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WEYL SUBMERSIONS OF WEYL MANIFOLDS

BY

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Abstract. We define Weyl submersions, for which we derive equations analogous to the Gauss and Codazzi equations for an isometric immersion. We obtain a necessary and sufficient condition for the total space of a Weyl submersion to admit an Einstein–Weyl structure. Moreover, we investigate the Einstein–Weyl structure of canonical variations of the total space with Einstein–Weyl structure.

1. Introduction. In [11], B. O'Neill introduced the notion of a Riemannian submersion and obtained equations analogous to the Gauss and Codazzi equations for an isometric immersion.

Let $\pi : (M, g) \to (M', g')$ be a Riemannian submersion. We denote by $\mathcal{V}$ the vector subbundle of the tangent bundle TM of M consisting of the tangent vectors to the fibers of π. $\mathcal{V}$ is called the vertical distribution of π. $\mathcal{H}$ will denote the complementary "horizontal" distribution in TM determined by the metric g of M. For $t > 0$, we define the *canonical variation* g_t of the Riemannian metric g on M by setting $g_t|\mathcal{V} = tg|\mathcal{V}$, $g_t|\mathcal{H} = g|\mathcal{H}$ and $g_t(\mathcal{V}, \mathcal{H}) = 0$ (cf. [2]).

For a Riemannian submersion $\pi : (M, g) \to (M', g')$ with totally geodesic fibers, in [2], the author gave a necessary and sufficient condition for the Riemannian manifold (M, g) to admit an Einstein structure. Moreover he proved the following: Let $\pi : (M, g) \to (M', g')$ be a Riemannian submersion with totally geodesic fibers. Assume that (M, g), (M', g') and the fiber are Einstein manifolds (i.e., $r = \lambda g$, $r' = \lambda' g'$, $\widehat{r} = \widehat{\lambda}\widehat{g}$) and the integrability tensor A^g is nonzero. Then the canonical variation g_t $(t \neq 1)$ of g is also Einstein if and only if $0 < \widehat{\lambda} \neq \frac{1}{2}\lambda'$.

Let M be a manifold with a conformal structure $[g]$ and a torsion-free affine connection D. A triplet $(M, [g], D)$ is called a *Weyl manifold* if $Dg = \omega \otimes g$ for a 1-form ω. The Ricci tensor of an affine connection D is not necessarily symmetric.

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A Weyl manifold is said to be *Einstein–Weyl* if the symmetrized Ricci tensor of the affine connection D is proportional to a representative metric g in $[g]$. The Einstein–Weyl equation is conformally invariant.

In [12], H. Pedersen and A. Swann proved the following: Let $\pi : (M, g) \to (M', g')$ be a principal circle bundle with totally geodesic fibers over a compact Einstein manifold (M', g') with positive scalar curvature and the integrability tensor $A^g \neq 0$. For the vertical 1-form ω and the canonical variation g_t of g, we define a torsion-free affine connection D^t by $D^t g_t = \omega \otimes g_t$. Then, for $0 < t \leq t_0$ where g_{t_0} is an Einstein metric, the canonical variation (M, g_t, D^t) admits an Einstein–Weyl structure.

On the other hand, in [8], [9] we studied the existence of Einstein–Weyl structures on the total space of Riemannian submersions with totally geodesic fibers of dimension one over Einstein manifolds and on almost contact metric manifolds.

In [1], N. Abe and K. Hasegawa defined an affine submersion with horizontal distribution. They computed the fundamental equations, without using the metric tensor.

In [3], D. M. J. Calderbank and H. Pedersen studied conformal submersions. In particular they investigated conformal submersions with one-dimensional fibers and the minimal Weyl derivative exact.

We consider a special case of conformal submersions. Let $(M, [\overline{g}], D)$ and $(M', [\widetilde{g}], D')$ be two Weyl manifolds. Let $\pi : M \to M'$ be a submersion. We say that $\pi : (M, [\overline{g}], D) \to (M', [\widetilde{g}], D')$ is a *Weyl submersion* if $\pi : M \to M'$ is a submersion which satisfies the following two conditions:

(i) for some metric $g' \in [\widetilde{g}]$ there exists $g \in [\overline{g}]$ such that $\pi_* : (\mathcal{H}_x, g_x|\mathcal{H}_x) \to (T_{\pi(x)}M', g'_{\pi(x)})$ is an isometry for every x in M, i.e., $\pi : (M, g) \to (M', g')$ is a Riemannian submersion,

(ii) for basic vector fields X and Y which are π-related to $\widetilde{X}$ and $\widetilde{Y}$, $\mathcal{H}D_X Y$ is a basic vector field which is π-related to $D'_{\widetilde{X}}\widetilde{Y}$.

