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SPECTRAL PROPERTIES OF ERGODIC DYNAMICAL SYSTEMS CONJUGATE TO THEIR COMPOSITION SQUARES

BY

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Abstract. Let S and T be automorphisms of a standard Borel probability space. Some ergodic and spectral consequences of the equation $ST = T^2S$ are given for T ergodic and also when $T^n = I$ for some $n > 2$. These ideas are used to construct examples of ergodic automorphisms S with oscillating maximal spectral multiplicity function. Other examples illustrating the theory are given, including Gaussian automorphisms having simple spectra and conjugate to their squares.

0. Introduction. Let T be an invertible measure-preserving transformation (*automorphism*) defined on a standard Borel probability space $(X, \mathcal{F}, \mu)$. We investigate spectral and ergodic consequences of the equation $ST = T^2S$ for automorphisms S and T. No examples of weakly mixing automorphisms conjugate to their squares and having simple spectrum have been published that we are aware of, and we give some Gaussian automorphisms having this property. Very few examples of transformations conjugate to their squares are known, and few general results are available indicating when this can happen. (After this paper was submitted, a preprint was received from O. N. Ageev (2005), who uses a category argument to show the existence of rank one transformations which are weakly mixing and conjugate to their squares. This answered a question mentioned in Goodson (2002, 1999) which had been open for some time. Many of the results of this paper are applicable to such examples.)

Maps having finite non-zero entropy cannot be conjugate to their squares because of the identity $h(T^2) = 2h(T)$. Consequently, we are only interested in maps having zero or infinite entropy.

We recall the basic properties of transformations conjugate to their squares in Section 1. In Section 2 we show that if $ST = T^2S$ where T is mixing with no Lebesgue component, then S is weakly mixing. On the other hand, if T is ergodic and has an eigenvalue which is a root of unity, S cannot be ergodic.

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In Section 3, using the spectral theorem for unitary operators we see that if $T^n = I$ (I = the identity map), for some $n \in \mathbb{Z}$, $n > 2$, then S^q has maximal spectral multiplicity equal to q on some subspace (for some $1 < q < n$), and we apply these ideas to give examples of automorphisms having an oscillating multiplicity function. These ideas are used to construct weakly mixing rank one transformations S for which S^2 has non-simple spectrum, answering a question of Thouvenot.

In Section 4, properties of the maximal spectral type of an automorphism which is conjugate to its square are considered. Section 5 gives examples of Gaussian automorphisms having simple spectrum and which are conjugate to their squares. Properties of the conjugating map are also studied.

Much of our exposition generalizes to the case where there are automorphisms S and T satisfying $ST = T^pS$ for some $p > 1$.

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1. Preliminaries. By a *dynamical system* we mean a 4-tuple $\mathcal{X} = (X, \mathcal{F}, \mu, T)$ consisting of an automorphism $T : (X, \mathcal{F}, \mu) \to (X, \mathcal{F}, \mu)$ defined on a non-atomic standard Borel probability space. Both the identity automorphism and the identity operator will be denoted by I. The group $\mathrm{Aut}(X)$ of all automorphisms of $(X, \mathcal{F}, \mu)$ becomes a completely metrizable topological group when endowed with the weak convergence of transformations ($T_n \to T$ if for all $A \in \mathcal{F}$, $\mu(T_n^{-1}(A) \triangle T^{-1}(A)) + \mu(T_n(A) \triangle T(A)) \to 0$ as $n \to \infty$). Denote by $C(T)$ the *centralizer* of T, i.e., the set of those members of $\mathrm{Aut}(X)$ which commute with T (more generally it is usual to define $C(T)$ to be those measure-preserving transformations which commute with T, but it will be convenient to assume that $C(T)$ is a group).

The spectral properties of T are those of the induced unitary operator defined by

$$\widehat{T} : L^2(X, \mu) \to L^2(X, \mu), \quad \widehat{T} f(x) = f(Tx), \quad f \in L^2(X, \mu).$$

Note that if $ST = T^2S$, then $\widehat{T}\widehat{S} = \widehat{S}\widehat{T}^2$.

Generally a unitary operator $U : H \to H$ on a separable Hilbert space H is said to have *simple spectrum* if there exists $h \in H$ such that $Z(h) = H$, where $Z(h)$ is the closed linear span of the vectors $U^n h$, $n \in \mathbb{Z}$.

In this case there exists a finite Borel measure σ_h defined on the unit circle S^1 in the complex plane for which

$$\langle U^n h, h\rangle = \int_{S^1} z^n \, d\sigma_h, \qquad n \in \mathbb{Z},$$

and such that U is unitarily equivalent to $V : L^2(S^1, \sigma_h) \to L^2(S^1, \sigma_h)$ defined by $Vf(z) = zf(z)$.

Let us mention some basic facts about automorphisms S and T satisfying $ST = T^2 S$ (see Goodson (2002)):

(i) $ST^n = T^{2n}S$ for $n \in \mathbb{Z}$ and $S^n T = T^{2^n} S^n$ for all $n \in \mathbb{Z}^+$.
(ii) If T is aperiodic, then S is aperiodic (i.e., $\mu(\{x \in X : S^n x = x\}) = 0$ for all $n \in \mathbb{Z}$).
(iii) If S is ergodic, then either T is aperiodic, or $T^n = I$ for some $n \in \mathbb{Z}$.
(iv) If S is prime, then T is weakly mixing or $T = I$.
(v) If T is ergodic with discrete spectrum, then S is ergodic (and hence mixing) if and only if T has no eigenvalues having finite order.
(vi) The entropy of T satisfies $h(T) = 0$ or $h(T) = \infty$.
(vii) The Bernoulli shift of infinite entropy and the time one map in the horocycle flow are conjugate to their squares.
(viii) If T has the weak closure property (see King (1986)), then the map $\Phi : C(T) \to C(T)$, $\Phi(S) = S^2$ is a group automorphism. Furthermore, $S\phi = \phi^2 S$ for all $\phi \in C(T)$, and any two conjugations between T and T^2 are isomorphic.
(ix) If S is rigid (there is a sequence $n_i \to \infty$ with $S^{n_i} \to I$ as $i \to \infty$), then T is rigid, or $T^m = I$ for some $m \in \mathbb{Z}^+$.

Recall that rank one maps and also Gaussian–Kronecker maps have the weak closure property (see King (1986) and Thouvenot (1987)).

2. Mixing and ergodic properties of S and T when $ST = T^2S$. We now look at how the equation $ST = T^2 S$ forces certain mixing properties on S and T when additional assumptions are made.

THEOREM 1. *Suppose that $ST = T^2 S$ where T is ergodic. If T has an eigenvalue which is a root of unity, then S is not ergodic.*

Proof. There exists $\lambda \in S^1$ and $n > 1$ with $f(Tx) = \lambda f(x)$, $\lambda^k \neq 1$ for $1 \le k < n$ and $\lambda^n = 1$. This implies that $f^n(Tx) = f^n(x)$, and T ergodic implies $f^n =$ constant a.e. Necessarily, n is odd since otherwise T would have -1 as an eigenvalue, contradicting the ergodicity of T^2. Thus $\lambda \neq \pm 1$, and in addition, we may assume that $f^n = 1$. In particular, $f^2(Tx) = \lambda^2 f^2(x)$ and also

$$f \circ S(Tx) = f(STx) = f(T^2 Sx) = f \circ T^2(Sx) = \lambda^2 f \circ S(x),$$

i.e., both f^2 and $f\circ S$ are eigenfunctions for T corresponding to the same eigenvalue λ^2. Since T is ergodic, there exists $c\in S^1$ for which

$$f\circ S(x)=cf^2(x)\quad \text{a.e.},$$

where $c^n=1$.

