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DIRICHLET FORMS ON QUOTIENTS OF SHIFT SPACES

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Abstract. We define thin equivalence relations $\sim$ on shift spaces $\mathscr{A}^\infty$ and derive Dirichlet forms on the quotient space $\Sigma = \mathscr{A}^\infty/{\sim}$ in terms of the nearest neighbour averaging operator. We identify the associated Laplace operator. The conditions are applied to some non-self-similar extensions of the Sierpiński gasket.

1. Introduction. A *fractal set* is commonly defined as a compact subset K of some Polish space Ω with a fixed metric d satisfying

$$K = \bigcup_{i\in\mathscr{A}} F_i(K),$$

where $F_i : \Omega \to \Omega$, $i \in \mathscr{A} = \{1, \ldots, s\}$ $(s \in \mathbb{N})$, are continuous maps ([10]). In many cases, for example when all F_i are contractions, the fractal set K has a representation as a (continuous) factor of the one-sided shift space $\mathscr{A}^\infty$ such that each diagram

$$\begin{array}{ccc} \mathscr{A}^\infty & \xrightarrow{S_i} & \mathscr{A}^\infty \\ {\scriptstyle\Pi}\downarrow & & \downarrow{\scriptstyle\Pi} \\ K & \xrightarrow{F_i} & K \end{array}$$

commutes, where Π is continuous and $S_i : \mathscr{A}^\infty \to \mathscr{A}^\infty$ maps an infinite sequence $x_1, x_2, \ldots$ to the sequence $i, x_1, x_2, \ldots$.

The fractal set K is called *self-similar* if for each $i \in \mathscr{A}$ and all $x, y \in \Omega$,

$$d(F_i(x), F_i(y)) = r_i d(x, y)$$

for some $r_i \in (0,1)$, and it is called *post-critically finite* if it is connected and there exists a finite boundary set $V_0 \subset K$ such that

$$F_i(K) \cap F_j(K) = F_i(V_0) \cap F_j(V_0) \subset \Omega \setminus V_0 \quad (i \neq j)$$

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and each point in V_0 is a fixed point for some map F_l. For post-critically finite self-similar fractal sets the factor map is an almost topological isomorphism in the sense of [1], hence all Bernoulli measures μ on $\mathscr{A}^\infty$ can be considered on the quotient space. For such fractal sets and L_2-spaces the construction of Dirichlet forms has been investigated by several authors (cf. [10]).

In what follows we consider the abstract setting of the diagram disregarding the motivation arising from fractal geometry. We are interested in deriving Dirichlet forms on $L_2(\mu)$ which are determined by the equivalence relation given by the preimage relation of Π, thus extending previous constructions to non-fractal settings, and continuing our previous investigations in [2]–[5] and [7].

In order to describe the general framework of the present paper, consider an equivalence relation on the space of all infinite sequences of letters from a finite alphabet. This relation does not need to be shift invariant, as it is the case for post-critically finite fractal sets. We only consider those relations for which the Bernoulli measure with uniform marginals can be regarded as a probability measure on the quotient space. The representation by sequence spaces defines balls in a natural way and operators averaging over midpoints of neighbouring balls (in the quotient topology). This will be used to construct Dirichlet forms $\mathscr{E}$ on the L_2-space of the Bernoulli measure μ.

We recall the definition of a Dirichlet form. Let Ω be a locally compact separable Hausdorff space and μ be a positive Radon measure on Ω for which $\mathrm{Supp}(\mu) = \Omega$. A *Dirichlet form* $\mathcal{E}$ on $L_2(\Omega,\mu)$ is a non-negative definite symmetric bilinear form defined on a dense linear subspace $\mathrm{Dom}[\mathcal{E}] \subset L_2(\Omega,\mu)$, for which the following properties hold:

(1) $\mathcal{E}$ is *closed*, i.e. $\mathrm{Dom}[\mathcal{E}]$ is complete with respect to the metric which is induced by the form $\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + \int_\Omega fg\,d\mu$, $f,g \in \mathrm{Dom}[\mathcal{E}]$,
(2) $\mathcal{E}$ is *Markovian*, i.e. for each $\varepsilon > 0$, there exists a real function ϕ_ε on $\mathbb{R}$ such that
   - $\phi_\varepsilon(t) = t$ on $[0,1]$, $-\varepsilon \le \phi_\varepsilon(t) \le 1+\varepsilon$ on $\mathbb{R}$ and $0 \le \phi_\varepsilon(\widetilde{t}\,) - \phi_\varepsilon(t) \le \widetilde{t} - t$ for all $t < \widetilde{t}$,
   - for $f \in \mathrm{Dom}[\mathcal{E}]$ also $\phi_\varepsilon \circ f \in \mathrm{Dom}[\mathcal{E}]$ and $\mathcal{E}(\phi_\varepsilon \circ f, \phi_\varepsilon \circ f) \le \mathcal{E}(f,f)$.

The construction closely follows Friedrichs' extension procedure (see [6]), starting with a sufficiently rich class of functions on which the above averaging operator is defined. We show that this symmetric form is Markovian and closable, thus extending to a Dirichlet form. In fact, we formulate this result for an abstract measure m which may be the sum of the Bernoulli measure μ and a boundary measure ν. Such a splitting admits a decomposition

$$\mathscr{E}_m(\phi,\psi) = \mathscr{E}_\mu(\phi,\psi) + \mathscr{E}_\nu(\phi,\psi)$$

of the corresponding Dirichlet forms for certain continuous functions, which may be regarded as a generalized Green's formula. In particular, we thus obtain the Dirichlet and Neumann extensions.

By the spectral theory for self-adjoint operators the Laplacian Δ associated to a Dirichlet form is well defined by

$$\mathscr{E}_\varrho(\phi,\psi) = -\int \Delta(\phi)\cdot\psi\,d\varrho, \qquad \varrho\in\{m,\mu,\nu\},$$

whence the right hand side represents the classical "differential" operator for $\varrho=\mu$ and the negative of the Neumann derivative for $\varrho=\nu$. Since the Beurling–Deny conditions must be satisfied, we also see that for all $t>0$ the operators $e^{-t\Delta}$ are positivity-preserving, and consequently, define a reversible Markov process ([6]) which may be considered as Brownian motion on the quotient space.

In order to illustrate these ideas and definitions, consider Ω the unit interval $[0,1]$ and let $\mathscr{A}=\{0,1\}$. The equivalence relation $x_1\ldots x_n0111\cdots\sim x_1\ldots x_n1000\ldots$ defines the dyadic representation of reals in $[0,1]$ ($\mathscr{A}^\infty/\sim$ is isomorphic to Ω). A Dirichlet form on the L_2-space of Lebesgue measure is given by

$$\mathscr{E}(u,v)=\int u'(x)v'(x)\,dx=-\int u''(x)v(x)\,dx+u'(1)v(1)-u'(0)v(0)$$

($u\in C^2([0,1])$, $v\in C^1([0,1])$) with associated Laplace operator $\Delta u=u''$. This also illustrates the idea behind the analysis for the Sierpiński gasket in N dimensions in Section 6.

In Section 2 we set up the notations, assumptions and necessary definitions. Section 3 contains the basic result (Theorem 1) which gives conditions for the existence of a Dirichlet form on the space of square-integrable functions on the quotient space. In Section 4 we prove a Gauss–Green formula for the Laplace operator defined by this Dirichlet form. An example of an equivalence relation is considered in Section 5. We construct a suitable dense set of functions satisfying the assumptions of the main theorem and for the existence of the "differentiable" form of the Laplace operator. Applications are given in Section 6. The method immediately applies to the Sierpiński gasket and we derive as a special case Kigami's Laplace operator ([9]). Other, new examples are also obtained in Section 6.

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2. Preliminaries. Consider a finite alphabet $\mathscr{A}:=\{1,\ldots,N\}$, $N\ge 2$. The space of finite words over $\mathscr{A}$ is defined by $\mathscr{T}=\{\mathbf{w}=w_1\ldots w_n : 1\le w_j\le N,\, n\in\mathbb{N}\}\cup\{\emptyset\}$, where $\emptyset$ is the empty word consisting of no letter. Set $\mathscr{T}_+=\mathscr{T}\setminus\{\emptyset\}$. For $0\le n\le\infty$, let $\mathscr{A}^n$ denote the collection of words consisting of n symbols. The space $\mathscr{T} \cup \mathscr{A}^\infty$ has a metrizable, natural topology. For $\mathbf{x} \in \mathscr{A}^n$ and $0 \le n \le \infty$, the *length* of a word is defined by $d(\mathbf{x}) := n$. If $\mathbf{w} = w_1 \ldots w_n$ is a finite word, set $\mathbf{w}^- = w_1 \ldots w_{n-1}$. We denote by $\tau(\mathbf{w}) := w_n$ the last letter of $\mathbf{w}$. The *product* of $\mathbf{w} = w_1 \ldots w_n \in \mathscr{T}$ and $\mathbf{x} = x_1 x_2 \ldots \in \mathscr{T} \cup \mathscr{A}^\infty$ is defined by $\mathbf{wx} := w_1 \ldots w_n x_1 x_2 \ldots \in \mathscr{T} \cup \mathscr{A}^\infty$. Moreover, the nth *power* of a letter is defined by

$$a^n := \begin{cases} \emptyset & \text{if } n = 0, \\ \overbrace{a \ldots a}^{n \text{ times}} & \text{otherwise}, \end{cases}$$

where $0 \le n \le \infty$ and $a \in \mathscr{A}$.

If $\sim$ is an equivalence relation, we denote by $\langle \mathbf{x} \rangle$ the equivalence class of $\mathbf{x} \in \mathscr{A}^\infty$ and by

$$\Pi : \mathscr{A}^\infty \to \Sigma = \mathscr{A}^\infty/{\sim}$$

the quotient map Π onto the quotient space Σ.

Definition 1. An equivalence relation $\sim$ on $\mathscr{A}^\infty$ is called *thin* if there is an embedding $\gamma : \mathscr{T}_+ \to \Sigma$ such that

(1) $\gamma(\mathscr{T}_+)$ contains all points in Σ which, as an equivalence class, are of cardinality ≥ 2.
(2) Equivalence classes are finite and of uniformly bounded cardinality with upper bound $R \in \mathbb{N}$.
(3) For every $\mathbf{x} = x_1 x_2 \ldots \in \mathscr{A}^\infty$ we have

$$\lim_{n \to \infty} \gamma(x_1 \ldots x_n) = \langle \mathbf{x} \rangle \in \Sigma.$$

If $\sim$ is a thin equivalence relation the quotient space is metrizable, and we fix a metric $\bar{d}$ to describe the quotient topology on Σ.

