increasing* if

$$\lim_{\lambda \to 1+} \limsup_{x \to \infty} \sup_{x \leq t \leq \lambda x} [f(t) - f(x)] \leq 0. \tag{1.4}$$

Clearly, f is slowly increasing if and only if the function $-f$ is slowly decreasing. In particular, the right-hand limit $\lim_{\lambda \to 1+}$ in (1.4) can be equivalently replaced by $\inf_{\lambda > 1}$.

We recall that a complex-valued function f is said to be *slowly oscillating* if

$$\lim_{\lambda \to 1+} \limsup_{x \to \infty} \sup_{x \leq t \leq \lambda x} |f(t) - f(x)| = 0. \tag{1.5}$$

Again, the right-hand limit $\lim_{\lambda \to 1+}$ in (1.5) can be equivalently replaced by $\inf_{\lambda > 1}$.

It is easy to check that (1.5) is satisfied if and only if for every $\varepsilon > 0$ there exist $x_0 = x_0(\varepsilon) > 0$ and $\lambda_0 = \lambda_0(\varepsilon) > 1$, the latter as close to 1 as we want, such that

$$|f(t) - f(x)| \leq \varepsilon \quad \text{whenever } x_0 \leq x \leq t \leq \lambda_0 x_0. \tag{1.6}$$

In particular, a real-valued function f is slowly oscillating if and only if it is both slowly decreasing and slowly increasing.

We recall that a function f is said to be *locally absolutely continuous* on $\mathbb{R}_+$, in symbols: $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$, if the derivative f' exists almost everywhere on $\mathbb{R}_+$, f' is locally integrable (in Lebesgue's sense) on $\mathbb{R}_+$, in symbols: $f' \in L_{\mathrm{loc}}(\mathbb{R}_+)$, and

$$f(t) = \int_0^t f'(y)\,dy, \qquad t \in \mathbb{R}_+. \tag{1.7}$$

It is easy to check that if a real-valued function $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$ satisfies *Landau's one-sided Tauberian condition*:

$$yf'(y) \geq -H \quad \text{for some constant } H > 0 \text{ and almost every } y \in \mathbb{R}_+ \tag{1.8}$$

(see [4] and also [3, pp. 124–126] for the discrete case), then f is slowly decreasing. Furthermore, if a complex-valued function $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$ satisfies *Hardy's two-sided Tauberian condition*:

$$y|f'(y)| \leq H \text{ for some constant } H \text{ and almost every } y \in \mathbb{R}_+ \tag{1.9}$$

(see [2] and also [3, p. 121] for the discrete case), then f is slowly oscillating.

We note that the discrete analogues of (1.8) and (1.9) are the following conditions:

$$\begin{matrix} \text{(i) } k(s_k - s_{k-1}) \geq -H, \\ \text{(ii) } k|s_k - s_{k-1}| \leq H, \end{matrix} \qquad k \geq k_0, \tag{1.10}$$

respectively, where $(s_k : k = 0, 1, \ldots)$ is a given sequence of real or complex numbers, while H and k_0 are positive constants.

2. Main results. In Theorems 1 and 2 below we prove nondiscrete analogues of [6, Lemmas 6 and 7], without using the so-called decomposition theorem (see [5, Thoerem 1]) in the proof.

THEOREM 1. *Assume f is a real-valued, measurable and slowly decreasing function on $\mathbb{R}_+$. If the statistical limit ℓ of f exists at ∞, then the ordinary limit of f also exists at ∞ and equals ℓ.*

THEOREM 2. *Assume f is a complex-valued, measurable and slowly oscillating function on $\mathbb{R}_+$. If the statistical limit ℓ of f exists at ∞, then the ordinary limit of f also exists at ∞ and equals ℓ.*

We note that the discrete versions of Theorems 1 and 2 in the special cases when the condition of slow decrease is replaced by (1.10i), and respectively when the condition of slow oscillation is replaced by (1.10ii), were proved in [1].

We recall that a function $f \in L_{\mathrm{loc}}(\mathbb{R}_+)$ is said to be *statistically summable $(C,1)$ at ∞ to ℓ* if

$$\operatorname*{st-lim}_{x\to\infty} \sigma(x) = \ell,$$

where

$$\sigma(x) := \frac{1}{x}\int_0^x f(t)\,dt, \qquad x > 0, \tag{2.1}$$

is the $(C,1)$ mean function of f. (See, for example, [3, p. 11] or [8, p. 26].)