In the case that $\pi : (M, [\overline{g}], D) \to (M', [\widetilde{g}], D')$ is a Weyl submersion for which $\pi_* : (\mathcal{H}_x, g_x|\mathcal{H}_x) \to (T_{\pi(x)}M', g'_{\pi(x)})$ is an isometry, we write $\pi : (M, g, D) \to (M', g', D')$.

In this paper, for a Weyl submersion, we derive equations analogous to the Gauss and Codazzi equations for an isometric immersion. For a Weyl submersion $\pi : (M, g, D) \to (M', g, D')$ with Weyl totally geodesic fibers, we obtain a necessary and sufficient condition for the Weyl manifold (M, g, D) to admit an Einstein–Weyl structure.

In Section 5, we give some examples of Weyl submersions. As an example with the 1-form ω vertical, we produce a Weyl submersion whose total space is a contact metric manifold with Weyl structure induced from the contact form. As examples with ω horizontal, we exhibit Weyl submersions whose total space is a warped product with Weyl structure and whose total space is a locally conformal cosymplectic manifold with Weyl structure.

In Section 6, for a Weyl submersion, we investigate the Einstein–Weyl structure of canonical variations of the total space with Einstein–Weyl structure. If $\pi : (M, g, D) \to (M', g', D')$ is a Weyl submersion, then $\pi : (M, g_t, D^t) \to (M', g', D')$ is also a Weyl submersion, where D, D' and D^t are the torsion-free affine connections such that $Dg = \omega \otimes g$, $D'g' = \omega' \otimes g'$ and $D^t g_t = \omega \otimes g_t$.

When the 1-form ω is vertical, we obtain the following result: Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion with Weyl totally geodesic fibers of dimension 1 and $\dim M = n + 1$. Let ξ be a unit vertical vector field and η its dual 1-form with respect to g. Assume that $\omega = f\eta$, where f is a function on M. We assume that (M', g') is an Einstein manifold with $r'(\widetilde{X}, \widetilde{Y}) = \lambda' g'(\widetilde{X}, \widetilde{Y})$ whose scalar curvature is positive and (M, g, D) is an Einstein–Weyl manifold with $r^D(E, F) + r^D(F, E) = \Lambda g(E, F)$ and $A^g \neq 0$. If there exists a positive $t \neq 1$ such that (M, g_t, D^t) is an Einstein–Weyl manifold, then $X(f) = 0$ and $0 < 2\xi(f) + f^2 \neq \frac{2}{n-1}\lambda'$, where X is any horizontal vector field. If f is constant, then (M, g_t, D^t) admits an Einstein–Weyl structure for $t = \frac{(n-1)f^2}{4\lambda' - (n-1)f^2}$.

Next, when the 1-form ω is horizontal, we obtain the following result: Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion with Weyl totally geodesic fibers over an Einstein–Weyl manifold (M', g', D') with $r^{D'}(\widetilde{X}, \widetilde{Y}) + r^{D'}(\widetilde{Y}, \widetilde{X}) = \Lambda' g'(\widetilde{X}, \widetilde{Y})$ and $A^D \neq 0$. Suppose ω is horizontal and Λ' is constant. We assume that the fibers $(\widehat{F}, \widehat{g})$ are Einstein manifolds with $\widehat{r}(U, V) = \widehat{\lambda}\widehat{g}(U, V)$ and (M, g, D) is an Einstein–Weyl manifold with $r^D(E, F) + r^D(F, E) = \Lambda g(E, F)$. Then there exists a positive $t \neq 1$ such that (M, g_t, D^t) is also an Einstein–Weyl manifold if and only if $0 < 4\widehat{\lambda} \neq \Lambda'$.

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2. Weyl manifolds. Let $(M, [g], D)$ be a Weyl manifold with $Dg = \omega \otimes g$. We assume $\dim M \geq 3$.

Let ∇ be the Levi-Civita connection of g. We define a vector field B by $g(X, B) = \omega(X)$. Then, since $Dg = \omega \otimes g$, we have

$$D_X Y = \nabla_X Y - \tfrac{1}{2}\omega(X)Y - \tfrac{1}{2}\omega(Y)X + \tfrac{1}{2}g(X, Y)B \tag{1}$$

for any vector fields X, Y on M.