Let $g(x)=\sum_{k=1}^{n-1}c^{k-1}f^k(x)$. Then

$$g(Sx)=\sum_{k=1}^{n-1}c^{k-1}f^k(Sx)=\sum_{k=1}^{n-1}c^{k-1}[cf^2(x)]^k=\sum_{k=1}^{n-1}c^{2k-1}f^{2k}(x)=g(x),$$

since n is odd and $c^n=1$, $f^n=1$. We see that $g(x)$ is non-constant since the functions f^k, $k=1,\ldots,n-1$, are orthogonal (because they correspond to distinct eigenvalues λ^k, $0<k<n$). It follows that S is not ergodic. ■

Recall that an ergodic transformation T is *totally ergodic* if it has no eigenvalues that are roots of unity. We immediately obtain:

COROLLARY 1. *If $ST=T^2S$ where T and S are ergodic, then T is totally ergodic.*

Suppose that $ST=T^2S$ where T has a Lebesgue component in its spectrum. If T has a Lebesgue component of multiplicity n, then it can be seen that T^2 must have a Lebesgue component of multiplicity $2n$, which is impossible unless $n=\infty$. In particular, a transformation with a finite Lebesgue component cannot be conjugate to its square. The next result shows that if $ST=T^2S$ where T is mixing, and S is not weakly mixing, then T has a countable Lebesgue component. We conjecture that if T is mixing and S is ergodic, then S is weakly mixing. This would have implications concerning Bernoulli shifts having infinite entropy as these are conjugate to their squares. Recall from Goodson and Ryzhikov (1997) that a finite rank mixing transformation (or in fact any locally rank one mixing transformation) is never conjugate to its square.

THEOREM 2. *If $ST=T^2S$ where T is mixing, then*:

(i) *If T has no Lebesgue component in its spectrum, then S is weakly mixing.*

(ii) *If S is ergodic, but not weakly mixing, then T has a countable Lebesgue component in its spectrum.*

(iii) *S cannot be ergodic with purely discrete spectrum (and in fact cannot be rigid).*

Proof. (i) Suppose that S is ergodic but not weakly mixing, and let Λ be the eigenvalue group of S with $\{f_\lambda:\lambda\in\Lambda\}$ the set of eigenfunctions of S in the orthogonal complement of the constant functions. Let $\lambda,\mu\in\Lambda$. Then

$$\langle\widehat{T}f_\lambda,f_\mu\rangle=\langle\widehat{T}\,\overline{\lambda}\widehat{S}f_\lambda,\overline{\mu}\widehat{S}f_\mu\rangle=\overline{\lambda}\mu\langle\widehat{S}^{-1}\widehat{T}\widehat{S}f_\lambda,f_\mu\rangle=\overline{\lambda}\mu\langle\widehat{T}^2f_\lambda,f_\mu\rangle.$$

In the same way we see that

$$\langle \widehat{T} f_\lambda, f_\mu\rangle = (\bar{\lambda}\mu)^n \langle \widehat{T}^{2^n} f_\lambda, f_\mu\rangle \quad \text{for all } n.$$

However, T is mixing, so as $n \to \infty$,

$$\langle \widehat{T}^{2^n} f_\lambda, f_\mu\rangle \to \langle f_\lambda, 1\rangle\langle 1, f_\mu\rangle = 0 \quad \text{for all } \lambda \text{ and } \mu$$

since $f_\lambda \perp \mathbb{C}$. Since $(\bar{\lambda}\mu)^n$ is a bounded sequence, we deduce that $\langle \widehat{T} f_\lambda, f_\mu\rangle = 0$. The same argument shows that $\langle \widehat{T}^n f_\lambda, f_\mu\rangle = 0$ for all $n \in \mathbb{Z}$, $n \neq 0$, and all $\lambda, \mu \in \Lambda$.

If $Z(f_\lambda)$ is the cyclic subspace generated by f_λ (with respect to T), then the restriction of $\widehat{T}$ to $Z(f_\lambda)$ has simple Lebesgue spectrum for each λ, and part (i) follows.

(ii) If S is ergodic but not weakly mixing, then $Z(f_\lambda) \perp Z(f_\mu)$ for all $\lambda \neq \mu$, and we conclude that $\widehat{T}$ has a Lebesgue component in its spectrum, which must be countable.

(iii) Recall that if S is ergodic with purely discrete spectrum, then S is rigid, so there is a sequence $n_i \to \infty$ as $i \to \infty$ with $S^{n_i} \to I$. Since $S^{n_i} T = T^{2^{n_i}} S^{n_i}$ for $i = 1, 2, \ldots$, we deduce that $T^{2^{n_i}-1} \to I$ as $i \to \infty$, so T is also rigid, and cannot be mixing. ∎

3. Spectral properties of S when T is of finite order. Suppose instead we have the situation $S\phi = \phi^2 S$ for some automorphism ϕ satisfying $\phi^n = I$ (some $n > 2$ necessarily odd). An *automorphism extension* is a map $S : X \times G \to X \times G$ of the form

$$S(x, g) = (S_0 x, \psi(x) + v(g)),$$

where $v : G \to G$ is a group automorphism, $S_0 : X \to X$ is an automorphism and $\psi : X \to G$ is measurable. S is itself an automorphism on the space $(X \times G, \mathcal{F} \otimes \mathcal{F}_G, \mu \times \lambda)$, where $\mathcal{F}_G$ are the Borel sets of G, and λ is Haar measure on the compact group G. We shall see that automorphism extension can have the property that $S\phi = \phi^2 S$ for some automorphism ϕ which satisfies $\phi^n = I$ for some $n > 2$ odd.

Let $\omega \in S^1$ be a primitive nth root of unity and write

$$H_k = \{f \in L^2(X, \mu) : f \circ \phi(x) = \omega^k f(x)\}, \quad 0 \le k \le n-1.$$

The subspaces H_k, being the eigenspaces of the unitary operator $\widehat{\phi}$, are $\widehat{\phi}$ invariant, pairwise orthogonal and

$$L^2(X, \mu) = \bigoplus_{k=0}^{n-1} H_k.$$

LEMMA 1. *$\widehat{S}^m H_k = H_{k\cdot 2^m \,(\mathrm{mod}\, n)}$ for $k = 0, 1, \ldots, n-1$ and $m \ge 1$. In particular, H_0 is $\widehat{S}$-invariant.*

Proof. Let $f \in H_k$. Then $\widehat{S}^m f = f \circ S^m$, and

$$f \circ S^m(\phi x) = f(\phi^{2^m} S^m x) = \omega^{k \cdot 2^m} f \circ S^m(x),$$

so that $f \circ S^m \in H_{k \cdot 2^m \,(\mathrm{mod}\, n)}$. ■

As a consequence, we deduce some results about the spectrum of the powers of S. Specifically we show that for some $q > 1$, the maximal spectral multiplicity of S^q on a subspace is divisible by q. Note that S can be ergodic in this theorem.

THEOREM 3. *Suppose that $S\phi = \phi^2 S$ where $\phi^n = I$ ($n \geq 3$ odd, $\phi^k \neq I$, $0 \leq k < n$). Then there exists $1 < q \leq n-1$ for which $\widehat{S}^q$ restricted to the ortho-complement of the subspace*

$$\{f \in L^2(X,\mu) : f \circ \phi = f\},$$

has a component of multiplicity divisible by q in its spectrum. Specifically, $q = \min\{m \in \mathbb{Z}^+ : 2^m = 1 \ (\mathrm{mod}\, n)\}$.

Proof. Let $f \in H_1$. Then by the lemma we have

$$\widehat{S} f \in H_2, \quad \widehat{S}^2 f \in H_{2^2}, \quad \ldots, \quad \widehat{S}^m f \in H_{2^m \,(\mathrm{mod}\, n)} \quad \text{and} \quad \widehat{S}^q f \in H_1.$$

Let $g \in L^2(X,\mu)$, and denote by $Z(g)$ the cyclic subspace of $L^2(X,\mu)$ generated by the unitary operator $\widehat{S}^q$, i.e., the closed linear span of all vectors of the form $\widehat{S}^{nq} g$, $n \in \mathbb{Z}$. Then for $f \in H_1$,

$$Z(f) \subseteq H_1, \quad Z(\widehat{S} f) \subseteq H_2, \quad \ldots, \quad Z(\widehat{S}^m f) \subseteq H_{2^m \,(\mathrm{mod}\, n)},$$

and this implies that $Z(\widehat{S}^r f) \perp Z(\widehat{S}^p f)$ for $r \neq p$, $1 \leq r, p \leq q-1$.