It is routine to show that if a function $f \in L_{\rm loc}(\mathbb{R}_+)$ is bounded almost everywhere on $\mathbb{R}_+$ and the statistical limit ℓ of f exists at ∞, then

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x |f(t) - \ell|\,dy = 0,$$

whence it follows that f is statistically summable $(C,1)$ at ∞ to ℓ. (See [5, Theorem 2].)

In the following, we study the reverse implication under so-called Tauberian conditions. Our Theorems 3 and 4 below are nondiscrete analogues of [6, Theorems 1 and 2].

THEOREM 3. *Assume $f \in L_{\rm loc}(\mathbb{R}_+)$ is a real-valued, slowly decreasing function. If f is statistically summable $(C,1)$ at ∞ to ℓ, then the ordinary limit of f exists at ∞ and equals ℓ.*

THEOREM 4. *Assume $f \in L_{\rm loc}(\mathbb{R}_+)$ is a complex-valued, slowly oscillating function. If f is statistically summable $(C,1)$ at ∞ to ℓ, then the ordinary limit of f exists at ∞ and equals ℓ.*

It is interesting to apply Theorems 3 and 4 in the particular case when $f \in {\rm AC}_{\rm loc}(\mathbb{R}_+)$. By (2.1) and (1.7), using Fubini's theorem, we obtain

$$\begin{aligned}\sigma(x) &:= \frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x}\int_0^x \Big\{\int_0^t f'(y)\,dy\Big\}\,dt \\ &= \int_0^x f'(y)\Big(1 - \frac{y}{x}\Big)\,dy, \qquad x > 0.\end{aligned}$$

Now, it is well known (see, e.g., [8, pp. 26-27]) that if the improper integral

$$\int_0^{\to\infty} f'(y)\,dy \tag{2.2}$$

is convergent, that is, if the finite limit

$$\lim_{x\to\infty}\int_0^x f'(y)\,dy = \ell \tag{2.3}$$

exists, then the ordinary limit

$$\lim_{x\to\infty}\sigma(x) = \ell \tag{2.4}$$

also exists. The reverse implication is not true in general. However, if the derivative f' of a real-valued function $f \in AC_{\text{loc}}(\mathbb{R}_+)$ is of constant sign, then the limits in (2.3) and (2.4) exist (or not) simultaneously.

The following two corollaries are immediate consequences of Theorems 3 and 4, respectively.

COROLLARY 1. *Assume* $f \in \mathrm{AC}_{\text{loc}}(\mathbb{R}_+)$ *is a real-valued function satisfying condition* (1.8). *If* f *is statistically summable* $(C,1)$ *at* ∞ *to* ℓ, *then the improper integral* (2.2) *is convergent to* ℓ.

COROLLARY 2. *Assume* $f \in \mathrm{AC}_{\text{loc}}(\mathbb{R}_+)$ *is a complex-valued function satisfying condition* (1.9). *If* f *is statistically summable* $(C,1)$ *at* ∞ *to* ℓ, *then the improper integral* (2.2) *is convergent to* ℓ.

3. Auxiliary results, including nondiscrete analogues of Vijayaraghavan's lemma. Our first lemma is interesting in itself and may be useful in other investigations.

LEMMA 1. *If the statistical limit* ℓ *of a function* f *exists at* ∞, *then for any* $\varepsilon > 0$ *and* $\lambda > 1$, *there exists an increasing sequence* $(b_n : n = 1, 2, \ldots)$ *of positive numbers tending to* ∞ *such that*

$$|f(b_n) - \ell| \le \varepsilon, \qquad n = 1, 2, \ldots, \tag{3.1}$$

and for some natural number $n_0 = n_0(\varepsilon, \lambda)$, *we have*

$$b_{n+1} < \lambda b_n, \qquad n = n_0 + 1, n_0 + 2, \ldots. \tag{3.2}$$

Proof. By definition (1.1) with $a := 0$, there exists $b_1 > 0$ such that (3.1) is satisfied for $n = 1$. There are two cases: (i) there exists some $b_2 \in (\sqrt{\lambda}\, b_1, \lambda b_1)$ for which (3.1) is satisfied for $n = 2$; (ii) there is no such b_2, that is, we have

$$|f(t) - \ell| > \varepsilon \quad \text{ for every } t \in (\sqrt{\lambda}\, b_1, \lambda b_1).$$

In the latter case, we choose some $b_2 \ge \lambda b_1$ for which (3.1) is satisfied for $n = 2$ (such a b_2 certainly exists, due to (1.1)).