The curvature tensor R^D of the affine connection D is defined by $R^D(X, Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z$. Let R be the curvature tensor field of the Levi-Civita connection ∇ of g. Then

$$(2)\quad \begin{aligned} &R^D(X,Y)Z \\ &= R(X,Y)Z - \tfrac{1}{2}\Big\{\big[(\nabla_X\omega)Z + \tfrac{1}{2}\omega(X)\omega(Z)\big]Y \\ &\quad - \big[(\nabla_Y\omega)Z + \tfrac{1}{2}\omega(Y)\omega(Z)\big]X + ((\nabla_X\omega)Y)Z - ((\nabla_Y\omega)X)Z \\ &\quad - g(Y,Z)\big(\nabla_X B + \tfrac{1}{2}\omega(X)B\big) + g(X,Z)\big(\nabla_Y B + \tfrac{1}{2}\omega(Y)B\big)\Big\} \\ &\quad - \tfrac{1}{4}|\omega|^2(g(Y,Z)X - g(X,Z)Y), \end{aligned}$$

where X, Y and Z are any vector fields on M.

By a simple calculation, we have

LEMMA 1 (cf. [10]).

(a) $g(R^D(X,Y)Z,H) + g(R^D(Y,X)Z,H) = 0,$

(b) $g(R^D(X,Y)Z,H) + g(R^D(X,Y)H,Z) = -2d\omega(X,Y)g(Z,H),$

(c) $g(R^D(X,Y)Z,H) + g(R^D(Y,Z)X,H) + g(R^D(Z,X)Y,H) = 0,$

(d) $$\begin{aligned} &g(R^D(X,Y)Z,H) - g(R^D(Z,H)X,Y) \\ &= d\omega(Y,X)g(Z,H) + d\omega(Z,H)g(Y,X) \\ &\quad + d\omega(Z,X)g(H,Y) + d\omega(H,Y)g(Z,X) \\ &\quad + d\omega(Y,Z)g(X,H) + d\omega(X,H)g(Y,Z), \end{aligned}$$

where $2d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$. ■

The Ricci tensor field r^D is defined as follows:

$$r^D(X,Y) = \mathrm{tr}(Z \mapsto R^D(Z,X)Y),$$

where $X, Y, Z \in T_x(M)$. Let $X_1, \ldots, X_n$ be an orthonormal basis of $T_x(M)$ with respect to g. By using (2), we get

$$(3)\quad \begin{aligned} r^D(X,Y) &= r(X,Y) + \tfrac{1}{2}(n-1)(\nabla_X\omega)Y \\ &\quad - \tfrac{1}{2}(\nabla_Y\omega)X + \tfrac{1}{4}(n-2)\omega(X)\omega(Y) \\ &\quad + g(X,Y)\Big(\tfrac{1}{2}\sum_{i=1}^{n} g(\nabla_{X_i}B, X_i) - \tfrac{1}{4}(n-2)|\omega|^2\Big). \end{aligned}$$

A Weyl manifold $(M,[g],D)$ is said to have an *Einstein–Weyl structure* if there exists a function Λ on M such that

$$(4)\qquad r^D(X,Y) + r^D(Y,X) = \Lambda g(X,Y).$$

Since D is not a metric connection, the Ricci tensor is not necessarily symmetric.

3. Weyl submersions. We denote the second fundamental form and integrability tensor of a Riemannian manifold by T^g and A^g respectively.

LEMMA 2. *Let $\pi : (M, g) \to (M', g')$ be a Riemannian submersion. Let D and D' be torsion-free affine connections such that $Dg = \omega \otimes g$, $D'g' = \omega' \otimes g'$. Then, for basic vector fields X and Y which are π-related to $\widetilde{X}$ and $\widetilde{Y}$, $\mathcal{H}D_X Y$ is basic vector field which is π-related to $D'_{\widetilde{X}}\widetilde{Y}$ if and only if $\omega(X) = \omega'(\widetilde{X}) \circ \pi$.*

Proof. Suppose that $\mathcal{H}D_X Y$ is a basic vector field which is π-related to $D'_{\widetilde{X}}\widetilde{Y}$. For basic vector fields X, Y, Z which are π-related to $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$, from $g(X, Y) = g'(\widetilde{X}, \widetilde{Y}) \circ \pi$, we obtain $(D_X g)(Y, Z) = (D'_{\widetilde{X}} g')(\widetilde{Y}, \widetilde{Z}) \circ \pi$. Thus we get $\omega(X) = \omega'(\widetilde{X}) \circ \pi$.