If $\varphi \in C(\Sigma)$ we write $\varphi_\gamma = \varphi \circ \gamma$. A thin equivalence relation defines an equivalence relation on $\mathscr{T}$ by setting

$$\mathbf{w} \sim \mathbf{v} \quad \text{if and only if} \quad d(\mathbf{w}) = d(\mathbf{v}) \text{ and } \gamma(\mathbf{w}) = \gamma(\mathbf{v}).$$

The equivalence class of $\mathbf{v}$ will be denoted by $\langle \mathbf{v} \rangle \subset \mathscr{T}$. For a finite word $\mathbf{w} \in \mathscr{A}^n$ define

$$[\mathbf{w}] = \{\xi \in \Sigma : \exists x \in \xi \text{ such that } \mathbf{w} = x_1 \ldots x_n\}.$$

For example, the Sierpiński equivalence relation is defined by $\mathbf{x} = x_1 x_2 \ldots \sim \mathbf{y} = y_1 y_2 \ldots$ if there is an $n \in \mathbb{N}$ such that $x_m = y_m$ for all $1 \le m \le n-1$, $x_{n+k} = y_n$ and $y_{n+k} = x_n$ for all $k \ge 1$, and the embedding is $\gamma(\mathbf{w}) = \langle w_1 \ldots w_{d(\mathbf{w})} w_{d(\mathbf{w})}^\infty \rangle$. This equivalence relation has been studied by Denker and Sato in [4] and by Denker and Koch in [2]. In case $N = 2$ it reduces to the dyadic representation of reals in the unit interval. Imai considered another equivalence relation for the pentakun in [7].

LEMMA 1. *If* $\sim$ *is a thin equivalence relation, then the Bernoulli measure* μ *on* $\mathscr{A}^\infty$ *with uniformly distributed* 1*-dimensional marginals is mapped to* $\{\langle\xi\rangle \in \Sigma : |\langle\xi\rangle| = 1\}$ *via the map* $x \mapsto \langle x\rangle$. *Denote this image measure also by* μ.

Let $\mathbf{w} \in \mathscr{T}_+$ and $\varphi \in C(\Sigma)$ and define the operator $D : C(\Sigma) \to C(\mathscr{T}_+)$ by

$$(D\varphi)(\mathbf{w}) := \frac{1}{N}\sum_{i\in\mathscr{A}} \varphi_\gamma(\mathbf{w}^- i) - \varphi_\gamma(\mathbf{w}).$$

Let $\varrho = (\varrho_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive reals converging to zero. Call a function $\psi \in C(\Sigma)$ *ϱ-continuous* if there exists a constant c_ψ satisfying

$$|\psi(\Pi(\mathbf{u}x)) - \psi(\Pi(\mathbf{u}y))| \le c_\psi \varrho_n, \quad d(\mathbf{u}) = n,$$

and let $\mathcal{D}_\varrho$ denote the subspace consisting of these functions.

3. Dirichlet forms on quotient spaces. In this section we consider a thin equivalence relation on $\mathscr{A}^\infty$ and derive conditions for the existence of certain Dirichlet forms on L_2-spaces of measures on Σ.

To begin with, note the following easy fact:

LEMMA 2. *For any two functions* $\varphi, \psi : \Sigma \to \mathbb{R}$,

$$\begin{aligned}\sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle\mathbf{w}\rangle|}\Big\{\sum_{\mathbf{v}\in\langle\mathbf{w}\rangle} (D\varphi)(\mathbf{v})\Big\}\psi_\gamma(\mathbf{w}) &= \sum_{\mathbf{w}\in\mathscr{A}^n} (D\varphi)(\mathbf{w})\psi_\gamma(\mathbf{w}) \\ &= -\sum_{\mathbf{w}\in\mathscr{A}^n} (D\varphi)(\mathbf{w})(D\psi)(\mathbf{w}).\end{aligned}$$

Proof. Since $\mathbf{v} \sim \mathbf{w} \Rightarrow \gamma(\mathbf{v}) = \gamma(\mathbf{w})$, the claim follows easily from

$$\begin{aligned}\sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle\mathbf{w}\rangle|}\Big\{\sum_{\mathbf{v}\in\langle\mathbf{w}\rangle} (D\varphi)(\mathbf{v})\Big\}\psi_\gamma(\mathbf{w}) &= \sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\in\langle\mathbf{w}\rangle} (D\varphi)(\mathbf{v})\psi_\gamma(\mathbf{v}) \\ &= \sum_{\mathbf{v}\in\mathscr{A}^n} (D\varphi)(\mathbf{v})\psi_\gamma(\mathbf{v}),\end{aligned}$$

and

$$\sum_{\mathbf{v}\in\mathscr{A}^n} (D\varphi)(\mathbf{v})\frac{1}{N}\sum_{a\in\mathscr{A}} \psi_\gamma(\mathbf{v}^- a) = 0. \ \blacksquare$$

For all $n \in \mathbb{N}$, let I_n denote a positive real constant and let $\varrho = (\varrho_n)_{n\in\mathbb{N}}$. Let m, m_n denote probability measures on Σ such that $m_n([\mathbf{w}]) \in (0,\infty]$ for $\mathbf{w} \in \mathscr{A}^n$ and m_n converges weakly to m. Define $\alpha : \mathscr{T}_+ \to \mathbb{R}_+$ by $\alpha(\mathbf{w}) = 1/m_{d(\mathbf{w})}([\mathbf{w}])$. For $\varphi \in C(\Sigma)$ and $n \in \mathbb{N}$, let

$$\|\varphi\|^2_{\alpha,n} := \sum_{\mathbf{w}\in\mathscr{A}^n} \varphi_\gamma(\mathbf{w})^2\alpha(\mathbf{w})^{-1}.$$

It follows that

$$\lim_{n\to\infty}\sum_{\mathbf{w}\in\mathscr{A}^n}\varphi_\gamma(\mathbf{w})^2\alpha(\mathbf{w})^{-1}=\|\varphi\|_{L_2(m)}.$$

Proposition 1. *Let $\mathcal{D}$ denote the set of all functions $\phi\in\mathcal{D}_\varrho$ such that*

$$\lim_{n\to\infty} I_n^2\sum_{\mathbf{w}\in\mathscr{A}^n,\,a\in\mathscr{A}}\alpha(\mathbf{w})(\phi_\gamma(\mathbf{w})-\phi_\gamma(\mathbf{w}^-a))^4<\infty, \tag{3.1}$$

$$\sup_{n\in\mathbb{N}} I_n^2\sum_{\mathbf{w}\in\mathscr{A}^n}\alpha(\mathbf{w})\biggl(\frac{1}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\in\langle\mathbf{w}\rangle}(D\phi)(\mathbf{v})\biggr)^2<\infty. \tag{3.2}$$

Then $\mathcal{D}$ is a linear subspace of $L_2(\mu)$, and for any function $\phi\in\mathcal{D}$ and any twice differentiable function $g:\mathbb{R}\to\mathbb{R}$ with bounded first and second derivatives, also $g\circ\phi\in\mathcal{D}$. Moreover, for all $\phi\in\mathcal{D}$ there exists a constant C_ϕ such that for any $\psi\in C(\Sigma)$ and $n\in\mathbb{N}$,

$$\biggl|I_n\sum_{\mathbf{w}\in\mathscr{A}^n}(D\phi)(\mathbf{w})\psi_\gamma(\mathbf{w})\biggr|\le C_\phi\|\psi\|_{\alpha,n}. \tag{3.3}$$

Proof. Clearly, $\mathcal{D}$ is a linear subspace.

Let $\phi\in\mathcal{D}$ and $g:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function with bounded first and second derivative. By the mean value property of g it follows for $\mathbf{v},\mathbf{w}\in\mathscr{T}$ that

$$|g(\phi_\gamma(\mathbf{w}))-g(\phi_\gamma(\mathbf{v}))|\le\|g'\|_\infty|\phi_\gamma(\mathbf{w})-\phi_\gamma(\mathbf{v})|,$$

hence $g\circ\phi\in\mathcal{D}_\varrho$, and moreover, condition (3.1) holds for $g\circ\phi$.

By Taylor expansion, for n large enough and for some $\eta:\mathscr{T}_+\times\mathscr{A}\to\mathbb{R}_+$ with $\sup_{\mathbf{v}\in\mathscr{A}^n,\,a\in\mathscr{A}}\eta(\mathbf{v},a)\to 0$ as $n\to\infty$ it follows that

$$\begin{aligned}
&I_n^2\sum_{\mathbf{w}\in\mathscr{A}^n}\frac{\alpha(\mathbf{w})}{|\langle\mathbf{w}\rangle|^2}\Bigl\{\sum_{\mathbf{v}\in\langle\mathbf{w}\rangle}(D(g\circ\phi))(\mathbf{v})\Bigr\}^2\\
&\qquad=I_n^2\sum_{\mathbf{w}\in\mathscr{A}^n}\frac{\alpha(\mathbf{w})}{|\langle\mathbf{w}\rangle|^2}\biggl[\sum_{\mathbf{v}\in\langle\mathbf{w}\rangle}g'(\phi_\gamma(\mathbf{w}))(D\phi)(\mathbf{v})\\
&\qquad\qquad+\frac{1}{2N}\sum_{i\in\mathscr{A}}g''(\phi_\gamma(\mathbf{w})+\eta(\mathbf{v},i))\{\phi_\gamma(\mathbf{v})-\phi_\gamma(\mathbf{v}^-i)\}^2\biggr]^2\\
&\qquad\le 2\|g'\|^2I_n^2\sum_{\mathbf{w}\in\mathscr{A}^n}\alpha(\mathbf{w})\biggl(\frac{1}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\in\langle\mathbf{w}\rangle}(D\phi)(\mathbf{v})\biggr)^2\\
&\qquad\qquad+\frac{I_n^2}{2}\|g''\|^2\frac{1}{N}\sum_{\mathbf{w}\in\mathscr{A}^n}\alpha(\mathbf{w})\sum_{i\in\mathscr{A}}\{\phi_\gamma(\mathbf{w})-\phi_\gamma(\mathbf{w}^-i)\}^4.
\end{aligned}$$

Now (3.2) for $g\circ\phi$ follows from (3.1) and (3.2) for ϕ.