Then we repeat the previous step by starting with b_2 in place of b_1, and so on. As a result, we obtain an increasing sequence $(b_n : n = 1, 2, \ldots)$ of positive numbers tending to ∞ such that (3.1) is satisfied for all n.

We claim that the case when

$$|f(t) - \ell| > \varepsilon \quad \text{ for every } t \in (\sqrt{\lambda}\, b_n, \lambda b_n) \tag{3.3}$$

cannot occur for infinitely many values of n. Otherwise, for infinitely many n we would have

$$\frac{1}{b_n} \left|\{x \in (0, b_n) : |f(x) - \ell| > \varepsilon\}\right| \ge \lambda - \sqrt{\lambda} > 0,$$

which clearly contradicts (1.1). If we denote by n_0 the largest value of n (perhaps $n_0 = 0$) for which inequality (3.3) occurs, then (3.2) is also satisfied. ■

Our Lemma 2 below can be considered to be a nondiscrete analogue of the famous Vijayaraghavan lemma (see [9, Lemma 6]), under less restrictive conditions.

LEMMA 2. *If a real-valued function f is such that condition* (1.3) *is satisfied only for $\varepsilon := 1$, where $x_0 > 0$ and $\lambda_0 > 1$, then there exists a positive constant B such that*

$$f(t) - f(x) \geq -B \ln \frac{t}{x} \quad \textit{for all } x_0 \leq x < \frac{t}{\lambda_0}. \tag{3.4}$$

Proof. For given $x_0 \leq x < t/\lambda_0$, we set

$$t_0 := t, \quad t_p := \frac{t_{p-1}}{\lambda_0}, \quad p = 1, \dots, q+1, \tag{3.5}$$

where q is determined by the condition

$$t_{q+1} \leq x < t_q. \tag{3.6}$$

By (1.3) and (3.6), we estimate as follows:

$$f(t) - f(x) = \sum_{p=1}^{q} [f(t_{p-1}) - f(t_p)] + [f(t_q) - f(x)] \geq -q - 1. \tag{3.7}$$

It is clear that

$$\lambda_0^q = \frac{t}{t_q} < \frac{t}{x}, \quad \text{or equivalently,} \quad q < \frac{1}{\ln \lambda_0} \ln \frac{t}{x}. \tag{3.8}$$

Combining (3.7) and (3.8) gives

$$f(t) - f(x) > -1 + \frac{1}{\ln \lambda_0} \ln \frac{t}{x}, \quad x_0 \leq x < \frac{t}{\lambda_0}. \tag{3.9}$$

Taking into account that $\lambda_0 < t/x$, we obtain (3.4) with $B := 2/\ln \lambda_0$. ■

The next lemma is the counterpart of Lemma 2 in the complex-valued case.

LEMMA 3. *If a complex-valued function f is such that condition* (1.6) *is satisfied only for $\varepsilon := 1$, where $x_0 > 0$ and $\lambda_0 > 1$, then there exists a positive constant B such that*

$$|f(t) - f(x)| \leq B \ln \frac{t}{x} \quad \textit{for all } x_0 \leq x < \frac{t}{\lambda_0}. \tag{3.10}$$

Proof. It runs along the same lines as the proof of Lemma 2. For given $x_0 \leq x < t/\lambda_0$, we consider $t_0, t_1, \dots, t_{q+1}$ defined by (3.5) and (3.6). By

(1.6) and (3.6), we estimate as follows:

$$|f(t)-f(x)| \le \sum_{p=1}^{q} |f(t_{p-1})-f(t_p)| + |f(t_q)-f(x)| \le q+1 \tag{3.11}$$

(cf. (3.7)). Combining (3.8) and (3.11) gives

$$|f(t)-f(x)| \le 1 + \frac{1}{\ln \lambda_0} \ln \frac{t}{x}, \qquad x_0 \le x < \frac{t}{\lambda_0}$$