Next, suppose that $\omega(X) = \omega'(\widetilde{X}) \circ \pi$. Then $\mathcal{H}B$ is a basic vector field corresponding to B', where $g(X, B) = \omega(X)$ and $g'(\widetilde{X}, B') = \omega'(\widetilde{X})$. From (1) and the properties of a Riemannian submersion, it follows that $\mathcal{H}D_X Y$ is a basic vector field which is π-related to $D'_{\widetilde{X}}\widetilde{Y}$. ∎

Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion. The fundamental tensors T^D and A^D are defined by

$$T^D_E F := \mathcal{H}D_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}D_{\mathcal{V}E}\mathcal{H}F, \tag{5}$$

$$A^D_E F := \mathcal{V}D_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}D_{\mathcal{H}E}\mathcal{V}F, \tag{6}$$

where E and F are any vector fields on M.

From the definitions and (1), using the properties of a Riemannian submersion, we have the following lemma.

LEMMA 3. *For any vector fields E, F on M, we have*

$$A^D_E F = A^g_E F + \tfrac{1}{2} g(\mathcal{H}E, \mathcal{H}F)\mathcal{V}B - \tfrac{1}{2}\omega(\mathcal{V}F)\mathcal{H}E, \tag{a}$$

$$T^D_E F = T^g_E F + \tfrac{1}{2} g(\mathcal{V}E, \mathcal{V}F)\mathcal{H}B - \tfrac{1}{2}\omega(\mathcal{H}F)\mathcal{V}E. \tag{b}$$

If X, Y are horizontal and U, V are vertical, then

$$A^D_X Y = A^g_X Y + \tfrac{1}{2} g(X, Y)\mathcal{V}B, \tag{c}$$

$$A^D_X Y = \tfrac{1}{2}\mathcal{V}[X, Y] + \tfrac{1}{2} g(X, Y)\mathcal{V}B, \tag{d}$$

$$A^D_X Y = -A^D_Y X + g(X, Y)\mathcal{V}B, \tag{e}$$

$$A^D_X U = \mathcal{H}D_U X + \mathcal{H}[X, U], \tag{f}$$

$$T^D_U V = T^g_U V + \tfrac{1}{2} g(U, V)\mathcal{H}B, \tag{g}$$

$$T^D_U V = T^D_V U, \tag{h}$$

$$T^D_U X = \mathcal{V}D_X U + \mathcal{V}[U, X]. \tag{i}$$

∎

From the definition, using $Dg = \omega \otimes g$, the following lemma can be proved as in the case of a Riemannian submersion.

LEMMA 4.

(a) *For vector fields E, F, a horizontal vector field X and a vertical vector field U,*

$$g(A_X^D E, F) = -g(E, A_X^D F), \qquad g(T_U^D E, F) = -g(E, T_U^D F).$$

(b) *T^D and A^D interchange the horizontal and vertical subspaces.* ∎

Now, for a Weyl submersion $\pi : (M, g, D) \to (M', g', D')$ we derive equations analogous to the Gauss and Codazzi equations of an immersion. Let $R^{D'}$ be the curvature tensor field of the affine connection D'. Let $R^{\widehat{D}}$ be the curvature tensor field of the induced affine connection $\widehat{D}$ on the fibers. From Lemmas 1, 3 and 4 we obtain the following theorem.

THEOREM 1. *Let X, Y, Z, H be horizontal vector fields on M which are π-related to $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{H}$ on M', and U, V, W, W' vertical vector fields on M. Then*

$$\begin{aligned}(7)\quad & g(R^D(X,Y)Z,H) \\ &= g'(R^{D'}(\widetilde{X},\widetilde{Y})\widetilde{Z},\widetilde{H})\circ\pi - g(A_Y^D Z, A_X^D H) + g(A_X^D Z, A_Y^D H) \\ &\quad + 2g(A_X^D Y, A_Z^D H) - g(X,Y)\omega(A_Z^D H),\end{aligned}$$

$$\begin{aligned}(8)\quad g(R^D(X,Y)Z,U) &= g((D_X A^D)_Y Z, U) - g((D_Y A^D)_X Z, U) \\ &\quad - g(A_X^D Y, T_U^D Z) + g(A_Y^D X, T_U^D Z),\end{aligned}$$