Furthermore,

$$\langle \widehat{S}^{nq}(\widehat{S}^r f), \widehat{S}^r f \rangle = \langle \widehat{S}^{nq} f, f \rangle \quad \text{for all } n \in \mathbb{Z},$$

so that $\sigma_{\widehat{S}^r f} \sim \sigma_f$ for $0 \leq r \leq q-1$ (spectral measures with respect to $\widehat{S}^q$), and hence $\widehat{S}^q$ has maximal spectral multiplicity divisible by q on the subspace

$$H_1 \oplus H_2 \oplus \cdots \oplus H_{2^{q-1}},$$

where $2^q = 1 \ (\mathrm{mod}\, n)$. ■

A slightly more general version of the above can be given, which is only of interest when T is not ergodic.

THEOREM 4. *Suppose that $ST = T^2 S$ and ω is a primitive nth root of unity for which the set $\Lambda = \{\omega^k : 0 \leq k < n\}$ consists of eigenvalues of T, but no member of $-\Lambda$ is an eigenvalue of T. If S has simple spectrum, then S^q has a component of multiplicity q in its spectrum, for some $1 < q \leq n-1$.*

The theorem is shown by the method of proof of the last theorem, together with the following lemma:

LEMMA 2. *Under the conditions of the last theorem, the subspaces*

$$H_k = \{f \in L^2(X,\mu) : \widehat{T}f(x) = \omega^k f(x)\}, \quad 0 \le k \le n-1,$$

are S^q-invariant for some $1 < q < n$.

Proof. We first show that $\widehat{S}H_k = H_{2k}$ for each k (reduced modulo n if necessary). Proceeding as before, we see that if $f \in H_k$ then $\widehat{S}f \in H_{2k}$, so that $\widehat{S}H_k \subseteq H_{2k}$, hence it suffices to show the reverse containment. Write

$$H'_k = \{f \in L^2(X,\mu) : \widehat{T}^2 f(x) = \omega^{2k} f(x)\}, \quad 0 \le k \le n-1.$$

Then clearly $H_k \subseteq H'_k$. On the other hand, if $f \in H'_k$, set $g = f - \omega^{-k}\widehat{T}f$; then $\widehat{T}g = -\omega^k g$.

But $-\omega^k$ is not an eigenvalue of $\widehat{T}$ so $g = 0$ and $\widehat{T}f = \omega^k f$, or $f \in H_k$ and we deduce that $H_k = H'_k$. Suppose now that $f \in H_{2k}$. Then

$$\widehat{T}^2\widehat{S}^{-1}f = \widehat{S}^{-1}\widehat{T}f = \omega^{2k}\widehat{S}^{-1}f,$$

so that $\widehat{S}^{-1}f \in H'_k = H_k$, or $\widehat{S}^{-1}H_{2k} \subseteq H_k$. We have shown that

$$\widehat{S}H_k \subseteq H_{2k} \subseteq \widehat{S}H_k, \quad \text{i.e.,} \quad H_{2k} = \widehat{S}H_k.$$

Now continue in this way to see that there is some q with $\widehat{S}^q H_k = H_k$. ∎

In this direction we also have:

THEOREM 5. *If $ST = T^2S$ where T is totally ergodic, but not weakly mixing, then S has a countable Lebesgue component in its spectrum.*

Proof. Since T is totally ergodic and not weakly mixing, there exists $f \in L^2(X,\mu)$, $f(Tx) = \lambda f(x)$, where $\lambda^n \ne 1$ for all $n \in \mathbb{Z}$. We see that for $n \ge 1$,

$$f \circ S^n(Tx) = f(S^nTx) = f(T^{2^n}S^n x) = \lambda^{2^n} f \circ S^n(x) \quad \text{for } n \ge 1,$$

so $f \circ S^n$ is also an eigenfunction of T, but corresponding to a different eigenvalue. Therefore $\langle \widehat{S}^n f, f\rangle = 0$ for all $n \ne 0$. Since this can be done for each of a countable collection of orthogonal eigenfunctions of T (using the fact that the eigenvalue group is $R(z) = z^2$ invariant), S must have a countable Lebesgue component in its spectrum. ∎

COROLLARY 2. *Suppose that $ST = T^2S$, where S and T are ergodic. If* $\mathrm{msm}(S) < \infty$, *then T is weakly mixing.*

Proof. The ergodicity of S and T implies that T has no eigenvalues of finite order. Since S cannot have a countable Lebesgue component, T cannot have eigenvalues of infinite order, so T must be weakly mixing. ∎

The following is also a straightforward consequence of Theorem 3 and (iii) of Section 1.

Corollary 3. *If $ST=T^2S$ where S^n has simple spectrum for all $n\in\mathbb{Z}^+$, then $T=I$ or T is aperiodic.*

Examples. 1. If T is a Bernoulli shift having infinite entropy, then T is conjugate to its square. If S is a conjugating map, then since T is mixing, Theorem 2 implies that S cannot be rigid, and we conjecture that S has to be weakly mixing in this case.

2. It is possible for S to be mixing with T non-ergodic and satisfying $ST=T^2S$. For example, there is an ergodic discrete spectrum map T_0 and a mixing map S_0 with $S_0T_0=T_0^2S_0$, so simply set $T=T_0\times T_0$ and $S=S_0\times S_0$.

3. Suppose that $ST=T^2S$ and T is ergodic with discrete spectrum. Suppose also that every eigenvalue λ of T satisfies $\lambda^n\neq 1$ for all $n\in\mathbb{Z}$, $n\neq 0$. Set

$$H_n=\{f\in L^2(X,\mu): f(Tx)=\lambda^n f(x)\},\quad n\in\mathbb{Z}^+,$$

i.e., H_n is just the one-dimensional eigenspace corresponding to the eigenvalue λ^n. Then as before, if $f\in H_1$, $\widehat{S}^m f\in H_{2^m}$, so that $\langle\widehat{S}^m f,f\rangle=0$ for all $m\neq 0$. These considerations show that S has a countable Lebesgue spectrum, and hence is mixing (see Goodson (2002)).

4. We give some examples for which $S\phi=\phi^2S$ with $\phi^n=I$ for some $n>2$ odd. Let G be a compact Abelian group and $\sigma: G\to G$ a group automorphism. Suppose that T is an ergodic automorphism of the Lebesgue space $(X,\mathcal{F},\mu)$ and $\psi: X\to G$ is a cocycle. Then we can define an automorphism extension $T_{\psi,\sigma}: X\times G\to X\times G$ by

$$T_{\psi,\sigma}(x,g)=(Tx,\psi(x)+\sigma(g)),\qquad x\in X,\ g\in G.$$

Set $S=T_{\psi,\sigma}$. Then S preserves product measure $\mu\times\lambda$, where λ is Haar measure on G. If $\phi(x,g)=(x,g+h)$ for some $h\in G$, then we can check that

$$S\phi(x,g)=S(x,g+h)=(Tx,\psi(x)+\sigma(g+h))$$

and

$$\phi^2S(x,g)=\phi^2(Tx,\psi(x)+\sigma(g))=(Tx,\psi(x)+\sigma(g)+2h),$$

so if $\sigma(h)=2h$ we have $S\phi=\phi^2S$. A special case of this is the following:

Proposition 1. *Suppose that $\sigma: G\to G$, $\sigma(g)=2g$ is a group automorphism. If $S(x,g)=(Tx,\psi(x)+\sigma(g))$ and $\phi(x,g)=(x,g+h)$, $h\neq 0$, then $S\phi=\phi^2S$.*

We can arrange ϕ to have any order simply by choosing G and h appropriately. For example, if we take $G=\mathbb{Z}_5$ and set $\phi(x,j)=(x,j+1)$, then the proposition is applicable. Now S can be chosen to be of rank one by choosing T, ψ and σ suitably. From Theorem 3, we deduce that S^4 has a component in its spectrum of multiplicity four on the orthogonal complement of the subspace $\{f\in L^2(X,\mu): f(x,j+1)=f(x,j)\}$ (also, $\widehat{S}^2$ has a component of multiplicity two). So the maximal spectral multiplicity of S^4 is equal to 4 (and we must also have rank 4 for S^4). Incidentally we see that S cannot be conjugate to S^2 as this would imply S conjugate to S^4.