(cf. (3.9)), whence (3.10) follows with $B := 2/\ln \lambda_0$. ■

LEMMA 4. *Under the assumptions of Lemma* 2, *there exists a positive constant* B_1 *such that*

$$\frac{1}{t} \int_{x_0}^{t} [f(t)-f(x)]\,dx \ge -B_1 \qquad \textit{whenever } t > \lambda_0 x_0. \tag{3.12}$$

Proof. It hinges on the crucial Lemma 2. By (1.3) and (3.4), we estimate as follows:

$$\begin{aligned}\int_{x_0}^{t} [f(t)-f(x)]\,dx &= \Big\{ \int_{x_0}^{t/\lambda_0} + \int_{t/\lambda_0}^{t} \Big\} [f(t)-f(x)]\,dx \\ &\ge -B \int_{x_0}^{t/\lambda_0} \ln \frac{t}{x}\,dx - \int_{t/\lambda_0}^{t} dx.\end{aligned} \tag{3.13}$$

Since

$$\int_{x_0}^{t/\lambda_0} \ln \frac{t}{x}\,dx \le \int_{0}^{t/\lambda_0} [\ln t - \ln x]\,dx = \frac{t(1+\ln \lambda_0)}{\lambda_0}, \tag{3.14}$$

from (3.13) it follows that

$$\frac{1}{t} \int_{x_0}^{t} [f(t)-f(x)]\,dx \ge -B\,\frac{1+\ln \lambda_0}{\lambda_0} - 1 \qquad \text{for every } t > \lambda_0 x_0. \tag{3.15}$$

This inequality proves (3.12) with $B_1 := B(1+\lambda_0+\ln \lambda_0)/\lambda_0$. ■

The counterpart of Lemma 4 in the complex-valued case reads as follows.

LEMMA 5. *Under the assumptions of Lemma* 3, *there exists a positive constant* B_1 *such that*

$$\frac{1}{t} \int_{x_0}^{t} |f(t)-f(x)|\,dx \le B_1 \qquad \textit{whenever } t > \lambda_0 x_0. \tag{3.16}$$

Proof. It runs along the same lines as the proof of Lemma 4. By (1.6) and (3.10), we estimate as follows:

$$\int_{x_0}^{t} |f(t)-f(x)|\,dx \le B\int_{x_0}^{t/\lambda_0} \ln\frac{t}{x}\,dx + \int_{t/\lambda_0}^{t} dx \tag{3.17}$$

(cf. (3.13)). Combining (3.14) and (3.17) gives

$$\frac{1}{t}\int_{x_0}^{t} |f(t)-f(x)|\,dx \le B\frac{1+\ln\lambda_0}{\lambda_0}+1 \qquad \text{for every } t>\lambda_0 x_0$$

(cf. (3.15)). This inequality proves (3.16) with $B_1 := B(1+\lambda_0+\ln\lambda_0)/\lambda_0$. ■

4. Proofs of Theorems 1–4

Proof of Theorem 1. It hinges on Lemma 1, according to which for any $\varepsilon>0$ and $\lambda>1$, there exists an increasing sequence $(b_n : n=1,2,\ldots)$ of positive numbers tending to ∞ such that conditions (3.1) and (3.2) are satisfied.

By the condition (1.3) of slow decrease, we have

$$f(t)-f(b_n) \ge -\varepsilon \qquad \text{whenever } x_0(\varepsilon) \le b_n < t < \lambda b_n. \tag{4.1}$$

Since $b_n \to \infty$ as $n\to\infty$, this is certainly the case if n is large enough, say $n>n_1$. From (3.1), (3.2) and (4.1) it follows that if $n>\max\{n_0,n_1\}$, where n_0 occurs in (3.2), then for every $t\in(b_n,b_{n+1}]$, we have

$$f(t)-\ell = [f(t)-f(b_n)]+[f(b_n)-\ell] \ge -2\varepsilon. \tag{4.2}$$

Taking into account that if $t\in(b_n,b_{n+1}]$, then $b_n<t\le b_{n+1}<\lambda t$, by (1.3), we can also conclude that

$$f(b_{n+1})-f(t)\ge -\varepsilon \qquad \text{whenever } t\in(b_n,b_{n+1}].$$

Combining this with (3.1) and (4.1) gives that if $n>\max\{n_0,n_1\}=:n_2$, then for every $t\in(b_n,b_{n+1}]$, we have

$$f(t)-\ell = [f(t)-f(b_{n+1})]+[f(b_{n+1})-\ell] \le 2\varepsilon. \tag{4.3}$$

Putting together (4.2) and (4.3) yields

$$|f(t)-\ell| \le 2\varepsilon \qquad \text{for every } t\in \bigcup_{n=n_2+1}^{\infty}(b_n,b_{n+1}] = (b_{n_2+1},\infty).$$