$$\begin{aligned}(9)\quad g(R^D(X,Y)U,Z) &= g((D_X A^D)_Y U, Z) - g((D_Y A^D)_X U, Z) \\ &\quad + g(A_X^D Y, T_U^D Z) - g(A_Y^D X, T_U^D Z),\end{aligned}$$

$$\begin{aligned}(10)\quad g(R^D(X,Y)U,V) &= g((D_U A^D)_X V, Y) - g((D_V A^D)_X U, Y) \\ &\quad - g(A_Y^D V, A_X^D U) + g(A_X^D V, A_Y^D U) - g(T_V^D X, T_U^D Y) \\ &\quad + g(T_U^D X, T_V^D Y) - g(Y, A_X^D U)\omega(V) + g(Y, A_X^D V)\omega(U) \\ &\quad + d\omega(Y,X)g(U,V) + d\omega(U,V)g(Y,X),\end{aligned}$$

$$\begin{aligned}(11)\quad g(R^D(U,X)Y,Z) &= -g((D_Y A^D)_Z X, U) + g((D_Z A^D)_Y X, U) \\ &\quad + g(A_Y^D Z, T_U^D X) - g(A_Z^D Y, T_U^D X) \\ &\quad - d\omega(U,X)g(Y,Z) - d\omega(Z,U)g(Y,X) - d\omega(U,Y)g(X,Z),\end{aligned}$$

$$\begin{aligned}(12)\quad g(R^D(U,X)Y,V) &= g((D_U A^D)_X Y, V) - g((D_X T^D)_U Y, V) \\ &\quad - g(T_U^D X, T_V^D Y) + g(A_X^D U, A_Y^D V) + g(A_X^D U, Y)\omega(V),\end{aligned}$$

$$\begin{aligned}(13)\quad g(R^D(U,X)V,Y) &= g((D_U A^D)_X V, Y) - g((D_X T^D)_U V, Y) \\ &\quad + g(T_U^D X, T_V^D Y) - g(A_X^D U, A_Y^D V) - g(A_X^D U, Y)\omega(V),\end{aligned}$$

$$\begin{aligned}(14)\quad g(R^D(U,X)V,W) &= g((D_V T^D)_W U, X) - g((D_W T^D)_V U, X) \\ &\quad + d\omega(X,U)g(V,W) + d\omega(W,X)g(V,U) + d\omega(X,V)g(U,W),\end{aligned}$$

$$(15)\quad g(R^D(U,V)X,Y) = g((D_U A^D)_X V, Y) - g((D_V A^D)_X U, Y) - g(A^D_Y V, A^D_X U) + g(A^D_X V, A^D_Y U) - g(T^D_V X, T^D_U Y) + g(T^D_U X, T^D_V Y) - g(Y, A^D_X U)\omega(V) + g(Y, A^D_X V)\omega(U),$$

$$(16)\quad g(R^D(U,V)X,W) = g((D_U T^D)_V X, W) - g((D_V T^D)_U X, W),$$

$$(17)\quad g(R^D(U,V)W,X) = g((D_U T^D)_V W, X) - g((D_V T^D)_U W, X),$$

$$(18)\quad g(R^D(U,V)W,W') = g(R^{\widehat{D}}(U,V)W,W') - g(T^D_V W, T^D_U W') + g(T^D_U W, T^D_V W'). \ \blacksquare$$

Let $K^D, K^{\widehat{D}}, K^{D'}$ be the sectional curvatures of the affine connections $D, \widehat{D}$ and D' respectively. We set $|X \wedge Y|^2 = g(X,X)g(Y,Y) - g(X,Y)^2$. Then we obtain the following

Corollary 1. *Let X, Y be horizontal vector fields on M which are π-related to $\widetilde{X}, \widetilde{Y}$ on M', and U, V vertical vector fields on M. Suppose $|X| = |Y| = |U| = |V| = 1$, $|X \wedge Y| = 1$, $|U \wedge V| = 1$. Then*

$$(19)\quad K^D(X,Y) = K^{D'}(\widetilde{X}, \widetilde{Y}) \circ \pi - 3|A^D_X Y|^2 - \tfrac{1}{4}|\mathcal{V}B|^2,$$

$$(20)\quad K^D(U,V) = K^{\widehat{D}}(U,V) + |T^D_U V|^2 - g(T^D_U U, T^D_V V),$$

$$(21)\quad K^D(X,U) = g((D_X T^D)_U U, X) - |T^D_U X|^2 + |A^D_X U|^2 + \tfrac{1}{2}(\omega(U)^2 + g(D_U \mathcal{V}B, U)) + g(A^D_X U, X)\omega(U). \ \blacksquare$$