Finally, we show (3.3). Let $\phi \in \mathcal{D}$ and $\psi \in C(\Sigma)$. By Lemma 2 and Hölder's inequality,

$$\begin{aligned}
&\Big| I_n \sum_{\mathbf{w}\in\mathscr{A}^n} (D\phi)(\mathbf{w})\psi_\gamma(\mathbf{w})\Big| \\
&\quad = \Big| I_n \sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle\mathbf{w}\rangle|} \sum_{\mathbf{v}\in\langle\mathbf{w}\rangle} (D\phi)(\mathbf{v})\psi_\gamma(\mathbf{w})\Big| \\
&\quad \le \bigg\{ I_n^2 \sum_{\mathbf{w}\in\mathscr{A}^n} \alpha(\mathbf{w}) \bigg(\frac{1}{|\langle\mathbf{w}\rangle|} \sum_{\mathbf{v}\in\langle\mathbf{w}\rangle} (D\phi)(\mathbf{v})\bigg)^2 \sum_{\mathbf{w}\in\mathscr{A}^n} \psi_\gamma(\mathbf{w})^2\alpha(\mathbf{w})^{-1}\bigg\}^{1/2}.
\end{aligned}$$

The claim follows immediately from (3.2) and Lemma 1. ∎

LEMMA 3. *If* (3.2) *holds for* $\phi \in C(\Sigma)$, *then*

$$\lim_{n\to\infty} I_n \sum_{\mathbf{w}\in\mathscr{A}^n,\, a\in\mathscr{A}} (\phi_\gamma(\mathbf{w}) - \phi_\gamma(\mathbf{w}a))^4 = 0. \tag{3.4}$$

Proof. A direct calculation shows

$$\begin{aligned}
&I_n \sum_{\mathbf{w}\in\mathscr{A}^n,\, a\in\mathscr{A}} (\phi_\gamma(\mathbf{w}) - \phi_\gamma(\mathbf{w}a))^4 \\
&\quad \le \sup_{\mathbf{w}\in\mathscr{A}^n,\, b\in\mathscr{A}} |\phi_\gamma(\mathbf{w}) - \phi_\gamma(\mathbf{w}^- b)|^2 I_n \\
&\qquad \times \sum_{\mathbf{w}\in\mathscr{A}^n,\, a\in\mathscr{A}} (\phi_\gamma(\mathbf{w})^2 - 2\phi_\gamma(\mathbf{w})\phi_\gamma(\mathbf{w}^- a) + \phi_\gamma(\mathbf{w}^- a)^2) \\
&\quad = -2N \sup_{\mathbf{w}\in\mathscr{A}^n,\, b\in\mathscr{A}} |\phi_\gamma(\mathbf{w}) - \phi_\gamma(\mathbf{w}^- b)|^2 I_n \sum_{\mathbf{v}\in\mathscr{A}^n} \phi_\gamma(\mathbf{v})(D\phi)(\mathbf{v}) \\
&\quad = -2N \sup_{\mathbf{w}\in\mathscr{A}^n,\, b\in\mathscr{A}} |\phi_\gamma(\mathbf{w}) - \phi_\gamma(\mathbf{w}^- b)|^2 I_n \sum_{\mathbf{v}\in\mathscr{A}^n} \phi_\gamma(\mathbf{v}) \frac{1}{|\langle\mathbf{v}\rangle|} \sum_{\mathbf{u}\sim\mathbf{v}} (D\phi)(\mathbf{u}) \\
&\quad \le 2N \sup_{\mathbf{w}\in\mathscr{A}^n,\, b\in\mathscr{A}} |\phi_\gamma(\mathbf{w}) - \phi_\gamma(\mathbf{w}^- b)|^2 \\
&\qquad \times \bigg\{ \|\phi_\gamma\|_{\alpha,n} I_n^2 \sum_{\mathbf{v}\in\mathscr{A}^n} \alpha(\mathbf{v}) \bigg(\frac{1}{|\langle\mathbf{v}\rangle|} \sum_{\mathbf{u}\sim\mathbf{v}} (D\phi)(\mathbf{u})\bigg)^2\bigg\}^{1/2}.
\end{aligned}$$

This last expression converges to 0 because of (3.2) and continuity of ϕ. ∎

Whenever the limit exists, define

$$\begin{aligned}
\mathscr{E}(\varphi, \psi) &= -\lim_{n\to\infty} I_n \sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle\mathbf{w}\rangle|} \Big\{ \sum_{\mathbf{v}\in\langle\mathbf{w}\rangle} (D\varphi)(\mathbf{v}) \Big\} \psi_\gamma(\mathbf{w}) \\
&= \lim_{n\to\infty} I_n \sum_{\mathbf{w}\in\mathscr{A}^n} (D\varphi)(\mathbf{w})(D\psi)(\mathbf{w}).
\end{aligned} \tag{3.5}$$

THEOREM 1. *Let $\mathcal{D}_1 \subset \mathcal{D}$ generate an $L_2(m)$-dense subspace. Assume that for every $\varphi \in \mathcal{D}_1$ and every $\psi \in \mathcal{D}$ the form $\mathscr{E}(\varphi, \psi)$ in (3.5) exists. Then $(\mathscr{E}, \mathcal{D}_1)$ extends to a Dirichlet form on $L_2(m)$ with domain containing $\mathcal{D}$.*

Proof. We first remark that the form $\mathscr{E}(\varphi, \psi)$ extends to finite linear combinations of functions from $\mathcal{D}_1$. Therefore we may as well assume that $\mathcal{D}_1$ is a dense, linear subspace of $L_2(m)$. By (3.3) in Proposition 1, we can also extend the form $\mathscr{E}(\cdot\,,\cdot)$ to $\mathcal{D} \times C(\Sigma)$. Since $\mathscr{E}$ is symmetric by Lemma 2, for a Cauchy sequence $\varphi_l \in \mathcal{D}_1$ $(l \geq 1)$, converging to $\varphi \in C(\Sigma)$ in $L_2(m)$, the sequence $\mathscr{E}(\varphi_l, \psi)$ $(\psi \in \mathcal{D})$ is convergent:

$$\begin{aligned}|\mathscr{E}(\varphi_l - \varphi_k, \psi)| &= |\mathscr{E}(\psi, \varphi_l - \varphi_k)| \\ &= -\lim_{n\to\infty} I_n \sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle \mathbf{w}\rangle|} \sum_{\mathbf{v}\sim\mathbf{w}} (D\psi)(\mathbf{v})(\varphi_l - \varphi_k)_\gamma(\mathbf{w}) \\ &\leq C_\psi \lim_{n\to\infty} \|\varphi_l - \varphi_k\|_{\alpha,n} = C_\psi \|\varphi_l - \varphi_k\|_{L_2(m)}\end{aligned}$$

shows that $\mathscr{E}(\varphi_k, \psi)$ is a Cauchy sequence. We define its limit as

$$\mathscr{E}(\varphi, \psi) = \lim_{n\to\infty} \mathscr{E}(\varphi_n, \psi).$$

Moreover, by (3.3), for $\varphi \in \mathcal{D}$ and $\psi_k \in \mathcal{D}_1$ converging to $\psi \in \mathcal{D}$ we deduce that for every $k \in \mathbb{N}$,

$$\limsup_{n\to\infty} \Big| I_n \sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle \mathbf{w}\rangle|} \sum_{\mathbf{v}\sim\mathbf{w}} (D\varphi)(\mathbf{v})(\psi - \psi_k)_\gamma(\mathbf{w}) \Big| \leq C_\varphi \|\psi - \psi_k\|_{L_2(m)}.$$

Therefore

$$\mathscr{E}(\varphi, \psi) = \lim_{n\to\infty} I_n \sum_{\mathbf{w}\in\mathscr{A}^n} (D\varphi)(\mathbf{w})(D\psi)(\mathbf{w}), \qquad \varphi, \psi \in \mathcal{D}.$$

It follows that we may assume $\mathcal{D}_1 = \mathcal{D}$.

Clearly, $\mathscr{E}$ is bilinear, non-negative definite and symmetric by Lemma 2.

In order to show that $\mathscr{E}$ extends to a Dirichlet form it is sufficient to show that $(\mathscr{E}, \mathcal{D})$ is Markovian and closable (see e.g. [6]).

We begin showing the Markov property (see Section 1). By Proposition 1 again, $\mathcal{D}$ contains all functions of the form $g \circ \varphi$ for every $\varphi \in \mathcal{D}$ and every $g \in C^2(\mathbb{R})$ with bounded first and second derivative, in particular those satisfying

(1) $0 \leq g' \leq 1$,
(2) $-\varepsilon \leq g(t) \leq 1 + \varepsilon$ for some $\varepsilon > 0$,
(3) $g(t) = t$ for $0 \leq t \leq 1$.

Using Taylor expansion for g around $\varphi_\gamma(\mathbf{v})$ for each $\mathbf{v} \in \mathscr{A}^n$, and since $\|g'\|_\infty \leq 1$, we have

$$
\begin{aligned}
I_n & \sum_{\mathbf{v}\in\mathscr{A}^n} (Dg\circ\varphi)(\mathbf{v})^2 \\
&= I_n \sum_{\mathbf{v}\in\mathscr{A}^n} \Big\{ g'(\varphi_\gamma(\mathbf{v}))(D\varphi)(\mathbf{v}) + \frac{1}{2N}\sum_{a\in\mathscr{A}} g''(\theta_{\mathbf{v},a})(\varphi_\gamma(\mathbf{v})-\varphi_\gamma(\mathbf{v}^-a))^2\Big\}^2 \\
&\le I_n \sum_{\mathbf{v}\in\mathscr{A}^n} g'(\varphi_\gamma(\mathbf{v}))^2\{(D\varphi)(\mathbf{v})\}^2 \\
&\quad + I_n\Big\{\sum_{\mathbf{v}\in\mathscr{A}^n} \{(D\varphi)(\mathbf{v})\}^2 g'(\varphi_\gamma(\mathbf{v}))^2 \\
&\quad \times \sum_{\mathbf{v}\in\mathscr{A}^n}\sum_{a\in\mathscr{A}} g''(\theta_{\mathbf{v},a})^2(\varphi_\gamma(\mathbf{v})-\varphi_\gamma(\mathbf{v}^-a))^4\Big\}^{1/2} \\
&\quad + \frac{I_n}{4N}\sum_{\mathbf{v}\in\mathscr{A}^n}\sum_{a\in\mathscr{A}} g''(\theta_{\mathbf{v},a})^2(\varphi_\gamma(\mathbf{v})-\varphi_\gamma(\mathbf{v}^-a))^4 \\
&\le I_n \sum_{\mathbf{v}\in\mathscr{A}^n} \{(D\varphi)(\mathbf{v})\}^2 + O\Big(\Big[I_n \sum_{\mathbf{w}\in\mathscr{A}^n}\sum_{a\in\mathscr{A}}(\varphi_\gamma(\mathbf{w})-\varphi_\gamma(\mathbf{w}^-a))^4\Big]^{1/2}\Big),
\end{aligned}
$$

where $\theta_{\mathbf{v},a}$ is some value in the interval determined by $\varphi(\mathbf{v})$ and $\varphi(\mathbf{v}^-a)$. Letting $n\to\infty$ and applying (3.4) of Lemma 3 shows that $\mathscr{E}(g\circ\varphi, g\circ\varphi)\le \mathscr{E}(\varphi,\varphi)$.

Finally, we show that $\mathscr{E}$ is closable, i.e. if $\phi_l\in\mathcal{D}$, $\mathscr{E}(\phi_l-\phi_k,\phi_l-\phi_k)\to 0$ $(k,l\to\infty)$ and $\|\phi_l\|_2\to 0$ $(l\to\infty)$, then $\mathscr{E}(\phi_l,\phi_l)\to 0$ $(l\to\infty)$. Let $\phi_l\in\mathcal{D}$, $\phi_l\to 0$ in $L_2(m)$ and $\psi\in\mathcal{D}$. By (3.3), for some constant $C_\psi>0$, we obtain

$$
\begin{aligned}
|\mathscr{E}(\psi,\phi_l)| &= \lim_{n\to\infty}\Big| I_n \sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle\mathbf{w}\rangle|}\Big\{\sum_{\mathbf{v}\in\langle\mathbf{w}\rangle}(D\psi)(\mathbf{v})\Big\}\phi_l(\gamma(\mathbf{w}))\Big| \\
&\le CC_\psi\|\phi_l\|_{L_2(m)}.
\end{aligned}
$$

The last expression tends to 0 as $l\to\infty$. It is well known that this property is sufficient to prove that $\mathscr{E}$ is closable. ∎

The following fact has been shown in the previous proof.

Corollary 1. *The form $\mathscr{E}$ is well defined on $\mathcal{D}\times C(\Sigma)$.*

4. Gauss–Green formula. We define the notion of boundaries in Σ, continue with the definition of the Laplace operator Δ_Σ on the quotient space Σ and finally prove the Gauss–Green formula for this operator and the Dirichlet form constructed in the last section.