Similar constructions may be made for other n odd. Take $n = 7$. Then since $2^3 = 1 \pmod 7$, we obtain an S with simple spectrum, for which S^3 has a component of multiplicity three.

5. Friedman, Gabriel and King (1988), and also Filipowicz, Kwiatkowski and Lemańczyk (1988), construct ergodic automorphisms having oscillating rank function. The example of Friedman *et al.* is an automorphism extension $S : X \times \mathbb{Z}_4 \to X \times \mathbb{Z}_4$ of the form

$$S(x, j) = (Tx, \psi(x) + \sigma(j)),$$

where $\sigma : \mathbb{Z}_4 \to \mathbb{Z}_4$, $\sigma(j) = -j$. The map S is shown to have the following properties:

- S is weakly mixing.
- S has rank one.
- $\operatorname{rank}(S^n) = \begin{cases} 1, & n \text{ odd}, \\ 2, & n \text{ even}, \ n \neq 4k. \end{cases}$

Note that $S\phi = \phi^3 S$ where $\phi : X \times \mathbb{Z}_4 \to X \times \mathbb{Z}_4$, $\phi(x, j) = (x, j + 1)$.

Since $\operatorname{msm}(\widehat{S}^n) \le \operatorname{rank}(S^n)$ we immediately see that $\operatorname{msm}(\widehat{S}^n) = 1$ (n odd), and ≤ 2 (n even, $n \neq 4k$). However, modifying the argument of Theorem 3 for this situation, if we set

$$H_k = \{f \in L^2(X, \mu) : f \circ \phi(x) = \omega^k f(x)\}$$

(where ω is a primitive 4th root of unity), we see that $\bigoplus_{k=0}^{3} H_k = L^2(X, \mu)$ and H_0 and H_2 are each invariant under $\widehat{S}$, and $\widehat{S}H_1 = H_3$, $\widehat{S}H_3 = H_1$ (using the method of Lemma 1). We deduce that $\widehat{S}^2$ has a component of multiplicity two in its spectrum. Furthermore, if n is odd, $\operatorname{msm}(\widehat{S}^{2n}) \ge \operatorname{msm}(\widehat{S}^2) = 2$. We deduce that

$$\operatorname{msm}(S^n) = \begin{cases} 1, & n \text{ odd}, \\ 2, & n \text{ even}, \ n \neq 4k. \end{cases}$$

The examples of Filipowicz, Kwiatkowski and Lemańczyk (1988) are Morse automorphisms with oscillating rank function and all having simple spectrum. The authors remark that the maximal spectral multiplicity can also oscillate.

6. Let $\alpha \in [0, 1)$ be irrational, and define S, T on $[0, 1) \times [0, 1)$ by $S(x, y) = (x + \alpha, x + 2y) \pmod 1$, and $T(x, y) = (x, y + \beta) \pmod 1$, where $\beta \in [0, 1)$. It is interesting to note that $ST = T^2 S$ and that T has finite order if β is rational, otherwise T has infinite order and also S is ergodic with a discrete component and a countable Lebesgue component. However, S is not an automorphism, so our theory is not applicable. In a similar manner, if we

define $S, T : S^1 \to S^1$ by $S(z) = z^2$, $T(z) = az$, then S and T are mixing and ergodic respectively (with respect to Lebesgue measure on the circle). In addition $ST = T^2S$, and S is not an automorphism. Since S is onto, T^2 is a factor of T, but they cannot be isomorphic since they do not have the same eigenvalue group. A similar situation holds for any ergodic discrete spectrum transformation T, and it may be of interest to ask more generally: when is T^2 a factor of T?

7. In Goodson (2002) it was shown that if T is a $\mathbb{Z}_n$-extension (n even) of some transformation T_0 where T has the weak closure property, then T cannot be conjugate to its square. The case where n is odd is not clear. Suppose that $ST = T^2S$ where T is a $\mathbb{Z}_3$-extension of some automorphism $T_0 : X \to X$. If T has the weak closure property and is ergodic, then $S\sigma = \sigma^2 S$ (see Goodson (2002)) where σ is the flip map $\sigma(x, g) = (x, g + 1)$. Since $\sigma^3 = I$, we deduce that S^2 has a component of even multiplicity in its spectrum (in fact on the ortho-complement of the subspace $\{f \in L^2(X, \mu) : f \circ \sigma = f\}$).

We ask whether it is generally true that if $ST = T^2S$ where T ergodic, then S^n has non-simple spectrum (for some n), possibly non-ergodic.

4. Properties of the spectral measure of T. We study the spectral measure of T when T is conjugate to its square. In particular we look at its maximal spectral type σ. We shall see that the maximal spectral type is quasi-invariant with respect to the transformation $R : S^1 \to S^1$, $R(z) = z^2$. We also show that if T has simple spectrum, then R is one-to-one and onto (a.e. σ) on the (non-closed) support of σ. To say that σ is *R-quasi-invariant* means that for any Borel set A in S^1, $\sigma(A) = 0$ if and only if $\sigma(R^{-1}(A)) = 0$.

PROPOSITION 2. *Suppose that $ST = T^2S$. Then σ, the maximal spectral type of $\widehat{T}$, is R-quasi-invariant, where $R : S^1 \to S^1$, $R(z) = z^2$.*

Proof. Suppose $\sigma = \sigma_h$ for some $h \in L^2(X, \mu)$. Then

$$\int z^n \, d\sigma_h = \langle \widehat{T}^n h, h\rangle = \langle \widehat{S}^{-1}\widehat{T}^n h, \widehat{S}^{-1}h\rangle = \langle \widehat{T}^{2n}\widehat{S}^{-1}h, \widehat{S}^{-1}h\rangle = \int z^{2n} \, d\sigma_{\widehat{S}^{-1}h}.$$

We deduce that $\int f(z) \, d\sigma_h = \int f(z^2) \, d\sigma_{\widehat{S}^{-1}h}$ for all f continuous, or that $\sigma_h(A) = \sigma_{\widehat{S}^{-1}h}(R^{-1}A)$ for all Borel subsets A of S^1.

Suppose that $\sigma(R^{-1}A) = 0$. Then $\sigma_h(R^{-1}A) = 0$, and so $\sigma_{\widehat{S}^{-1}h}(R^{-1}A) = 0$ because $\sigma_{\widehat{S}^{-1}h} \ll \sigma_h$ as σ is the maximal spectral type. It follows that $\sigma_h(A) = 0$, or that $\sigma \ll \sigma R^{-1}$.