Next, for a Weyl submersion $\pi : (M, g, D) \to (M', g', D')$ we derive some properties of DA^D and DT^D. From Lemmas 1, 3, 4 and Theorem 1, using $Dg = \omega \otimes g$ we have the following

Lemma 5. *Let E be a vector field on M. For horizontal vector fields X, Y, Z and vertical vector fields U, V, W, we have*

(a) $g((D_E A^D)_X Y, U) = -g((D_E A^D)_X U, Y),$

(b) $g((D_E T^D)_U V, X) = -g((D_E T^D)_U X, V),$

(c) $g((D_E T^D)_U V, X) = g((D_E T^D)_V U, X),$

(d) $g((D_E A^D)_X Y, U) = -g((D_E A^D)_Y X, U) + (\omega(E)\omega(U) + g(D_E \mathcal{V}B, U))g(X,Y),$

(e) $g((D_U A^D)_X Y, V) + g((D_V A^D)_X Y, U) = g((D_Y T^D)_U V, X) - g((D_X T^D)_U V, Y) + d\omega(X,Y)g(V,U) + d\omega(U,V)g(X,Y) - g(A^D_X U, Y)\omega(V) + g(A^D_Y V, X)\omega(U) + (\omega(U)\omega(V) + g(D_V \mathcal{V}B, U))g(X,Y),$

$$\text{(f)}\quad g((D_X A^D)_Y Z, U) = g((\nabla_X A^g)_Y Z, U) + \tfrac{1}{2}\omega(X) g(Y, Z) g(\mathcal{V}B, U) + \tfrac{1}{2} g(Y, Z) g(D_X \mathcal{V}B, U) + \tfrac{1}{2}\omega(Y) g(A^g_X Z, U) - \tfrac{1}{2} g(X, Y) g(A^g_B Z, U) + \tfrac{1}{2}\omega(X) g(A^g_Y Z, U) + \tfrac{1}{2}\omega(Z) g(A^g_Y X, U) - \tfrac{1}{2} g(X, Z) g(A^g_Y B, U). \blacksquare$$

Now we suppose that $\dim M = m + n$ and $\dim M' = n$. Let $X_1, \dots, X_n$ be an orthonormal basis of $\mathcal{H}_x$ and $V_1, \dots, V_m$ an orthonormal basis of $\mathcal{V}_x$ with respect to g. From Theorem 1, we get immediately the following

PROPOSITION 1. *Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion. Let $r^D, r^{\widehat{D}}, r^{D'}$ be the Ricci curvatures of the affine connections $D, \widehat{D}$ and D' respectively. For horizontal vector fields X, Y which are π-related to $\widetilde{X}, \widetilde{Y}$, and vertical vector fields U, V, we derive the Ricci curvature*:

$$\text{(22)}\quad r^D(X, Y) = r^{D'}(\widetilde{X}, \widetilde{Y}) \circ \pi - 3 \sum_{i=1}^{n} g(A^D_X X_i, A^D_Y X_i) + \sum_{j=1}^{m} g(A^D_X V_j, A^D_Y V_j) - \sum_{j=1}^{m} g(T^D_{V_j} X, T^D_{V_j} Y) + \sum_{j=1}^{m} \{ g((D_{V_j} A^D)_X Y, V_j) - g((D_X T^D)_{V_j} Y, V_j) \} - \frac{n+2}{2} g(A^D_X Y, \mathcal{V}B) + g(X, Y) g(\mathcal{V}B, \mathcal{V}B),$$

$$\text{(23)}\quad r^D(U, V) = r^{\widehat{D}}(U, V) - \sum_{j=1}^{m} g(T^D_{V_j} V_j, T^D_U V) + \sum_{i=1}^{n} g(A^D_{X_i} U, A^D_{X_i} V) + \sum_{i=1}^{n} g((D_{X_i} T^D)_U V, X_i) - \sum_{i=1}^{n} g((D_U A^D)_{X_i} V, X_i) - \sum_{i=1}^{n} g(T^D_U X_i, T^D_V X_i) + \sum_{j=1}^{m} g(T^D_V V_j, T^D_U V_j) - \frac{n}{2}\omega(U)\omega(V),$$