Definition 2. Let $\sim$ be a thin equivalence relation with embedding γ. A set $\partial\Sigma\subset\Sigma$ is called a *boundary* if it has the following properties:

(1) $\partial\Sigma$ is compact and nowhere dense.
(2) There is a set $\mathscr{T}_* \subset \mathscr{T}$ such that $\gamma(\mathscr{T}_*)$ is dense in $\partial\Sigma$ and satisfies $\langle \mathbf{w}\rangle \in \mathscr{T}_*$ for $\mathbf{w} \in \mathscr{T}_*$.
(3) For each $n \in \mathbb{N}$ the set

$$U_n = \{\xi \in \Sigma : \exists x \in \xi \text{ such that } x_1 \dots x_n \in \mathscr{T}_*\}$$

is a neighbourhood of $\partial\Sigma$.

Note that $\bigcap_{n\in\mathbb{N}} U_n = \partial\Sigma$, since for $\Sigma \ni \eta \in [\mathbf{w}^n] \subset U_n$ there is some $\mathbf{x} \in \eta$ with $x_1 \dots x_n = \mathbf{w}^n$, whence

$$\bar{d}(\eta, \gamma(\mathbf{w}^n)) \le \bar{d}(\langle \mathbf{x}\rangle, \gamma(\mathbf{w}^n)) \to 0.$$

Define

$$\mathscr{A}_*^n = \{\mathbf{u} \in \mathscr{A}^n : \forall a, b \in \mathscr{A},\ \mathbf{w} \in \mathscr{T}_*,\ \mathbf{v} \sim \mathbf{u} \Rightarrow \mathbf{v}^- b \not\sim \mathbf{w}^- a\}.$$

LEMMA 4.

(1) *Let μ denote the Bernoulli measure with uniformly distributed one-dimensional marginals, and suppose that $\mu(\partial\Sigma) = 0$. Then $V_n = \bigcup_{\mathbf{v}\notin\mathscr{A}_*^n} [\mathbf{v}]$ satisfies $U_n \subset V_n$ and* $\lim_{n\to\infty} \mu(V_n) = 0$.
(2) *If $\mathbf{u} \in \mathscr{A}_*^n$ and $\mathbf{v} \sim \mathbf{u}$, then $\mathbf{v} \in \mathscr{A}_*^n$.*
(3) *If $\mathbf{u} \in \mathscr{A}_*^n$ and $a \in \mathscr{A}$, then $\mathbf{u}^- a \notin \mathscr{T}_*$.*

Proof. Since

$$V_n = \bigcup_{\mathbf{w}\in\mathscr{A}^n\cap\mathscr{T}_*} \bigcup_{a\in\mathscr{A}} \bigcup_{\mathbf{v}\sim\mathbf{w}^- a} \bigcup_{b\in\mathscr{A}} \bigcup_{\mathbf{u}\sim\mathbf{v}^- b} [\mathbf{u}]$$

one immediately deduces

$$\mu(V_n) \le |\mathscr{A}^n \cap \mathscr{T}_*| N^{-n} N^2 R^2 \le N^2 R^2 \mu\Big(\bigcup_{v\in\mathscr{A}^n\cap\mathscr{T}_*} [v]\Big) \to 0$$

as $\mu(\partial\Sigma) = 0$ and μ is regular.

(2) and (3) are immediate consequences of the definition of $\mathscr{A}_*^n$. ∎

Let $\mathcal{D}_1$ be a dense set of ϱ-continuous functions $\psi \in C(\Sigma)$ with some fixed sequence $\varrho = (I_n^{-1})_{n\in\mathbb{N}}$.

DEFINITION 3. Suppose $\partial\Sigma$ is a boundary for the equivalence relation $\sim$, and m is a Borel measure on Σ. Define $\mu(A) = m(A \setminus \partial\Sigma)$ and $\nu(A) = m(A \cap \partial\Sigma)$. We call $m = \mu + \nu$ the *splitting* of m.

(1) Let $\varrho = (I_n^{-1})_{n\in\mathbb{N}}$. The *Laplace operator* Δ_Σ on Σ is defined on $\mathrm{Dom}(\Delta_\Sigma) \subset \mathcal{D}_\varrho \subset L_2(\mu)$ such that $\Delta_\Sigma \varphi = \psi$ if

$$\lim_{n\to\infty} I_n N^n \frac{1}{|\langle \mathbf{w}\rangle|} \sum_{\mathbf{v}\sim\mathbf{w}} (D\varphi)(\mathbf{v}) = \psi(\eta) \tag{4.1}$$

uniformly in $\mathbf{w} \in \mathscr{A}_*^n$ converging to some $\eta \in \Sigma$.

(2) Let $\phi \in C(\Sigma)$. A function $d\phi/dn \in L_2(\nu)$ is called the *Neumann derivative* of ϕ if for every $x \in \xi \in \partial\Sigma$ (in fact the following limit is independent of the choice of a representative in ξ),

$$\frac{d\phi}{dn}(\xi) = -\lim_{n\to\infty} \frac{I_n|\mathscr{T}_* \cap \mathscr{A}^n|}{|\langle x_1 \ldots x_n\rangle|} \sum_{\mathbf{v}\in\langle x_1\ldots x_n\rangle} (D\varphi)(\mathbf{v}).$$

THEOREM 2. *Let $\partial\Sigma$ be a boundary and m be a Borel measure on Σ with splitting $m = \mu + \nu$. Assume that μ is the Bernoulli measure on Σ with uniformly distributed one-dimensional marginals, and that ν is the weak limit of the measures*

$$\nu_n = \frac{1}{|\mathscr{T}_* \cap \mathscr{A}^n|} \sum_{\mathbf{w}\in\mathscr{T}_*\cap\mathscr{A}^n} \delta_{\mathbf{w}}.$$

Let $C_0 \subset C(\Sigma)$ be a class of functions with the following properties:

(a) C_0 (*considered as a subset of* $L_2(\mu)$) *is* $L_2(\mu)$*-dense and contained in* $\mathrm{Dom}(\Delta_\Sigma)$.
(b) C_0 (*considered as a subset of* $L_2(\nu)$) *is* $L_2(\nu)$*-dense and the Neumann derivative exists for* $\xi \in \partial\Sigma$ *and* $\phi \in C_0$.
(c) *Letting* $\mathscr{T}_*^n = \mathscr{A}^n \setminus (\mathscr{T}_* \cup \mathscr{A}_*^n)$, *the sequence* K_n $(n \in \mathbb{N})$ *defined by*

$$K_n^{-1} := |\mathscr{T}_*^n| \sum_{\mathbf{w}\in\mathscr{T}_*^n} \left(\frac{I_n}{|\langle\mathbf{w}\rangle|} \sum_{\mathbf{v}\sim\mathbf{w}} (D\phi)(\mathbf{v}) \right)^2$$

diverges to ∞.

If for each $\varphi \in C_0$,

$$\sup_{n\in\mathbb{N}} \sum_{\mathbf{w}\in\mathscr{A}_*^n,\, a\in\mathscr{A}} N^n(\varphi_\gamma(\mathbf{w}) - \varphi_\gamma(\mathbf{w}^- a))^2 + \sum_{\mathbf{w}\notin\mathscr{A}_*^n,\, a\in\mathscr{A}} |\mathscr{T}_* \cap \mathscr{A}^n|(\varphi_\gamma(\mathbf{w}) - \varphi_\gamma(\mathbf{w}^- a))^2 < \infty,$$

then

$$\mathscr{E}(\phi,\psi) = -\int \Delta_\Sigma \phi\psi \, d\mu + \int \frac{d\phi}{dn}\, \psi \, d\nu$$

($\phi \in C_0$, $\psi \in C(\Sigma)$) *extends to a Dirichlet form on* $L_2(m)$.

Proof. Let $\phi \in C_0$ and $\psi \in C(\Sigma)$. Since by Lemma 4, $\mathscr{A}_*^n$ contains complete equivalence classes, it follows that

$$\sum_{\mathbf{w}\in\mathscr{A}^n} (D\phi)(\mathbf{w})\psi_\gamma(\mathbf{w}) = \sum_{\mathbf{w}\in\mathscr{A}^n} \frac{1}{|\langle\mathbf{w}\rangle|} \sum_{\mathbf{v}\sim\mathbf{w}} (D\phi)(\mathbf{v})\psi_\gamma(\mathbf{w}),$$

$$\sum_{\mathbf{w}\notin\mathscr{A}_*^n} (D\phi)(\mathbf{w})\psi_\gamma(\mathbf{w}) = \sum_{\mathbf{w}\notin\mathscr{A}_*^n} \frac{1}{|\langle\mathbf{w}\rangle|} \sum_{\mathbf{v}\sim\mathbf{w}} (D\phi)(\mathbf{v})\, \psi_\gamma(\mathbf{w}).$$

Therefore

$$\begin{aligned}\Big| I_n \sum_{\mathbf{w}\in\mathscr{A}^n} &(D\phi)(\mathbf{w})\psi_\gamma(\mathbf{w})\\ &- \sum_{\mathbf{w}\notin\mathscr{A}_*^n} I_n(D\phi)(\mathbf{w})\psi_\gamma(\mathbf{w}) - N^{-n}\sum_{\mathbf{w}\in\mathscr{A}^n} \varDelta_\Sigma\phi(\gamma(\mathbf{w}))\psi_\gamma(\mathbf{w})\Big|\\ &\le N^{-n}\sum_{\mathbf{w}\in\mathscr{A}_*^n} |I_nN^n(D\phi)(\mathbf{w}) - \varDelta_\Sigma\phi(\gamma(\mathbf{w}))|\,|\psi_\gamma(\mathbf{w})|\\ &\quad + N^{-n}\sum_{\mathbf{w}\notin\mathscr{A}_*^n} |\varDelta_\Sigma\phi(\gamma(\mathbf{w}))\psi_\gamma(\mathbf{w})|.\end{aligned}$$

This bound tends to zero as $n\to\infty$, showing that

$$\begin{aligned}\mathscr{E}(\phi,\psi) &= -\lim_{n\to\infty} I_n \sum_{\mathbf{w}\in\mathscr{A}^n}(D\phi)(\mathbf{w})\psi_\gamma(\mathbf{w})\\ &= -\int \varDelta_\Sigma\phi(\xi)\psi(\xi)\,\mu(d\xi) - \lim_{n\to\infty} I_n\sum_{\mathbf{w}\in\mathscr{T}_*^n}\frac{1}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\sim\mathbf{w}}(D\phi)(\mathbf{v})\psi_\gamma(\mathbf{w})\\ &\quad - \lim_{n\to\infty} I_n|\mathscr{T}_*\cap\mathscr{A}^n|\sum_{\mathbf{w}\in\mathscr{T}_*\cap\mathscr{A}^n}\frac{1}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\sim\mathbf{w}}(D\phi)(\mathbf{v})\psi_\gamma(\mathbf{w})\,\frac{1}{|\mathscr{T}_*\cap\mathscr{A}^n|}\\ &= -\int \varDelta_\Sigma\phi(\xi)\psi(\xi)\,\mu(d\xi) + \int\frac{d\phi}{dn}(\xi)\psi(\xi)\,\nu(d\xi).\end{aligned}$$