On the other hand, we can see from the above that $\sigma_{\widehat{S}h}(A) = \sigma_h(R^{-1}A)$ for all Borel sets A, so that if $\sigma_h(A) = 0$, then $\sigma_{\widehat{S}h}(A) = 0$ since σ_h is the maximal spectral type, so that $\sigma_h(R^{-1}A) = 0$ and thus $\sigma R^{-1} \ll \sigma$ and σ is R-quasi-invariant. ∎

REMARK. Let $h \in L^2(X,\mu)$. Then

$$\begin{aligned}\widehat{S}^{-1}Z(h) &= \operatorname{span}\{\widehat{S}^{-1}\widehat{T}^n h : n \in \mathbb{Z}\} = \operatorname{span}\{\widehat{T}^{2n}(\widehat{S}^{-1}h) : n \in \mathbb{Z}\}\\ &\subseteq \operatorname{span}\{\widehat{T}^n(\widehat{S}^{-1}h) : n \in \mathbb{Z}\} = Z(\widehat{S}^{-1}h).\end{aligned}$$

The unitary operator $\widehat{T}$ can be represented in the usual way: there are functions $h_i \in L^2(X,\mu)$ and corresponding spectral measures σ_{h_i}, $i \in \mathbb{Z}^+$, so that $L^2(X,\mu)$ can be written as the direct sum of cyclic subspaces:

$$L^2(X,\mu) = Z(h_1) \oplus \cdots \oplus Z(h_n) \oplus \cdots, \qquad \sigma_{h_1} \gg \cdots \gg \sigma_{h_n} \gg \cdots,$$

where $\langle \widehat{T}^n h_i, h_i\rangle = \int_{S^1} z^n \, d\sigma_{h_i}(z)$. This representation is essentially unique in the sense that any other such representation leads to equivalent measures in the spectral sequence. In the case that $\widehat{S}^{-1}Z(h_i) = Z(\widehat{S}^{-1}h_i)$ for $i = 1, 2, \ldots$ we can conclude that each of the measures in the spectral sequence is R-quasi-invariant. However, this is not generally true as the latter subspace may be much larger. We can say the following:

PROPOSITION 3. *If (R,σ) is one-to-one, then $\widehat{S}^{-1}Z(h) = Z(\widehat{S}^{-1}h)$ for each $h \in L^2(X,\mu)$, and each measure σ_{h_i} in the spectral sequence is R-quasi-invariant.*

Proof. We know that $\widehat{S}^{-1}Z(h) \subseteq Z(\widehat{S}^{-1}h)$, so think of $\widehat{S}^{-1}$ as a map $\widehat{S}^{-1} : Z(h) \to Z(\widehat{S}^{-1}h)$. Now identify $Z(h)$ with $L^2(S^1, \sigma_h)$, and identify $Z(\widehat{S}^{-1}h)$ with $L^2(S^1, \sigma_{\widehat{S}^{-1}h})$. Then this identification gives a map $\widetilde{R}$ corresponding to $\widehat{S}^{-1}$:

$$\widetilde{R} : L^2(S^1,\sigma_h) \to L^2(S^1, \sigma_{\widehat{S}^{-1}h}), \qquad \widetilde{R}f(z) = f(z^2).$$

Because $\widehat{S}^{-1}$ is well defined and an isometry, the same is true of $\widetilde{R}$. The theorem will follow if we show that $\widetilde{R}$ is onto. We know that $R(z) = z^2$ is one-to-one with respect to both σ_h and $\sigma_{\widehat{S}^{-1}h}$ since both $\sigma_h \ll \sigma$ and $\sigma_{\widehat{S}^{-1}h} \ll \sigma$ where σ is the maximal spectral type. It follows that $\widetilde{R}$ is onto, and the first part of the proposition follows.

Furthermore, in this case

$$\begin{aligned}L^2(X,\mu) &= \widehat{S}^{-1}L^2(X,\mu) = \widehat{S}^{-1}Z(h_1) \oplus \cdots \oplus \widehat{S}^{-1}Z(h_n) \oplus \cdots\\ &= Z(\widehat{S}^{-1}h_1) \oplus \cdots \oplus Z(\widehat{S}^{-1}h_n) \oplus \cdots.\end{aligned}$$

The uniqueness of the spectral sequence now implies that the measures σ_{h_i} and $\sigma_{\widehat{S}^{-1}h_i}$ are equivalent. The result follows since $\sigma_h(A) = \sigma_{\widehat{S}^{-1}h}(R^{-1}A)$ for all h and Borel subsets A of S^1. ∎

We now obtain more detailed information about the case when T has simple spectrum. We show that R has to be one-to-one and onto the support of σ, and we give conditions for R to be measure preserving. In the lemma, σ_h is the maximal spectral type of U corresponding to some $h \in L^2(X,\mu)$.

LEMMA 3. *Let $U : H \to H$ be a unitary operator on a separable Hilbert space H which has simple spectrum and suppose there exists an operator $P : H \to H$ satisfying $U^2P = PU$. Then P is unitarily equivalent to an operator $\widetilde{P} : L^2(S^1, \sigma_h) \to L^2(S^1, \sigma_h)$ defined by*

$$\widetilde{P}f(z) = f(z^2)k(z), \quad f \in L^2(S^1, \sigma_h), \tag{1}$$

for some $k \in L^2(S^1, \sigma_h)$ and $h \in H$ where $Z(h) = H$.

Proof. We can represent U as $Vf(z) = zf(z)$, $f \in L^2(S^1, \sigma_h)$, where σ_h satisfies $\langle U^n h, h\rangle = \int_{S^1} z^n \, d\sigma_h$, $n \in \mathbb{Z}$; here h is a cyclic vector for U. We have $(V^2 f)(z) = z^2 f(z)$.

If W is the operator giving rise to this unitary equivalence, $W : Z(h) \to L^2(S^1, \sigma_h)$, $W(U^n h) = p_n$ (where $p_n(z) = z^n$, $n \in \mathbb{Z}$), we may suppose that

$$V^2\widetilde{P}f(z) = \widetilde{P}Vf(z), \quad f \in L^2(S^1, \sigma_h),$$

where $\widetilde{P} = WPW^{-1}$. Let $k = \widetilde{P}(p_0)$. Then

$$\widetilde{P}p_1(z) = \widetilde{P}Vp_0(z) = V^2\widetilde{P}p_0(z) = z^2k(z),$$

and in general

$$\widetilde{P}p_k(z) = z^{2k}k(z), \quad k \in \mathbb{Z}.$$

Thus

$$\widetilde{P}\Big(\sum_{k=-n}^{n} a_k z^k\Big) = k(z)\Big(\sum_{k=-n}^{n} a_k z^{2k}\Big)$$

and consequently

$$\widetilde{P}f(z) = k(z)f(z^2) \quad \text{for each } f \in L^2(S^1, \sigma_h). \ \blacksquare$$

THEOREM 6. *Let U be a unitary operator defined on a separable Hilbert space H, having simple spectrum. Then U is unitarily equivalent to U^2 if and only if the maximal spectral type σ of U is R-quasi-invariant ($R(z) = z^2$) and where R is one-to-one and onto a.e. σ.*

Proof. If U is unitarily equivalent to its square via a unitary operator P ($PU = U^2P$), we can represent U as $V : L^2(S^1, \sigma) \to L^2(S^1, \sigma)$, $Vf(z) = zf(z)$, and P as $\widetilde{P}f(z) = k(z)f(z^2)$ for some $k \in L^2(S^1, \sigma)$. This map is one-to-one and onto, so the same is true of $R(z) = z^2$, a.e. σ.

Conversely, suppose that the maximal spectral type σ of U is R-quasi-invariant, and R is one-to-one and onto a.e. σ. Represent U as $Vf(z) = zf(z)$ as above, and set $\widetilde{R}f(z) = k(z)f(z^2)$, where $k(z) = \sqrt{d\sigma \circ R/d\sigma}$ is the positive square root of the Radon–Nikodym derivative (well defined because σ is R-quasi-invariant). The hypotheses imply that $\widetilde{R} : L^2(S^1, \sigma) \to L^2(S^1, \sigma)$ is well defined, an isometry and onto a.e. σ. In addition we can check that $\widetilde{R}Vf(z) = V^2\widetilde{R}f(z)$. $\blacksquare$

COROLLARY 4. *Suppose that $ST = T^2S$, where T has simple spectrum. Then the map $R(z) = z^2$ defined on the non-closed support of σ (where σ is of the maximal spectral type of T) is one-to-one and onto and $k(z) \neq 0$ a.e. σ. In addition, (R, σ) is a measure preserving dynamical system if and only if $|k(z)| = 1$ a.e. σ. (Here k is the map defined in Lemma* 3.)