Since $\varepsilon>0$ is arbitrary, this means that the ordinary limit of f exists at ∞ and equals ℓ. ■

Proof of Theorem 2. By Lemma 1, for any $\varepsilon>0$ and $\lambda>1$, there exists an increasing sequence $(b_n : n=1,2,\ldots)$ of positive numbers tending to ∞ such that conditions (3.1) and (3.2) are satisfied.

By the condition (1.6) of slow oscillation, we have

$$|f(t) - f(b_n)| \le \varepsilon \quad \text{whenever } x_0(\varepsilon) \le b_n < t < \lambda b_n \tag{4.4}$$

(cf. (4.1)). Since $b_n \to \infty$ as $n \to \infty$, this is certainly the case if n is large enough, say $n > n_1$. From (3.1), (3.2) and (4.4) it follows that

$$|f(t) - \ell| \le |f(t) - f(b_n)| + |f(b_n) - \ell| \le 2\varepsilon$$

for every $t \in \bigcup_{n=n_2+1}^{\infty} (b_n, b_{n+1}] = (b_{n_2+1}, \infty)$, where $n_2 := \max\{n_0, n_1\}$ and n_0 occurs in (3.2). Since $\varepsilon > 0$ is arbitrary, this means that the ordinary limit of $f(t)$ exists at ∞ and equals ℓ. ∎

Proof of Theorem 3. It hinges on Lemma 4 and Theorem 1.

First, we prove that if the condition (1.3) of slow decrease is satisfied for a single $\varepsilon > 0$, say $\varepsilon := 1$, then we have

$$\liminf_{x \to \infty} \frac{f(x)}{x} \ge 0. \tag{4.5}$$

Indeed, from (1.3) with $\varepsilon = 1$ it follows that for $p = 1, 2, \dots$ we have

$$f(\lambda_0^p x_0) - f(x_0) \ge -p, \quad \text{where } x_0 := x_0(1) > 0 \text{ and } \lambda_0 := \lambda_0(1) > 1.$$

This means that

$$\frac{f(\lambda_0^p x_0)}{\lambda_0^p x_0} \ge \frac{f(x_0)}{\lambda_0^p x_0} - \frac{p}{\lambda_0^p x_0} \to 0 \quad \text{as } p \to \infty.$$

Now, (4.5) is obvious.

Second, we prove that if a real-valued function $f \in L_{\text{loc}}(\mathbb{R}_+)$ is slowly decreasing, then so is its $(C, 1)$ mean function $\sigma(x)$ defined in (2.1). To this end, let some $0 < \varepsilon < 1$ be given, and let

$$x_0 \le x \le t \le \lambda_0 x, \tag{4.6}$$

where $x_0 := x_0(\varepsilon)$ and $\lambda_0 := \lambda_0(\varepsilon)$ occur in (1.3) and this time λ_0 is chosen so that

$$1 < \lambda_0 \le 1 + \frac{\varepsilon}{B_1}. \tag{4.7}$$

By definition (2.1), we may write

$$\begin{aligned} \sigma(t) - \sigma(x) &:= \frac{1}{t}\int_0^t f(y)\,dy - \frac{1}{x}\int_0^x f(y)\,dy \\ &= -\frac{t-x}{tx}\int_0^x f(y)\,dy + \frac{1}{t}\int_x^t f(y)\,dy \end{aligned} \tag{4.8}$$

$$= \frac{t-x}{tx}\int_0^x [f(x)-f(y)]\,dy + \frac{1}{t}\int_x^t [f(y)-f(x)]\,dy$$

$$= \frac{t-x}{tx}\Big\{\int_0^{x_0} + \int_{x_0}^x\Big\}[f(x)-f(y)]\,dy + \frac{1}{t}\int_x^t [f(y)-f(x)]\,dy$$

$$= \frac{t-x}{tx}\,x_0 f(x) - \frac{t-x}{tx}\int_0^{x_0} f(y)\,dy + \frac{t-x}{tx}\int_{x_0}^x [f(x)-f(y)]\,dy$$

$$+ \frac{1}{t}\int_x^t [f(y)-f(x)]\,dy =: I_1 + I_2 + I_3 + I_4, \qquad \text{say.}$$