Setting $\alpha(\mathbf{w}) = N^n$ for $\mathbf{w}\in\mathscr{A}_*^n$, $\alpha(\mathbf{w}) = |\mathscr{T}_*\cap\mathscr{A}^n|$ for $\mathbf{w}\in\mathscr{T}_*\cap\mathscr{A}^n$ and $\alpha(\mathbf{w}) = K_n|\mathscr{T}_*\cap\mathscr{A}^n|$ otherwise, we find that m_n converges weakly to m (since $|\mathscr{T}_*\cap\mathscr{A}^n|$ and $|(\mathscr{A}_*^n)^{\mathrm{c}}|$ are of the same order). Formula (3.2) follows from

$$\begin{aligned}I_n^2 \sum_{\mathbf{w}\in\mathscr{A}^n}&\alpha(\mathbf{w})\Big(\frac{1}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\sim\mathbf{w}}(D\phi)(\mathbf{v})\Big)^2\\ &= N^{-n}\sum_{\mathbf{w}\in\mathscr{A}_*^n}\Big(\frac{I_nN^n}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\sim\mathbf{w}}(D\phi)(\mathbf{v})\Big)^2\\ &\quad + \frac{1}{|\mathscr{T}_*\cap\mathscr{A}^n|}\sum_{\mathbf{w}\in\mathscr{T}_*\cap\mathscr{A}^n}\Big(\frac{I_n|\mathscr{T}_*\cap\mathscr{A}^n|}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\sim\mathbf{w}}(D\phi)(\mathbf{v})\Big)^2\\ &\quad + K_n|\mathscr{T}_*\cap\mathscr{A}^n|\sum_{\mathbf{w}\in\mathscr{T}_*^n}\Big(\frac{I_n}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{v}\sim\mathbf{w}}(D\phi)(\mathbf{v})\Big)^2 < \infty,\end{aligned}$$

where we use (a)–(c) and the fact that $|\mathscr{T}_*\cap\mathscr{A}^n|$ and $|(\mathscr{A}_*^n)^{\mathrm{c}}|$ are of the same order.

Formula (3.1) for $\varphi\in C_0$ follows immediately from the assumption, since φ is ϱ-continuous.

We have shown that $\mathcal{D}_1 = C_0$ satisfies the assumptions in Theorem 1. Hence $\mathscr{E}$ extends to a Dirichlet form on $L_2(m)$. ■

The form $\mathscr{E}$ in the last theorem may be considered also as a form on $L_2(\mu)$ since continuous functions in C_0 are dense in $L_2(\mu)$ and $L_2(\nu)$. We therefore obtain

Corollary 2 (Gauss–Green formula). *Let* $\varphi, \psi \in \mathrm{Dom}(\Delta_\Sigma) \subset L_2(\mu)$. *Then*

$$\int_\Sigma \{\varphi(\xi)(\Delta_\Sigma\psi)(\xi) - (\Delta_\Sigma\varphi)(\xi)\psi(\xi)\}\,\mu(d\xi) = \frac{1}{2}\int_{\partial\Sigma}\left(\varphi(\xi)\,\frac{d\psi}{dn}(\xi) - \frac{d\varphi}{dn}(\xi)\psi(\xi)\right)\nu(d\xi).$$

5. An example. We consider the equivalence relation $\sim$ on Σ defined by $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{x} = \mathbf{y}$ or if there exist $a \neq b \in \mathscr{A}$ and $n \geq 0$ such that

$$\mathbf{x} = x_1 \dots x_n ab^\infty \quad \text{and} \quad \mathbf{y} = x_1 \dots x_n ba^\infty,$$

together with the embedding

$$\gamma(w_1 \dots w_n) = \langle w_1 \dots w_n w_n^\infty \rangle.$$

The boundary is defined by $\mathscr{T}_* = \{\mathbf{w} = a^n : a \in \mathscr{A},\ n \geq 0\}$, hence $\partial\Sigma = \{a^\infty : a \in \mathscr{A}\}$.

As an example consider the case $N = 2$. Then Σ can be identified with the unit interval and the boundary is $\{0, 1\}$. By the definition of D we deduce for a twice continuously differentiable function φ that

$$(D\varphi)(\mathbf{w}ij) = \frac{1}{2}\,(\varphi(y) - \varphi(x)),$$

where $x = \mathbf{w}ijjj\dots$ and $y = \mathbf{w}iiii\dots$. It follows that

$$2^{d(\mathbf{w})+2} D(\mathbf{w}ij) = \frac{\varphi(y) - \varphi(x)}{|x-y|} \to \pm\varphi'(\xi)$$

as $n \to \infty$ and $x \to \xi$. Moreover,

$$\text{(5.1)}\qquad 2^{2d(\mathbf{w})+3}((D\varphi)(\mathbf{w}ij) + (D\varphi)(\mathbf{w}ji)) = 2^{d(\mathbf{w})+1}\left(\frac{\varphi(y) - \varphi(x)}{|x-y|} + \frac{\varphi(y') - \varphi(x')}{|x'-y'|}\right),$$

where $y' = \mathbf{w}jjjj\dots$ and $x' = \mathbf{w}jiii\dots$. As $n \to \infty$ and $x \to \xi \in [0,1]$, it follows that $x' \to \xi$. Moreover, either $y > x$ and $y' < x'$ or vice versa, whence (5.1) converges to $\varphi''(\xi)$. This proves that the construction in this section in case $N = 2$ recovers the usual Laplace operator on the L_2-space of Lebesgue measure on the unit interval.

In this section we construct a set of functions satisfying conditions (3.1), (3.2) and (3.5). In addition the functions belong to $\mathcal{D}_\varrho$ for some ϱ and finite linear combinations thereof are dense in $L_2(\mu)$ and in $L_2(m)$, where $m=\mu+\nu$ with the counting measure $\nu=\sum_{a\in\mathscr{A}}\varepsilon_{a^\infty}$ on $\partial\Sigma$.

We begin with two numbers $\alpha,\beta\in(0,1)$ satisfying $2\alpha+N\beta=1$ and define $I_n=(1-\alpha-\beta)^{-n}$. The first lemmas are straightforward calculations and their proofs are left to the reader.

LEMMA 5. *Let $p_a\in\mathbb{R}$ $(a\in\mathscr{A})$. Define recursively $f(a)=p_a$ for $a\in\mathscr{A}$, and for $\mathbf{w}\in\mathscr{T}$ and $i,j\in\mathscr{A}$ let*

$$f(\mathbf{w}ij)=\begin{cases} f(\mathbf{w}i) & \text{if } i=j,\\ \alpha\sum_{\mathbf{v}\sim\mathbf{w}ij}f(\mathbf{v}^-)+\beta\sum_{c\in\mathscr{A}}f(\mathbf{w}c) & \text{if } i\neq j.\end{cases}$$

Then

(1) *There exists a function $\varphi=\varphi^f\in C(\Sigma)$ such that*
   (a) $\varphi(\langle\mathbf{w}a^\infty\rangle)=f(\mathbf{w}a)$ *for all* $a\in\mathscr{A}$, $\mathbf{w}\in\mathscr{T}$;
   (b) $|\varphi(\langle\mathbf{v}\xi_1\xi_2\ldots\rangle)-\varphi(\langle\mathbf{v}\eta_1\eta_2\ldots\rangle)|\le 2(1-\alpha-\beta)^{d(\mathbf{v})}\|\varphi\|_\infty$, *where* $\xi_k,\eta_k\in\mathscr{A}$, $k\ge 1$.

(2) *We have*

$$(D\varphi)(\mathbf{w}ij)=\begin{cases}\left(\frac{N-1}{N}-\frac{N-2}{N}\alpha\right)(D\varphi)(\mathbf{w}i) & \text{if } i=j,\\ \alpha(D\varphi)(\mathbf{w}j)-\beta(D\varphi)(\mathbf{w}i) & \text{if } i\neq j.\end{cases}$$

(3) *If $i\neq j$ then*

$$(D\phi)(\mathbf{w}ij)+(D\phi)(\mathbf{w}ji)=0.$$

COROLLARY 3. *Let $\varphi\in C(\Sigma)$ satisfy, for all $\mathbf{w}\in\mathscr{T}$, and $i,j\in\mathscr{A}$,*

$$\varphi_\gamma(\mathbf{w}ij)=\begin{cases}\varphi_\gamma(\mathbf{w}i)+O((1-\alpha-\beta)^{d(\mathbf{w})}) & \text{if } i=j,\\ \alpha\{\varphi_\gamma(\mathbf{w}i)+\varphi_\gamma(\mathbf{w}j)\} & \\ \qquad+\beta\sum_{c\in\mathscr{A}}\varphi_\gamma(\mathbf{w}c)+O((1-\alpha-\beta)^{2d(\mathbf{w})}) & \text{if } i\neq j.\end{cases}$$

Then

(1) $|\varphi(\langle\mathbf{v}\xi_1\xi_2\ldots\rangle)-\varphi(\langle\mathbf{u}\mathbf{v}\eta_1\eta_2\ldots\rangle)|=O((1-\alpha-\beta)^{d(\mathbf{v})}\|\varphi\|_\infty)$, *where* $\xi_k,\eta_k\in\mathscr{A}$ *for all* $k\in\mathbb{N}$.

(2) *We have*

$$(D\varphi)(\mathbf{w}ij)=\begin{cases}\left(\frac{N-1}{N}-\frac{N-2}{N}\alpha\right)(D\varphi)(\mathbf{w}i)+O((1-\alpha-\beta)^{d(\mathbf{w})}) & \text{if } i=j,\\ \alpha(D\varphi)(\mathbf{w}j)-\beta(D\varphi)(\mathbf{w}i)+O((1-\alpha-\beta)^{d(\mathbf{w})}) & \text{if } i\neq j,\end{cases}$$

where i, $j\in\mathscr{A}$ and $\mathbf{w}\in\mathbf{u}\mathscr{T}$.

LEMMA 6. *Let $\varphi,\psi\in C(\Sigma)$ satisfy the condition of Lemma 5(2) for all $\mathbf{w}\in\mathscr{T}$ and $i,j\in\mathscr{A}$. Then, for $n\ge 2$,*

$$I_n \sum_{\mathbf{w}\in\mathscr{A}^n} (D\varphi)(\mathbf{w})(D\psi)(\mathbf{w})$$
$$= I_{n-1} \sum_{\mathbf{w}\in\mathscr{A}^{n-1}} (D\varphi)(\mathbf{w})(D\psi)(\mathbf{w})\left(1+\frac{N-2}{N}\{(N+2)\alpha-1\}^2\right).$$

If (2) *in Corollary* 3 *holds then this equality remains valid up to the order of* $(1-\alpha-\beta)^{d(\mathbf{w})}$.

Corollary 4. *Let* φ,ψ *satisfy the assumptions in Lemma* 6. *Then*

$$\lim_{n\to\infty} I_n \sum_{\mathbf{w}\in\mathscr{A}^n} (D\varphi)(\mathbf{w})(D\psi)(\mathbf{w})$$

exists if and only if $\alpha = (N+2)^{-1}$ $(=\beta)$.