Proof. We have seen that we can represent $\widetilde{S}$ by $\widetilde{S}^{-1}f(z) = k(z)f(z^2)$. The first part then follows from the previous theorem. To see that $|k(z)| = 1$, use the fact that $\widetilde{S}$ is unitary, giving $\langle \widetilde{S}^{-1}f, \widetilde{S}^{-1}1\rangle = \langle f, 1\rangle$ (where $k(z) = \widetilde{S}^{-1}1$) for all $f \in L^2(S^1, \sigma)$. It follows that

$$\int_{S^1} f(z^2)|k(z)|^2\, d\sigma(z) = \int_{S^1} f(z)\, d\sigma(z)$$

for all $f \in L^2(S^1, \sigma)$. So $|k(z)| = 1$ if and only if R is measure preserving. ∎

COROLLARY 5. *If T is ergodic with discrete spectrum and eigenvalue group $\Lambda = e(T)$, then T is conjugate to T^2 if and only if the map $R : \Lambda \to \Lambda$ $R(\lambda) = \lambda^2$, is a group automorphism.*

Proof. The support of the spectral measure σ is $e(T)$, and R has to be one-to-one and onto on this set, and the result follows. ∎

Recall that a non-singular transformation (R, σ) is *ergodic* if $R^{-1}A = A$ implies that $\sigma(A) = 0$ or $\sigma(A^c) = 0$. R is ergodic if and only if for all measurable functions f, $f(Rz) = f(z)$ a.e. σ implies $f =$ constant a.e. σ.

In the case that (R, σ) is $R(z) = z^2$ and σ is the maximal spectral type of some ergodic transformation T, 1 is always an eigenvalue of T, so 1 is an atom of σ and also an invariant set for R. When we talk about the ergodicity of R, we mean on the support of σ, excluding the invariant set $\{1\}$.

We state a lemma, which is of independent interest:

LEMMA 4. *Suppose that $ST = T^2S$ and let σ be the maximal spectral type of $\widehat{T}$.*

(i) *If S is not weakly mixing, there exists $f \perp \mathbb{C}$ for which σ_f is an $R(z) = z^2$ invariant measure.*
(ii) *If (R, σ) is ergodic, and S is not weakly mixing, then σ is of the type of an R-invariant measure.*
(iii) *If there are no R-invariant measures $\nu \ll \sigma$, then S is weakly mixing.*

Proof. (i) Suppose there exists $f \perp \mathbb{C}$ with $\widehat{S}f = \lambda f$ and $\lambda \in S^1$. Then

$$\langle \widehat{T}^n f, f\rangle = \langle \widehat{T}^n \overline{\lambda}\widehat{S}f, \overline{\lambda}\widehat{S}f\rangle = \langle \widehat{S}\widehat{T}^{2n}f, \widehat{S}f\rangle = \langle \widehat{T}^{2n}f, f\rangle,$$

so that $\int_{S^1} z^n\, d\sigma_f(z) = \int_{S^1} z^{2n}\, d\sigma_f(z)$ for all $n \in \mathbb{Z}$. It follows that σ_f is $R(z) = z^2$ invariant.

(ii) If (R, σ) is ergodic with $\sigma_f \ll \sigma$, then σ_f is ergodic, and we deduce that $\sigma_f \sim \sigma$, so that σ is the type of an R-invariant measure.

(iii) If there are no R-invariant measures $\nu \ll \sigma$, then $\widehat{S}$ can have no eigenfunctions, so S is both ergodic and weakly mixing. ∎

In the next theorem we assume the existence of a conjugating map S which is ergodic with discrete spectrum. It is an open question whether or not such S can exist for some aperiodic T. It is hoped that this theorem will throw light on the existence or otherwise of such transformations.

THEOREM 7. *Suppose that $ST = T^2S$ and let σ be of the maximal spectral type of $\widehat{T}$, and set $R(z) = z^2$. If S is ergodic with discrete spectrum then*:

(i) *σ can be chosen to be a finite R-invariant measure, which is not the type of Lebesgue measure. In addition, T is rigid, and if T is ergodic, then T is weakly mixing (but not mixing).*

(ii) *If T has simple spectrum, then*

$$L^2(X,\mu) = \bigoplus_n Z(f_n),$$

where the set $\{f_n : n \in \mathbb{Z}\}$ is a subfamily of the eigenfunctions of S. Each $Z(f_n)$ is both $\widehat{T}$- and $\widehat{S}$-invariant, and the corresponding spectral measures σ_{f_n} constitute an ergodic decomposition for the measure σ with respect to R.

Proof. (i) There is a complete orthonormal basis for $L^2(X,\mu)$, $\{f_n : n \in \mathbb{Z}\}$, consisting of eigenvectors of S, i.e., $\widehat{S}f_n(x) = \lambda_n f_n(x)$ for each $n \in \mathbb{Z}$ and some $\lambda_n \in S^1$. As in the proof of the above lemma, we see that

$$\langle \widehat{T}^m f_n, f_n\rangle = \langle \widehat{T}^{2m} f_n, f_n\rangle$$

for all $m, n \in \mathbb{Z}$. We deduce that

$$\int_{S^1} z^m \, d\sigma_{f_n}(z) = \int_{S^1} z^{2m} \, d\sigma_{f_n}(z)$$

for all $m, n \in \mathbb{Z}$ (spectral measures with respect to $\widehat{T}$). This tells us that each of the measures σ_{f_n} is $R(z) = z^2$ invariant, and since the f_n's constitute a complete orthonormal basis, it can be shown that σ must be the type of an R-invariant measure. The fact that S is rigid implies that T is rigid, and rigidity is incompatible with the existence of an absolutely continuous component.

If T is ergodic, then since S is also ergodic with $\mathrm{msm}(S) < \infty$, Corollary 2 applies to show that T must be weakly mixing. (In the case that S is totally ergodic, Corollary 3 implies that $T^m \neq I$ for all $m \neq 1$.)

(ii) $\{f_n : n \in \mathbb{Z}^+\}$ is the set of eigenfunctions of S with corresponding eigenvalues $\{\lambda_n : n \in \mathbb{Z}^+\}$. As in the proof of (i) above we see that

$$\int z^k \, d\sigma_{n,m}(z) = \overline{\lambda}_n \lambda_m \int z^{2k} \, d\sigma_{n,m}(z),$$

where $\sigma_{n,m} = \sigma_{f_n,f_m}$, so that $\sigma_n = \sigma_{n,n}$ is an $R(z) = z^2$ invariant measure. We also have

$$\begin{aligned}\langle \widehat{T}^k f_n, f_m\rangle &= \langle W\widehat{T}^k f_n, Wf_m\rangle = \langle V^k W f_n, W f_m\rangle = \langle V^k r_n(z), r_m(z)\rangle \\ &= \int z^k r_n(z)\overline{r}_m(z)\, d\sigma(z),\end{aligned}$$

where $W : L^2(X,\mu) \to L^2(S^1,\sigma)$ is the isometry which sends $\widehat{T}^n h$ to z^n, and $r_n(z) = Wf_n$.

This implies that

$$d\sigma_{n,m}(z) = r_n(z)\overline{r}_m(z)d\sigma(z),$$

and in particular $d\sigma_n(z) = |r_n(z)|^2 d\sigma(z)$, so that $r_n(z)$ and σ_n have the same support (say A_n). Set

$$H_n = \{f \in L^2(X,\mu) : \widehat{S}f = \lambda_n f\}$$

and $\widetilde{H}_n = WH_n$. Since S is ergodic, H_n and $\widetilde{H}_n$ are one-dimensional subspaces with $r_n(z) \in \widetilde{H}_n$, for each $n \in \mathbb{Z}^+$. Consequently, $r_n(z)$ is the essentially unique function with the property that

$$r_n(z^2)k(z) = \lambda_n r_n(z) \quad \text{for } z \in A_n.$$

Clearly each A_n is an R-invariant set. Suppose that $\sigma(A_n \cap A_m) > 0$. Then

$$\chi_{A_n\cap A_m}(z^2) r_n(z^2)k(z) = \lambda_n \chi_{A_n\cap A_m}(z) r_n(z),$$

contradicting the fact that $\widetilde{H}_n$ is one-dimensional (unless $A_n = A_m$).