It follows from (4.5) that

$$\liminf_{x\to\infty} I_1 \ge 0. \tag{4.9}$$

Since $f \in L_{\mathrm{loc}}(\mathbb{R}_+)$, we have

$$\lim_{x\to\infty} I_2 = 0. \tag{4.10}$$

By (4.6), we see that $(t-x)/t \le (\lambda_0 - 1)x$. Using this fact and (4.7), an application of Lemma 4 yields

$$I_3 \ge -(\lambda_0 - 1)B_1 \ge -\varepsilon. \tag{4.11}$$

Finally, (1.3) applies (again due to (4.6) and (4.7)) and gives

$$I_4 \ge -\varepsilon. \tag{4.12}$$

Putting (4.8)–(4.12) together yields

$$\sigma(t) - \sigma(x) \ge -4\varepsilon \qquad \text{whenever } x \le t \le \lambda_0 x,$$

provided that x is large enough, where we have also taken into account the limit relations in (4.9) and (4.10). Thus, we have proved that $\sigma(x)$ is also slowly decreasing.

Third, making use of Theorem 1 yields the existence of the ordinary limit of $\sigma(x)$ with the same value ℓ as $x \to \infty$. Applying Schmidt's classical Tauberian theorem (see [7]) yields the ordinary convergence of the function $f(x)$ itself as $x \to \infty$. ∎

Proof of Theorem 4. It hinges on Lemma 5 and Theorem 2.

Similarly to (4.5), it follows from the condition (1.6) of slow oscillation that

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0. \tag{4.13}$$

Now, we prove that if a function $f \in L_{\mathrm{loc}}(\mathbb{R}_+)$ is slowly oscillating, then so is its $(C,1)$ mean function defined in (2.1). To this end, let some $0 < \varepsilon < 1$

be given and consider those x and t for which conditions (4.6) and (4.7) are satisfied. By (4.8), we estimate as follows:

$$\begin{aligned}(4.14)\qquad |\sigma(t)-\sigma(x)| &\le \frac{t-x}{tx}\,x_0|f(x)| + \frac{t-x}{tx}\int_0^{x_0}|f(y)|\,dy \\ &\quad + \frac{t-x}{tx}\int_{x_0}^{x}|f(x)-f(y)|\,dy + \frac{1}{t}\int_x^t|f(y)-f(x)|\,dy \\ &=: J_1+J_2+J_3+J_4, \qquad \text{say.}\end{aligned}$$

It follows from (4.13) that

$$\lim_{x\to\infty} J_1 = 0. \tag{4.15}$$

Since $f \in L_{\mathrm{loc}}(\mathbb{R}_+)$, we have

$$\lim_{x\to\infty} J_2 = 0. \tag{4.16}$$

By (4.6), we see that $(t-x)/t \le (\lambda_0 - 1)x$. Using this fact and (4.7), and applying Lemma 5 yields

$$|J_3| \le (\lambda_0 - 1)B_1 \le \varepsilon. \tag{4.17}$$

Finally, (1.6) applies (again due to (4.6) and (4.7)) and gives

$$|J_4| \le \varepsilon. \tag{4.18}$$

Putting (4.14)–(4.18) together yields

$$|\sigma(t)-\sigma(x)| \le 4\varepsilon \qquad \text{whenever } x \le t \le \lambda_0 x,$$

provided that x is large enough, where we have also taken into account the limit relations in (4.15) and (4.16). Thus, we have proved that $\sigma(x)$ is also slowly oscillating.

Consequently, by Theorem 2, $\sigma(x)$ converges to ℓ in the ordinary sense as $x \to \infty$. Applying Schmidt's classical Tauberian theorem (see [7]) yields the ordinary convergence of the function $f(x)$ itself as $x \to \infty$. ■

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