In what follows we fix $\alpha=\beta=1/(N+2)$. For every $b\in\mathscr{A}$ let f_b denote the function defined in Lemma 5 for the boundary conditions $p_a=0$, $a\in\mathscr{A}$, $a\neq b$, and $p_b=1$. Let $\overline{\varphi}_b := \varphi^{f_b}$, as defined in (1) of Lemma 5, and consider this function defined on $\mathscr{A}^\infty$ by setting

$$\varphi_b(\mathbf{x}) = \overline{\varphi}_b(\Pi(\mathbf{x})) \quad (\mathbf{x}\in\mathscr{A}^\infty).$$

Fix a word $\mathbf{u}\in\mathscr{T}_+$ and define $\varphi_{\mathbf{u}}:\mathscr{A}^\infty\to\mathbb{R}$ by

$$\varphi_{\mathbf{u}}(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x}\notin[\mathbf{u}^-],\\ \varphi_{\tau(\mathbf{u})}(\mathbf{y}) - \frac{1}{2}\mathbb{I}_{\mathbf{x}=\mathbf{u}\tau(\mathbf{u})^\infty} & \text{if } \mathbf{x}=\mathbf{u}^-\mathbf{y}.\end{cases}$$

It is also easy to see that for $\mathbf{w}\in\mathscr{T}$ and $i,j\in\mathscr{A}$ with $d(\mathbf{w})+2\neq d(\mathbf{u})$,

$$\varphi_{\mathbf{u}}(\mathbf{w}ij^\infty) = \begin{cases} \varphi_{\mathbf{u}}(\mathbf{w}i^\infty) & \text{if } i=j,\\ \alpha\{\varphi_{\mathbf{u}}(\mathbf{w}i^\infty)+\varphi_{\mathbf{u}}(\mathbf{w}j^\infty)+\sum_{c\in\mathscr{A}}\varphi_{\mathbf{u}}(\mathbf{w}c^\infty)\} & \text{if } i\neq j.\end{cases}$$

Let $\mathscr{T}_{\neq}$ denote the set of all words in $\mathscr{T}$ for which the last two letters are different or which are of length ≤ 1. A function $\Phi:\mathscr{A}^\infty\to\mathbb{R}$ of the form

$$\Phi = \sum_{\mathbf{u}\in\mathscr{T}_{\neq}} \beta(\mathbf{u})(N+2)^{-d(\mathbf{u})}\varphi_{\mathbf{u}}$$

is well defined. Such a function defines a function on Σ if for each $\mathbf{u}\in\mathscr{T}_{\neq}$ we have $\beta(\mathbf{v})=\beta(\mathbf{u})$ for all $\mathbf{v}\in\langle\mathbf{u}\rangle$, because $\sum_{\mathbf{v}\in\langle\mathbf{u}\rangle}\varphi_{\mathbf{v}}$ is constant on the set $\{\mathbf{v}\tau(\mathbf{v})^\infty : \mathbf{v}\in\langle\mathbf{u}\rangle\}$. As a function on Σ, Φ is continuous as long as the coefficients β are uniformly bounded. In fact it has the same modulus of continuity as $\varphi_{\mathbf{u}}$.

We need to construct the coefficients $\beta(\mathbf{u})$ such that

(i) $\beta(\mathbf{u})$ are bounded and constant on equivalence classes.

(ii) Conditions (3.1), (3.2) and (3.5) are satisfied for the associated function Φ.

(iii) For a set $[\mathbf{u}]\cup[\mathbf{u}^\#]$ and $\varepsilon>0$, the associated function Φ satisfies

$$\|\Phi - \mathbb{I}_{[\mathbf{u}]\cup[\mathbf{u}^\#]}\|_{L_2(\mu)} < \varepsilon.$$

First note that (3.5) has been shown in Corollary 4 for $\varphi_{\mathbf{v}}$, hence also for finite linear combinations and also for Φ by the absolute convergence of the series. We start with the equation

$$\beta(\mathbf{u}) = -\frac{N}{|\langle\mathbf{u}\rangle|}\sum_{\mathbf{v}\in\langle\mathbf{u}\rangle} z(\mathbf{v})$$

and need to determine $z(\mathbf{v})$. Note that (i) is satisfied if the coefficients $z(\mathbf{v})$ are uniformly bounded. In the following calculations we suppress the index γ, thus for a function $g:\Sigma\to\mathbb{R}$ we write $g(\mathbf{w})$ for $g(\gamma(\mathbf{w}))$.

A direct calculation shows that for $k\geq 2$, $\mathbf{v}\in\mathscr{A}^n$ and $a,c\neq b$,

$$\begin{aligned}
&\Phi_\gamma(\mathbf{v}ab^{k-1}c) - \Phi_\gamma(\mathbf{v}ab^k)\\
&\quad= \sum_{\mathbf{u}\in\mathscr{T}_{\neq},\, d(\mathbf{u})\leq d(\mathbf{v})+1} \frac{\beta(\mathbf{u})}{(N+2)^{d(\mathbf{u})}}\left(\varphi_{\mathbf{u}}(\mathbf{v}ab^{k-1}c) - \varphi_{\mathbf{u}}(\mathbf{v}ab^k)\right)\\
&\qquad+\sum_{j=0}^{k-1}\sum_{b\neq d\in\mathscr{A}} \frac{\beta(\mathbf{v}ab^jd)}{(N+2)^{n+j+2}}\,\varphi_{\mathbf{v}ab^jd}(\mathbf{v}ab^{k-1}c)\\
&\quad= \sum_{\mathbf{u}\in\mathscr{T}_{\neq},\, d(\mathbf{u})\leq d(\mathbf{v})+1} \frac{\beta(\mathbf{u})}{(N+2)^{d(\mathbf{u})+1}}\Big(\varphi_{\mathbf{u}}(\mathbf{v}ab^{k-1}) + \varphi_{\mathbf{u}}(\mathbf{v}ab^{k-2}c)\\
&\qquad+\sum_{d\in\mathscr{A}}\varphi_{\mathbf{u}}(\mathbf{v}ab^{k-2}d) - (N+2)\varphi_{\mathbf{u}}(\mathbf{v}ab^{k-1})\Big)\\
&\qquad+\sum_{j=0}^{k-2}\sum_{b\neq d\in\mathscr{A}} \frac{\beta(\mathbf{v}ab^jd)}{(N+2)^{n+j+3}}\Big(\varphi_{\mathbf{v}ab^jd}(\mathbf{v}ab^{k-1})\\
&\qquad+\varphi_{\mathbf{v}ab^jd}(\mathbf{v}ab^{k-2}c) + \sum_{e\in\mathscr{A}}\varphi_{\mathbf{v}ab^jd}(\mathbf{v}ab^{k-2}e)\Big)(N+2)^{-n-k-1}\beta(\mathbf{v}ab^{k-1}c)\\
&\quad= (N+2)^{-1}(\Phi(\mathbf{v}ab^{k-2}c) - \Phi(\mathbf{v}ab^{k-1}) + N(D\Phi)(\mathbf{v}ab^{k-1}))\\
&\qquad+\frac{\beta(\mathbf{v}ab^{k-1}c)}{(N+2)^{n+k+1}}.
\end{aligned}$$

Summing over $c\neq b$ yields

$$\begin{aligned}
&(N+2)^{d(\mathbf{v})+k+1}(D\Phi)(\mathbf{v}ab^k)\\
&= N(N+2)^{d(\mathbf{v})+k}(D\Phi)(\mathbf{v}ab^{k-1}) + \frac{1}{N}\sum_{c\neq b}\beta(\mathbf{v}ab^{k-1}c)\\
&= N(N+2)^{d(\mathbf{v})+k}(D\Phi)(\mathbf{v}ab^{k-1}) - \frac{1}{2}\sum_{c\neq b}[z(\mathbf{v}ab^{k-1}c) + z(\mathbf{v}ab^{k-2}cb)].
\end{aligned}$$

Setting

$$z(\mathbf{v}ab^k) := (N+2)^{d(\mathbf{v})+k+1}(D\Phi)(\mathbf{v}ab^k)$$

we obtain

$$z(\mathbf{v}ab^{k-1}) = \frac{1}{N}\, z(\mathbf{v}ab^k) + \frac{1}{2N} \sum_{c\neq b} [z(\mathbf{v}ab^{k-1}c) + z(\mathbf{v}ab^{k-2}cb)],$$

which is a convex combination determining $z(\mathbf{v}ab^{k-1})$.

We sketch the construction of functions solving the associated Dirichlet problem. Consider functions ψ on Σ which are of the form

$$(\psi \circ \Pi)(\xi) = \sum_{\substack{\mathbf{v}\in\mathscr{T} \\ \tau(\mathbf{v})\neq\tau(\mathbf{v}^-)}} \alpha_{\mathbf{v}}(N+2)^{-d(\mathbf{v})}\varphi_{\mathbf{v}}(\xi),$$

where

$$\alpha_{\mathbf{v}} = \frac{-1}{N+2}\Big[(N+6)z(\mathbf{v}) + 3 \sum_{\substack{c\in\mathscr{A} \\ c\notin\{\tau(\mathbf{v}),\tau(\mathbf{v}^{\#})\}}} \{z(\mathbf{v}^-c) + z(\mathbf{v}^{\#-}c)\} + \sum_{\substack{c,d\in\mathscr{A},\, c\neq d \\ c,d\notin\{\tau(\mathbf{v}),\tau(\mathbf{v}^{\#})\}}} z((\mathbf{v}^-)^-cd)\Big], \tag{5.2}$$

and where $z : \mathscr{T} \to \mathbb{R}$ is some continuous function. Note that the series

$$\sum_{\substack{\mathbf{v}\in\mathscr{T} \\ \tau(\mathbf{v})\neq\tau(\mathbf{v}^-)}} \alpha_{\mathbf{v}}(N+2)^{-d(\mathbf{v})}$$

converges absolutely, hence $\psi \in C(\Sigma)$. We know that $\Delta_\Sigma\psi = z$ (see [5]).