This same argument shows that A_n can have no non-trivial invariant subsets, so that (R, A_n, σ_n) is ergodic and measure preserving.

We have shown that $A_n = A_m$ or $A_n \cap A_m = \emptyset$ a.e. σ. In the latter case we must have $\sigma_{n,m} = 0$, so that

$$\langle \widehat{T}^k f_n, f_m\rangle = 0 \quad \text{for all } k \text{ when } n \neq m.$$

We deduce that in this case $Z(f_n) \perp Z(f_m)$ (with respect to $\widehat{T}$) and $\sigma_n \perp \sigma_m$. Note also that since each f_n is an eigenfunction of S, each $Z(f_n)$ is $\widehat{S}^{-1}$-invariant, and we see that the restrictions of $\widehat{T}$ and $\widehat{S}$ to $Z(f_n)$ are unitary operators with the property that $\widehat{T}\widehat{S} = \widehat{S}\widehat{T}^2$. We remark that it can be shown that $Z(f_n) \perp Z(f_n^2)$ for each $n \in \mathbb{Z}$. ∎

In the preprint Ageev (2005), it is mentioned that typically the rank one transformations conjugate to their squares have the property that (R,σ) is an ergodic measure preserving transformation. It follows that the hypothesis of (ii) below is non-vacuous. Bernoulli shifts having infinite entropy satisfy the conditions of (i).

THEOREM 8. *Suppose that $ST = T^2S$ and let σ be of the maximal spectral type of $\widehat{T}$, and set $R(z) = z^2$.*

(i) *If T is mixing and σ is measure preserving for R, then $\widehat{T}$ has a countable Lebesgue component. In fact σ is the type of Lebesgue measure.*
(ii) *If T has simple spectrum and (R,σ) is ergodic, then S has at most two ergodic components. If S is not ergodic then there is an $\widehat{S}$-invariant function $f_0 \perp \mathbb{C}$ such that σ_{f_0} is R-invariant, ergodic and of the maximal spectral type of $\widehat{T}$.*
(iii) *If T has the weak closure property and simple spectrum, and there exists a non-trivial $\phi \in C(T)$ with $\phi^n = I$ for some $n \in \mathbb{Z}^+$, then (R,σ) is not ergodic.*

Proof. (i) There exists $f \perp \mathbb{C}$ such that $\sigma_f = \sigma$, so that σ_f is R-invariant. Thus

$$\int z^n \, d\sigma_f = \int z^{2n} \, d\sigma_f \quad \text{for all } n \in \mathbb{Z},$$

or $\langle \widehat{T}^n f, f\rangle = \langle \widehat{T}^{2n} f, f\rangle$ for all $n \in \mathbb{Z}$. We therefore have

$$\langle \widehat{T} f, f\rangle = \langle \widehat{T}^2 f, f\rangle = \cdots = \langle \widehat{T}^{2^n} f, f\rangle \to \langle f,1\rangle\langle 1,f\rangle \quad \text{as } n \to \infty,$$

since T is mixing. We deduce that $\langle \widehat{T} f, f\rangle = 0$, and similarly, $\langle \widehat{T}^n f, f\rangle = 0$ for all $n \neq 0$. It follows that $\sigma = \sigma_f = \lambda$, Lebesgue measure, so that $\widehat{T}$ must have a Lebesgue component in its spectrum. This component is countable, for if we suppose $\widehat{T}$ has a Lebesgue component of multiplicity $m \in \mathbb{Z}^+$, then $\widehat{T}^2$ has a Lebesgue component of multiplicity $2m$, contradicting the conjugacy between T and T^2.

(ii) Denote by H_0 the subspace

$$H_0 = \{f \in L^2(X,\mu) : \widehat{S} f = f\},$$

and let $W : L^2(X,\mu) \to L^2(S^1,\sigma)$ be the isometry which sends $\widehat{T}^n h$ to z^n (where $Z(h) = L^2(X,\mu)$). As usual, represent $\widehat{S}$ as an operator on $L^2(S^1,\sigma)$, by $\widetilde{S}^{-1} g(z) = g(z^2)k(z)$. Then we can set

$$\widetilde{H}_0 = WH_0 = \{f \in L^2(S^1,\sigma) : \widetilde{S} f(z) = f(z) \text{ a.e. } \sigma\}.$$

Now $1 \in H_0$, so $W1 = r(z) \in \widetilde{H}_0$, and $r(z^2)k(z) = r(z)$ a.e. σ.

But $\widehat{T}1 = 1$, so $VW1 = W\widehat{T}1 = W1$ and $Vr(z) = r(z)$, or $zr(z) = r(z)$ a.e. σ. This implies that $r(z)$ is supported on the set $\{1\}$ and $k(1) = 1$, so $r(z) = c\chi_{\{1\}}$ for some constant c, i.e., $\chi_{\{1\}} \in \widetilde{H}_0$.

Let us suppose that (R,σ) is ergodic and let $f \in \widetilde{H}_0$ and $B = \operatorname{supp}(f) = \{z \in \operatorname{supp}(\sigma) : f(z) \neq 0,\ z \neq 1\}$. Then B is an R-invariant set, and R ergodic implies that $\sigma(B) = 0$ or $\sigma(B^{\mathrm{c}}) = 0$ (we know that $k(z) \neq 0$ a.e. σ). Suppose that $\sigma(B^{\mathrm{c}}) = 0$, i.e., $f \neq 0$ a.e. σ (otherwise every $f \in \widetilde{H}_0$ is zero a.e. σ except possibly at $z = 1$, so $\widetilde{H}_0$ is one-dimensional, which implies S is ergodic), and let $g \in \widetilde{H}_0$. Then $g/f(z^2) = g/f(z)$ a.e. σ, so that $g/f = $ constant a.e. σ, i.e., $\chi_B \cdot \widetilde{H}_0$ is a one-dimensional subspace of $L^2(S^1, \sigma)$. It follows that H_0 is at most a two-dimensional subspace.

Suppose that S is not ergodic and $f_0 \in H_0$, with $f_0 \perp \mathbb{C}$. Then σ_{f_0} is $R(z) = z^2$ invariant because $\langle \widehat{T}^n f_0, f_0 \rangle = \langle \widehat{T}^{2n} f_0, f_0 \rangle$ for all $n \in \mathbb{Z}$.

In addition, $\sigma_{f_0} \ll \sigma$, and σ ergodic (with respect to R) implies σ_{f_0} is ergodic, so $\sigma_{f_0} \sim \sigma$ and hence $Z(f_0) = \mathbb{C}^{\perp}$.

(iii) As usual, represent $\widehat{T}$ by $Vf(z) = zf(z)$, $\widehat{S}$ using $\widetilde{S}^{-1} f(z) = k(z) f(z^2)$, and $\widehat{\phi}$ as $\widetilde{\phi} f(z) = h(z) f(z)$.

Since T has the weak closure property, $S\phi = \phi^2 S$, so $\widetilde{S}^{-1}\widetilde{\phi} = \widetilde{\phi}^2 \widetilde{S}^{-1}$. This gives $h^2(z) = h(z^2)$ for all z. Set

$$g(z) = h(z) + h^2(z) + \cdots + h^{n-1}(z).$$

Then

$$g(z^2) = \sum_{i=1}^{n-1} h^i(z^2) = g(z),$$

since we must have $h^n(z) = 1$ and necessarily n is odd. We see that g is non-constant, for if not, $\sum_{i=1}^{n-1} \widehat{\phi}^i = cI$ for some constant c, and this implies that $\widehat{\phi}$ is the identity operator, a contradiction. ∎

5. Gaussian–Kronecker automorphisms conjugate to their composition squares. We give some examples of weakly mixing transformations having simple spectrum which are conjugate to their squares. The construction involves Gaussian automorphisms. These are used because spectral isomorphism gives rise to isomorphism.