Let $\mathcal{D}_1$ denote the space of functions constructed above. We show that $\mathcal{D}$ is a dense linear subspace. By definition, $\mathcal{D}_1$ is linear and each $\varphi \in \mathcal{D}_1$ satisfies

$$\begin{aligned}\varphi_\gamma(\mathbf{w}ij) &= \sum_{\substack{\mathbf{v}\in\mathscr{T},\, d(\mathbf{v})\neq d(\mathbf{w})+2 \\ \tau(\mathbf{v})\neq\tau(\mathbf{v}^-)}} \alpha_{\mathbf{v}}(N+2)^{-d(\mathbf{v})}(\widetilde{\varphi}_{\mathbf{v}}\circ\gamma)(\mathbf{w}ij) \\ &\quad + O((N+2)^{-d(\mathbf{w})+2}) \\ &= \frac{1}{N+2}\Big(\varphi_\gamma(\mathbf{w}i) + \varphi_\gamma(\mathbf{w}j) + \sum_{c\in\mathscr{A}} \varphi_\gamma(\mathbf{w}c)\Big) + O(I^{-1}_{d(\mathbf{w})+2}).\end{aligned}$$

Therefore, by Corollary 3, φ is a ϱ-continuous function, where $\varrho_n = I_n^{-n}$. According to [5, the remarks after Lemma 5.7], for any $z : \mathscr{T} \to \mathbb{R}$ as above and for $k \in \mathbb{N}$,

$$-\frac{1}{2}(N+2)^{d(\mathbf{v})+k+2}\{(D\varphi)(\mathbf{v}ab^{k+1}) + (D\varphi)(\mathbf{v}ba^{k+1})\}$$
$$= Nz(\mathbf{v}ab^k) - \frac{2}{N+2}\Big[\sum_{\substack{c\in\mathscr{A}\\ c\neq b}} z(\mathbf{v}ab^k c) + \sum_{\substack{c\in\mathscr{A}\\ c\neq a}} z(\mathbf{v}ba^k c)\Big]$$
$$- \frac{1}{2(N+2)}\Big[\sum_{\substack{c\in\mathscr{A}\\ c\neq b}}\ \sum_{\substack{d\in\mathscr{A}\\ d\neq b,\, d\neq c}} z(\mathbf{v}ab^{k-1}cd) + \sum_{\substack{c\in\mathscr{A}\\ c\neq a}}\ \sum_{\substack{d\in\mathscr{A}\\ d\neq a,\, d\neq c}} z(\mathbf{v}ba^{k-1}cd)\Big]$$
$$\leq 3(N+2)\|z\|_\infty.$$

This shows that (3.2) holds for $\varphi \in \mathcal{D}_1$ when defining $\alpha(\mathbf{w}) = N^n$ for $\mathbf{w}$ not a monomial and $= 1$ otherwise. Condition (3.1) can be shown in the following way. Let $\mathcal{B}$ denote all words in $\mathscr{A}^{n-1}$ which are not monomials. Then it suffices to estimate as in the proof of Lemma 3:

$$I_n^2 N^n \sum_{\mathbf{v}\in\mathcal{B},\, a,b\in\mathscr{A}} (\varphi_\gamma(\mathbf{v}b) - \varphi_\gamma(\mathbf{v}a))^2 = 2NI_n^2 N^n \sum_{\mathbf{w}\notin\mathscr{T}_*} ((D\varphi)(\mathbf{w}))^2 < \infty.$$

In order to show that the functions in $\mathcal{D}_1$ are dense in $L_2(\mu)$, note that by (5.2) we can choose z in an arbitrary manner, e.g. vanishing on words not starting with $\mathbf{v}$. The associated function φ will vanish outside of $[(\mathbf{v}^-)^-]$. Therefore, given $\mathbf{u}$ and m define $z(\mathbf{u}v_1\ldots v_m) = 1$ if $v_1,\ldots,v_m \in \mathscr{A}$ contains the letter $\tau(\mathbf{u})$ (similarly for $z(\mathbf{u}^\# v_1\ldots v_m)$), and $z(\mathbf{w}) = 0$ if $\mathbf{w}$ does not start with $\mathbf{u}$ or $\mathbf{u}^\#$. As $m \to \infty$ these functions approach a multiple of $\mathbb{I}_{[\mathbf{u}]\cup[\mathbf{u}^\#]}$ in $L_2(\mu)$. Hence $\mathcal{D}_1$ is dense in $L_2(\mu)$. The solution of the Dirichlet problem for $\psi \in C(\Sigma)$ also shows that this class of functions is dense in $L_2(m)$.

6. Applications. In this section we discuss further applications of the results in Sections 3–5. However, we only sketch the proofs, since they are similar to those described before.

6.1. *Application to the Sierpiński gasket.* Fix $N \geq 2$. The Sierpiński gasket $\mathcal{S}$ is described geometrically in the following way. Let $\triangle := \triangle(p_1,\ldots,p_N)$ denote the non-degenerate regular simplex generated by N points $p_1,\ldots,p_N \in \mathbb{R}^{N-1}$. For each fixed $i_0 \in \mathscr{A}$, the midpoints $p_{j,i_0} := (p_{i_0} + p_j)/2$ ($j \in \mathscr{A}$) define a corresponding simplex $\triangle(i_0) := \triangle(p_{1,i_0},\ldots,p_{N,i_0}) \subset \triangle$ and the affine mappings $f_{i_0} : \triangle \to \triangle(i_0)$ satisfying $f_{i_0}(p_i) = p_{i,i_0}$. It is well known that the Sierpiński gasket $\mathcal{S}$ can be represented as a limit set of the semigroup generated by the f_i, hence the natural identification space is a symbolic sequence space over N symbols with identification by the equivalence relation

$$\mathbf{x} = (x_k)_{k\in\mathbb{N}} \sim_{\mathcal{S}} \mathbf{y} = (y_k)_{k\in\mathbb{N}}$$

if $\mathbf{x} = \mathbf{y}$ or if there exists some $n \in \mathbb{N}$ such that $x_{n+l} = y_n$ and $y_{n+l} = x_n$ for every $l \geq 1$ and $x_l + y_l$ for $l < n$. The Bernoulli measure μ is known to be a multiple of the Hausdorff measure of the Sierpiński gasket with respect to the Euclidean norm.

The situation described in Section 5 applies directly in this situation. If ν denotes the counting measure on the vertices of $\mathcal{S}$, we let $m = \mu + \nu$. It follows that

$$\mathcal{E}(\varphi, \psi) = \lim_{n\to\infty} \left(\frac{N+2}{N}\right)^n \sum_{\mathbf{w}\in\mathscr{A}^n} (D\varphi)(\mathbf{w})(D\psi)(\mathbf{w})$$

defines a Dirichlet form on $L_2(m)$ with Laplace operator (on $L_2(\mu)$)

$$\Delta_{\mathcal{S}}\varphi(\xi) = \lim_{n\to\infty} (N+2)^n \frac{1}{|\langle\mathbf{w}\rangle|} \sum_{\mathbf{v}\sim\mathbf{w}} (D\varphi)(\mathbf{w})$$

for all $\mathbf{w}$, where $\mathbf{w}$ converges to ξ as $n \to \infty$ under the condition that each equivalent $\mathbf{v}$ has two different letters in $\mathbf{v}^-$.

6.2. *The case of $\mathcal{S} \times \mathcal{B}^\infty$.* Consider two alphabets $\mathcal{A} = \{1, \ldots, N\}$ and $\mathcal{B} = \{1, \ldots, B\}$. Points in the sequence space $(\mathcal{A} \times \mathcal{B})^\infty$ are written in the form $\mathbf{x} = (x^1; x^2) = (x_1^1 x_2^1 \ldots; x_1^2 x_2^2 \ldots)$. Define an equivalence relation by $\mathbf{x} \sim \mathbf{y}$ if and only if $x^1 \sim_{\mathcal{S}} y^1$ and $x^2 = y^2$, where $\sim_{\mathcal{S}}$ denotes the equivalence relation of $\mathcal{S}$. Finite words are as well written in the form $\mathbf{w} = (w^1, w^2)$, where w^1 and w^2 have the same length $m = d(\mathbf{w})$ and the embedding is defined by

$$\gamma(\mathbf{w}) = \langle (w^1 (w_n^1)^\infty; w^2 (w_n^2)^\infty) \rangle,$$

where $\langle\cdot\rangle$ denotes the equivalence class.

Define $\mathscr{T}_*$ to be the set of all words of the form $\mathbf{w} = (w^1, w^2)$, where w^1 is a monomial and $w^2 \in \mathcal{B}^{d(\mathbf{w})}$. Note that $\gamma(\mathscr{T}_*)$ is compact, nowhere dense and consists of all points of the form $\langle\mathbf{x}\rangle = \langle(x^1, x^2)\rangle$, where x^1 is a monomial and $x^2 \in \mathcal{B}^\infty$. We let ν be a measure on $\partial\Sigma$ which is the sum of the Bernoulli measures on $\{a^\infty\} \times \mathcal{B}^\infty$. Notice that the equivalence relation is not thin, but the Bernoulli measure μ on $(\mathcal{A} \times \mathcal{B})^\infty$ satisfies

$$\mu(\{\mathbf{x} : \langle\mathbf{x}\rangle = 1\}) = 1$$

and can be transported to the quotient space.

Let φ be the function constructed in Section 5 for the Sierpiński gasket $\mathcal{S}$ and let v^2 be any finite word over the alphabet $\mathcal{B}$. We denote by $\mathbb{I}_{[v^2]}$ the indicator function of the set of all $\xi \in \mathcal{B}^\infty$ which begin with the word v^2. Then for any word $\mathbf{w}$ of length larger than $d(v^2)$,

$$D(\varphi \otimes \mathbb{I}_{[v^2]})(\mathbf{w}) = D(\varphi)(w^1)\mathbb{I}_{\{v^2\}}(w^2),$$

since all or none of the $(w^2)^- b$ belong to $[v^2]$.

Since the Bernoulli measure μ is the product measure of the Bernoulli measures on $\mathcal{A}^\infty/\sim$ and $\mathcal{B}^\infty$, finite linear combinations of functions of the form $\varphi \otimes \mathbb{I}_{[v^2]}$ are dense in $L_2(\mu)$, and we show that (3.1), (3.2) and (3.5) hold with $I_n = \left(\frac{N+2}{BN}\right)^n$. As before, for fixed $\varphi \otimes \mathbb{I}_{[v^2]}$, if n is large enough,

$$\lim_{n\to\infty} I_n(NB)^n \sum_{\mathbf{w}\in(\mathcal{A}\times\mathcal{B})^n} \sum_{\substack{\mathbf{u}\in\langle\mathbf{w}\rangle\\ a=(a^1,a^2)\in\mathcal{A}\times\mathcal{B}}} (\varphi\otimes\mathbb{I}_{[v^2]}(\gamma(\mathbf{u})) - \varphi\otimes\mathbb{I}_{[v^2]}(\gamma(\mathbf{u}^-a)))^4$$
$$\le \lim_{n\to\infty} B^{2n+1} I_n^2 N^n \sum_{\mathbf{w}\in\mathcal{A}^n} \sum_{u^1\in\langle w^1\rangle,\, a^1\in\mathcal{A}} (\varphi_\gamma(u^1) - \varphi_\gamma(u^1)^-a^1)^4,$$

which is bounded by the results in Section 5. This shows (3.1).

Similarly,

$$\sup_{n\in\mathbb{N}} I_n^2 (NB)^n \sum_{\mathbf{w}\in(\mathcal{A}\times\mathcal{B})^n} \left(\frac{1}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{u}\sim\mathbf{w}} (D\varphi\otimes\mathbb{I}_{[v^2]})(\mathbf{u})\right)^2$$
$$\le \sup_{n\in\mathbb{N}} I_n^2 N^n B^{2n} \sum_{w^1\in\mathcal{A}^n} \left(\frac{1}{|\langle w^1\rangle|}\sum_{u^1\sim w^1} (D\varphi)(u^1)\right)^2 < \infty$$

shows (3.2), and finally

$$\begin{aligned}
&\mathscr{E}(\varphi\otimes\mathbb{I}_{[v^2]},\psi)\\
&= -\lim_{n\to\infty} I_n \sum_{\mathbf{w}\in(\mathcal{A}\times\mathcal{B})^n} \frac{1}{|\langle\mathbf{w}\rangle|}\sum_{\mathbf{u}\sim\mathbf{w}} (D\varphi\otimes\mathbb{I}_{[v^2]})(\mathbf{u})\psi_\gamma(\mathbf{w})\\
&= -\lim_{n\to\infty} I_n B^n \sum_{\mathbf{w}\in\mathcal{A}^n} \frac{1}{|\langle w^1\rangle|}\sum_{u^1\sim w^1} (D\varphi)(u^1) B^{-n} \sum_{w^2\in\mathcal{B}^n} \psi_\gamma(w^1,w^2)\mathbb{I}_{\{v^2\}}(w^2)\\
&= \mathscr{E}^1(\varphi, E^2(\psi\mathbb{I}_{[v^2]})),
\end{aligned}$$

where $\mathscr{E}^1$ denotes the form associated to the first coordinate according to Section 5 and where E^2 denotes integration with respect to the second coordinate. Theorem 1 also applies in this situation and we obtain extensions of the Sierpiński gasket.

Note that products of fractals and their analysis have been considered in the literature. We refer to [11].

6.3. *Non-Sierpiński cases.* In this subsection we briefly sketch how our results can be applied in other cases than the Sierpiński relation.

Let $\mathscr{A} = \{1,2,3,4,5,6,7\}$. Consider the equivalence relation $\mathbf{x} = x_1x_2\ldots \sim \mathbf{y} = y_1y_2\ldots$ defined by $\mathbf{x} = \mathbf{y}$ or there exists an n such that

$$x_i = y_i \quad \forall i < n, \qquad x_{n+k} = y_{n+k} = c \quad \forall k \ge 1, \qquad c \ne x_n \sim_c y_n \ne c,$$

where $a \sim_c b$ is given by the following table of classes:

$c = 1$	$\{2,3,5\}$ and $\{4,6,7\}$
$c = 2$	$\{1,5,7\}$ and $\{3,4,6\}$
$c = 3$	$\{1,2,6\}$ and $\{4,5,7\}$
$c = 4$	$\{1,5,6\}$ and $\{2,3,7\}$
$c = 5$	$\{1,3,4\}$ and $\{2,6,7\}$
$c = 6$	$\{1,3,7\}$ and $\{2,4,5\}$
$c = 7$	$\{1,2,4\}$ and $\{3,5,6\}$

The embedding is as before, i.e. $\gamma(\mathbf{w}) = \langle \mathbf{w}\tau(\mathbf{w})^\infty \rangle$.

The first lemma is immediate from the definitions and its proof is omitted.

LEMMA 7. *Let $f(a) = p_a \in \mathbb{R}$ for $a \in \mathscr{A}$ and define*

$$f(\mathbf{w}ab) = \frac{1}{17}\sum_{c\sim_b a} f(\mathbf{w}c) + \frac{2}{17}\sum_{d\in\mathscr{A}} f(\mathbf{w}d), \qquad a \neq b,$$

$$f(\mathbf{w}a^2) = f(\mathbf{w}a), \qquad a \in \mathscr{A}.$$

Then there exists a function $\varphi \in C(\Sigma)$ with $\varphi(\mathbf{w}a^\infty) = f(\mathbf{w}a)$. Furthermore, φ is $\{(14/17)^n : n \geq 1\}$-continuous and satisfies

$$\sum_{c\sim_b a} (D\varphi)(\mathbf{w}cb) = 0, \qquad a \neq b,$$

$$(D\varphi)(\mathbf{w}a^2) = \frac{14}{17}(D\varphi)(\mathbf{w}a), \qquad a \in \mathscr{A}.$$

The following arguments are similar to those used for the Sierpiński relation before. We sketch the construction of a suitable set $\mathcal{D}_1$. For every $b \in \mathscr{A}$ let f_b and let $\overline{\varphi}_b$ denote the functions being defined in Lemma 7 for the boundary conditions $p_a = 0$, $a \in \mathscr{A}$, $a \neq b$, and $p_b = 1$. Consider a second function defined on $\mathscr{A}^\infty$ by setting

$$\varphi_b(\mathbf{x}) = \overline{\varphi}_b(\Pi(\mathbf{x})) \qquad (\mathbf{x} \in \mathscr{A}^\infty).$$

Fix a word $\mathbf{u} \in \mathscr{T}_+$ and define $\varphi_{\mathbf{u}} : \mathscr{A}^\infty \to \mathbb{R}$ by

$$\varphi_{\mathbf{u}}(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \notin [\mathbf{u}^-], \\ \varphi_{\tau(\mathbf{u})}(\mathbf{y}) - \frac{1}{2}\mathbb{I}_{\mathbf{x}=\mathbf{u}\tau(\mathbf{u})^\infty} & \text{if } \mathbf{x} = \mathbf{u}^-\mathbf{y}. \end{cases}$$

Let $\mathscr{T}_{\neq}$ denote the set of all words in $\mathscr{T}$ for which the last two letters are different or which are of length ≤ 1. Let $\beta = 2/17$ and recall that $N = 7$. A function $\Phi : \mathscr{A}^\infty \to \mathbb{R}$ of the form

$$\Phi = \sum_{\mathbf{u}\in\mathscr{T}_{\neq}} \theta(\mathbf{u})\beta^{-d(\mathbf{u})}\varphi_{\mathbf{u}}$$

is well defined on Σ and continuous as long as the coefficients θ are uniformly bounded and satisfy $\theta(\mathbf{v}) = \theta(\mathbf{u})$ for all $\mathbf{v} \in \langle \mathbf{u} \rangle$, because $\sum_{\mathbf{v}\in\langle\mathbf{u}\rangle} \varphi_{\mathbf{v}}$ is constant on the set $\{\mathbf{v}\tau(\mathbf{v})^\infty : \mathbf{v} \in \langle \mathbf{u} \rangle\}$.

We need to construct the coefficients $\theta(\mathbf{u})$ such that

(i) $\theta(\mathbf{u})$ are bounded and constant on equivalence classes.

(ii) Conditions (3.1), (3.2) and (3.5) are satisfied for the associated function Φ.

(iii) For a set $U = \bigcup_{\mathbf{v}\sim\mathbf{u}}[\mathbf{v}]$ and $\varepsilon > 0$ there exist coefficients $\theta(\mathbf{u})$ such that the associated function Φ satisfies $\|\Phi - \mathbb{I}_U\|_{L_2(\mu)} < \varepsilon$.

First note that (3.5) follows immediately from Lemma 7 for $\varphi_{\mathbf{v}}$, hence also for finite linear combinations and also for Φ by the absolute convergence of the series.

As before, we start with the equation

$$\theta(\mathbf{u}) = -\frac{N}{|\langle\mathbf{u}\rangle|}\sum_{\mathbf{v}\in\langle\mathbf{u}\rangle} z(\mathbf{v}).$$

Note that (i) is satisfied if the coefficients $z(\mathbf{v})$ are uniformly bounded.

A direct calculation shows that for $k \geq 2$, $\mathbf{v} \in \mathscr{A}^n$ and $a, c \neq b$,

$$\begin{aligned}
&\Phi_\gamma(\mathbf{v}ab^{k-1}c) - \Phi_\gamma(\mathbf{v}ab^k)\\
&\quad= \sum_{\mathbf{u}\in\mathscr{T}_{\neq},\, d(\mathbf{u})\leq d(\mathbf{v})+1} \frac{\theta(\mathbf{u})}{\beta^{d(\mathbf{u})}}\left(\varphi_{\mathbf{u}}(\mathbf{v}ab^{k-1}c) - \varphi_{\mathbf{u}}(\mathbf{v}ab^k)\right)\\
&\qquad+\sum_{j=0}^{k-1}\sum_{b\neq d\in\mathscr{A}} \frac{\theta(\mathbf{v}ab^j d)}{\beta^{n+j+2}}\varphi_{\mathbf{v}ab^j d}(\mathbf{v}ab^{k-1}c)\\
&\quad= \sum_{\mathbf{u}\in\mathscr{T}_{\neq},\, d(\mathbf{u})\leq d(\mathbf{v})+1} \frac{\theta(\mathbf{u})}{\beta^{d(\mathbf{u})+1}}\Big(\alpha\sum_{d\sim_c b}\varphi_{\mathbf{u}}(\mathbf{v}ab^{k-2}d)\\
&\qquad+\beta\sum_{d\in\mathscr{A}}\varphi_{\mathbf{u}}(\mathbf{v}ab^{k-2}d) - \varphi_{\mathbf{u}}(\mathbf{v}ab^{k-1})\Big)\\
&\qquad+\sum_{j=0}^{k-2}\sum_{b\neq d\in\mathscr{A}} \frac{\theta(\mathbf{v}ab^j d)}{\beta^{n+j+3}}\Big(\varphi_{\mathbf{v}ab^j d}(\mathbf{v}ab^{k-1})\\
&\qquad+\varphi_{\mathbf{v}ab^j d}(\mathbf{v}ab^{k-2}c) + \sum_{e\in\mathscr{A}}\varphi_{\mathbf{v}ab^j d}(\mathbf{v}ab^{k-2}e)\Big)\\
&\qquad+\beta^{-n-k-1}\theta(\mathbf{v}ab^{k-1}c)\\
&\quad= \beta^{-1}\Big(\sum_{b\neq d\sim_c b}\frac{1}{2}\left[\Phi(\mathbf{v}ab^{k-2}c) - \Phi(\mathbf{v}ab^{k-1})\right] + N(D\Phi)(\mathbf{v}ab^{k-1})\Big)\\
&\qquad+\frac{\theta(\mathbf{v}ab^{k-1}c)}{\beta^{n+k+1}}.
\end{aligned}$$

Summing over $c \neq b$ yields

$$\begin{aligned}\beta^{-d(\mathbf{v})-k-1}&(D\Phi)(\mathbf{v}ab^k)\\ &= N\beta^{-d(\mathbf{v})-k}(D\Phi)(\mathbf{v}ab^{k-1}) + \frac{1}{N}\sum_{c\neq b}\theta(\mathbf{v}ab^{k-1}c)\\ &= N(N+2)^{d(\mathbf{v})+k}(D\Phi)(\mathbf{v}ab^{k-1}) - \frac{1}{3}\sum_{c\neq b}\sum_{d\sim_c b} z(\mathbf{v}ab^{k-2}d).\end{aligned}$$

Setting $z(\mathbf{v}ab^k) := \beta^{-d(\mathbf{v})-k-1}(D\Phi)(\mathbf{v}ab^k)$ we obtain

$$z(\mathbf{v}ab^{k-1}) = \frac{1}{N}\, z(\mathbf{v}ab^k) + \frac{1}{3N}\sum_{c\neq b}\sum_{d\sim_c b} z(\mathbf{v}ab^{k-1}d),$$

which is a convex combination determining $z(\mathbf{v}ab^{k-1})$.

The remaining arguments are now carried out similar as before. We summarize them in

THEOREM 3. *The form*

$$\mathscr{E}(\phi,\psi) = \lim_{n\to\infty}\left(\frac{17}{14}\right)^n \sum_{\mathbf{w}\in\mathscr{A}^n}(D\phi)(\mathbf{w})(D\psi)(\mathbf{w})$$

exists for $\phi \in \mathcal{D}_1$ and $\psi \in C(\Sigma)$. The associated Laplace operator is given by

$$\Delta_\Sigma\phi(\eta) = \lim_{\mathbf{w}\to\eta,\, d(\mathbf{w})=n}\beta^{-n}\,\frac{1}{3}\sum_{c\sim_{\tau(\mathbf{w})}\tau(\mathbf{w}^-)}(D\phi)(\mathbf{w}^{--}c\tau(\mathbf{w})).$$

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