Let $K \subseteq S^1$ be a *Kronecker set*, i.e., for every continuous complex-valued function $f(z)$ of absolute value 1 defined on K and for all $\varepsilon > 0$ there exists $n \in \mathbb{Z}$ such that

$$\sup_{z \in K} |f(z) - z^n| < \varepsilon.$$

Let σ_0 be a continuous symmetric measure ($\sigma_0(A) = \sigma_0(\overline{A})$ for all Borel sets $A \subset S^1$) whose support is $K \cup \overline{K}$. We call σ_0 a *Gaussian–Kronecker* measure. Given a symmetric measure σ, there is a corresponding Gaussian automorphism T_σ. We call σ the *spectral measure* of T_σ (which is distinct from the maximal spectral type, which is the measure

$$e^{\sigma} - \delta_1 = \sigma + \frac{\sigma^{(2)}}{2!} + \frac{\sigma^{(3)}}{3!} + \cdots ,$$

where $\sigma^{(n)}$ is the n-fold convolution product of σ with itself and δ_1 is a normalized measure supported at the point $z = 1$).

Let us recall some of the properties of Gaussian automorphisms and Gaussian–Kronecker measures (see Cornfeld, Fomin and Sinai (1980) and Lemańczyk, Parreau, Thouvenot (2000) for the properties of Gaussian automorphisms, and Rudin (1962) for the properties of Kronecker sets):

(i) If σ is a Gaussian–Kronecker measure, then T_σ has a simple and continuous spectrum. In fact this is true for any σ which has no rational relations (except for the symmetry relation), in particular for measures supported on symmetrized Kronecker sets.
(ii) All Gaussian automorphisms with the same continuous spectral measure are isomorphic.
(iii) Any conjugation between Gaussian automorphisms having simple spectrum is Gaussian.
(iv) Every Gaussian–Kronecker map has the weak closure property.
(v) If σ is Gaussian–Kronecker, then $\sigma \perp \sigma * \delta_z$ for all $z \in S^1 \setminus \{1\}$. Consequently, the map $R(z) = z^n$ is one-to-one a.e. σ on $K \cup \overline{K}$, for all $n \in \mathbb{Z}^+$.
(vi) If σ has an absolutely continuous component, then T_σ has a countable Lebesgue component in its spectrum.
(vii) Gaussian automorphisms having countable Lebesgue spectrum are isomorphic.

Denote by $\widehat{R}\sigma_0$ the image of σ_0 under R, and define a new measure by

$$\sigma = \sum_{k=-\infty}^{\infty} \frac{1}{2^{|k|}}\, \widehat{R}^k \sigma_0,$$

where $R(z) = z^2$. Then σ is clearly an R-quasi-invariant measure and it can be seen that R is one-to-one and onto, σ a.e., and that this is also true for the measure e^σ on $S^1 \setminus \{1\}$. In fact, since σ_0 has no rational relations (except for the symmetry relation) on $K \cup \overline{K}$, the same is true for σ, and $K \cap R^n K = \emptyset$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then (i) above ensures that there is a Gaussian automorphism T_σ having simple spectrum, and whose spectral measure is σ with maximal spectral type $e^\sigma - \delta_1$. Furthermore, both T_σ and T_σ^2 are Gaussian automorphisms, whose spectral isomorphism follows from Theorem 6. It then follows from (ii) that T_σ is conjugate to T_σ^2.

We summarize the above with a theorem.

THEOREM 9. *The Gaussian automorphism T_σ constructed above has a simple and continuous spectrum. In addition T_σ is conjugate to its square, and any conjugating map S is Gaussian* (*possibly non-ergodic*) *with a countable Lebesgue component in its spectrum.*

Proof. It suffices to prove that the conjugating map has a countable Lebesgue component. Denote by $H \subset L^2_{\mathbb{R}}(X, \mu)$ the Gaussian subspace for T_σ

where $L^2_{\mathbb{R}}(X,\mu)$ are the real functions in $L^2(X,\mu)$. For $f \in H$, denote by σ_f the usual spectral measure of f with respect to the maximal spectral type e^{σ}. Then in this case we see that $\sigma_f \ll \sigma$.

For A a Borel subset of S^1, we set

$$H_A = \{f \in H : \mathrm{supp}(\sigma_f) \subseteq A\}.$$

Now for the Kronecker set K defined above, $\sigma(K) > 0$, and we let $f \in H_{K\cup\overline{K}}$. As before, $\int_{S^1} z^n \, d\sigma_f(z) = \int_{S^1} z^{2n} \, d\sigma_{\widehat{S}^{-1}f}$, and this implies that $\sigma_{\widehat{S}f}(B) = \sigma_f(R^{-1}B)$ for any Borel set $B \subseteq S^1$.

It follows that $\mathrm{supp}(\sigma_{\widehat{S}f}) \subseteq R(K \cup \overline{K})$. In addition we have $\sigma_{\widehat{S}f} = \sigma_f R^{-1} \ll \sigma R^{-1}$, and since the latter measure is equivalent to σ, we have $\sigma_{\widehat{S}f} \ll \sigma$, which implies $\widehat{S}f \in H$ (using the fact that $\sigma \perp \sigma^{(n)}$ for all $n > 1$). We have shown that $\widehat{S}f \in H_{R(K\cup\overline{K})}$, and more generally, $\widehat{S}^n f \in H_{R^n(K\cup\overline{K})}$ for $n \in \mathbb{Z}$. Clearly the subspaces $H_{R^n(K\cup\overline{K})}$, $n \in \mathbb{Z}$, are pairwise orthogonal (since the sets $R^n(K \cup \overline{K})$, $n \in \mathbb{Z}$, are pairwise disjoint), so that

$$\langle \widehat{S}^n f, f\rangle = 0 \quad \text{for all } n \in \mathbb{Z},\ n \neq 0,$$

i.e., S must have a Lebesgue component, which must be countably Lebesgue as S is Gaussian (using property (vi)). ■

EXAMPLE. Set $T = T_\sigma \times T_\sigma$ (where T_σ is as above). Then $\widetilde{S}T = T^2\widetilde{S}$ where $\widetilde{S} = S \times S$. Generally, the Cartesian square $T_\sigma \times T_\sigma$ of a Gaussian automorphism T_σ has infinite multiplicity.

REFERENCES

O. N. Ageev (2005), *Spectral rigidity of group actions, applications to the case* $\mathrm{gr}(t, s : ts = st^2)$, Proc. Amer. Math. Soc. 134, 1331–1338.

I. Cornfeld, V. Fomin and Ya. Sinai (1980), *Ergodic Theory*, Springer.

I. Filipowicz, J. Kwiatkowski and M. Lemańczyk (1988), *Approximation of* $\mathbb{Z}_2$*-cocycles and shift dynamical systems*, Publ. Mat. 32, 91–110.

N. Friedman, P. Gabriel and J. King (1988), *An invariant for rigid rank one transformations*, Ergodic Theory Dynam. Systems 8, 53–72.

G. R. Goodson and V. V. Ryzhikov (1997), *Conjugations, joinings and direct products of locally rank one dynamical systems*, J. Dynam. Control Systems 3, 321–341.

G. R. Goodson (2002), *Ergodic dynamical systems conjugate to their composition squares*, Acta Math. Univ. Comenian. 71, 201–210.

G. R. Goodson (1999), *A survey of recent results in the spectral theory of ergodic dynamical systems*, J. Dynam. Control Systems 5, 173–226.

J. King (1986), *The commutant is the weak closure of the powers, for rank one transformations*, Ergodic Theory Dynam. Systems 6, 363–384.

M. Lemańczyk, F. Parreau and J.-P. Thouvenot (2000), *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Fund. Math. 164, 253–293.

W. Rudin (1962), *Fourier Analysis on Groups*, Wiley.

J.-P. Thouvenot (1987), *The metrical structure of some Gaussian processes*, in: Proc. Conf. on Ergodic Theory and Related Topics II (Georgenthal, 1986), Teubner-Texte Math. 94, Teubner, 1987, 195–198.

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