$$\text{(24)}\quad r^D(X, U) = \sum_{i=1}^{n} g((D_{X_i} A^D)_X U, X_i) - \sum_{i=1}^{n} g((D_X A^D)_{X_i} U, X_i) + \sum_{i=1}^{n} g(A^D_{X_i} X, T^D_U X_i) - \sum_{i=1}^{n} g(A^D_X X_i, T^D_U X_i) + \sum_{j=1}^{m} g((D_U T^D)_{V_j} V_j, X) - \sum_{j=1}^{m} g((D_{V_j} T^D)_U V_j, X) + m\, d\omega(X, U),$$

$$(25)\quad r^D(U,X) = \sum_{i=1}^{n} g((D_{X_i}A^D)_X U, X_i) - \sum_{i=1}^{n} g((D_X A^D)_{X_i} U, X_i) + \sum_{i=1}^{n} g(A^D_{X_i} X, T^D_U X_i) - \sum_{i=1}^{n} g(A^D_X X_i, T^D_U X_i) + \sum_{j=1}^{m} g((D_{V_j} T^D)_U X, V_j) - \sum_{j=1}^{m} g((D_U T^D)_{V_j} X, V_j) + nd\omega(U,X). \ \blacksquare$$

We introduce some notations. For horizontal vector fields X, Y and vertical vector fields U, V, we define

$$g(A^D_X, A^D_Y) = \sum_{i=1}^{n} g(A^D_X X_i, A^D_Y X_i) = \sum_{j=1}^{m} g(A^D_X V_j, A^D_Y V_j),$$

$$g(A^D_X, T^D_U) = \sum_{i=1}^{n} g(A^D_X X_i, T^D_U X_i) = \sum_{j=1}^{m} g(A^D_X V_j, T^D_U V_j),$$

$$g(A^D U, A^D V) = \sum_{i=1}^{n} g(A^D_{X_i} U, A^D_{X_i} V),$$

$$g(T^D X, T^D Y) = \sum_{j=1}^{m} g(T^D_{V_j} X, T^D_{V_j} Y),$$

$$g(T^D_U, T^D_V) = \sum_{i=1}^{n} g(T^D_U X_i, T^D_V X_i),$$

$$(\widetilde{\delta} T^D)(U,V) = \sum_{i=1}^{n} g((D_{X_i} T^D)_U V, X_i)$$

and for any tensor field E on M,

$$\breve{\delta} E = -\sum_{i=1}^{n} (D_{X_i} E)_{X_i}, \qquad \widehat{\delta} E = -\sum_{j=1}^{m} (D_{V_j} E)_{V_j},$$

$$\delta E = \breve{\delta} E + \widehat{\delta} E, \qquad \breve{\delta}^g E = -\sum_{i=1}^{n} (\nabla_{X_i} E)_{X_i}.$$

We set

$$N = \sum_{j=1}^{m} T^D_{V_j} V_j, \qquad N^g = \sum_{j=1}^{m} T^g_{V_j} V_j,$$

$$|A^D|^2 = \sum_{i=1}^{n} g(A^D_{X_i}, A^D_{X_i}) = \sum_{j=1}^{m} g(A^D V_j, A^D V_j),$$

$$|T^D|^2 = \sum_{i=1}^{n} g(T^D X_i, T^D X_i) = \sum_{j=1}^{m} g(T^D_{V_j}, T^D_{V_j}).$$

Since $\breve{\delta} N = -\sum_{i=1}^{n} g(D_{X_i} N, X_i)$, we obtain $2\sum_{j=1}^{m}(\widetilde{\delta} T^D)(V_j, V_j) = -2\breve{\delta} N + 2\omega(N)$.

Now a straightforward computation gives

COROLLARY 2. *Let $s^D, s^{\widehat{D}}, s^{D'}$ be the scalar curvatures of the affine connections $D, \widehat{D}$ and D' respectively. Then*

$$\begin{aligned} s^D = {} & s^{D'} \circ \pi + s^{\widehat{D}} - |A^D|^2 - |T^D|^2 - |N|^2 - 2\breve{\delta} N + 2\omega(N) \\ & + \frac{n(4-n)}{4} |\mathcal{V} B|^2 + n \sum_{j=1}^{m} g(D_{V_j} \mathcal{V} B, V_j). \ \blacksquare \end{aligned} \tag{26}$$

For a Riemannian submersion $\pi : (M, g) \to (M', g')$, we say that $\mathcal{H}$ satisfies the *Yang–Mills condition* if $g((\breve{\delta}^g A^g)X, U) - g(A^g_X, T^g_U) = 0$, where X is any horizontal vector field and U is any vertical vector field (cf. [2]).

Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion. Using Lemma 5(f), we get

$$g(A^D_X, T^D_U) = g(A^g_X, T^g_U) + \tfrac{1}{2}\omega(T^D_U X) + \tfrac{1}{2}\omega(A^D_X U) + \tfrac{1}{4}\omega(X)\omega(U)$$

and

$$\begin{aligned} g((\breve{\delta} A^D)X, U) = {} & g((\breve{\delta}^g A^g)X, U) - \tfrac{1}{2}(D_X \omega)(U) \\ & + \frac{n-4}{2}\omega(A^D_X U) + \frac{n-3}{4}\omega(X)\omega(U). \end{aligned}$$

Thus we have the following

LEMMA 6. *Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion. Then*

$$\begin{aligned} & g((\breve{\delta} A^D)X, U) - g(A^D_X, T^D_U) \\ & \quad = g((\breve{\delta}^g A^g)X, U) - g(A^g_X, T^g_U) - \tfrac{1}{2}(D_X\omega)(U) - \tfrac{1}{2}\omega(T^D_U X) \\ & \qquad + \frac{n-5}{2}\omega(A^D_X U) + \frac{n-4}{4}\omega(X)\omega(U), \end{aligned}$$

where X is any horizontal vector field and U any vertical vector field. $\blacksquare$

4. Einstein–Weyl manifolds. Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion. We set $Dg = \omega \otimes g$ and $D'g' = \omega' \otimes g'$. From Proposition 1 and Lemma 5, we have the following

PROPOSITION 2. *Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion. Assume that* $\dim M = m + n$ *and* $\dim M' = n$. *For horizontal vector fields X, Y which are π-related to $\widetilde{X}, \widetilde{Y}$, and vertical vector fields U, V, we have*

(27)
$$
\begin{aligned}
r^D(X,Y)+r^D(Y,X) &= r^{D'}(\widetilde{X},\widetilde{Y})\circ\pi + r^{D'}(\widetilde{Y},\widetilde{X})\circ\pi - 4g(A^D_X, A^D_Y)\\
&\quad - 2g(T^DX, T^DY) + \sum_{j=1}^{m} g((D_XT^D)_{V_j}V_j, Y)\\
&\quad + \sum_{j=1}^{m} g((D_YT^D)_{V_j}V_j, X)\\
&\quad + \left\{\frac{-n+4}{2}\,|\mathcal{V}B|^2 + \sum_{j=1}^{m} g(D_{V_j}\mathcal{V}B, V_j)\right\} g(X,Y),
\end{aligned}
$$

(28)
$$
\begin{aligned}
r^D(U,V)+r^D(V,U) &= r^{\widehat{D}}(U,V) + r^{\widehat{D}}(V,U) - 2g(N, T^D_UV) + 2g(A^DU, A^DV)\\
&\quad + 2(\widetilde{\delta}T^D)(U,V) + \frac{n}{2}\,\{g(D_U\mathcal{V}B, V) + g(D_V\mathcal{V}B, U)\},
\end{aligned}
$$

(29)
$$
\begin{aligned}
&r^D(X,U)+r^D(U,X)\\
&= 2\Big\{g((\widehat{\delta}T^D)U, X) + \sum_{j=1}^{m} g((D_UT^D)_{V_j}V_j, X) - g((\breve{\delta}A^D)X, U)\\
&\quad - 2g(A^D_X, T^D_U) + \omega(T^D_UX)\Big\} + (n-2)\{\omega(X)\omega(U) + g(D_X\mathcal{V}B, U)\}\\
&\quad + (n-m)d\omega(U,X). \ \blacksquare
\end{aligned}
$$

Now we consider a Weyl submersion $\pi : (M, g, D) \to (M', g', D')$ with one-dimensional Weyl totally geodesic fibers (i.e. $T^D = 0$), where $Dg = \omega\otimes g$ and $D'g' = \omega'\otimes g'$. From Proposition 2, we obtain the following theorem.

THEOREM 2. *Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion with Weyl totally geodesic fibers of dimension 1 and $\dim M = n+1$. Let ξ be a unit vertical vector field and η its dual 1-form with respect to g. Assume that $\omega = \widetilde{\omega}+\widehat{\omega}$, where $\widetilde{\omega} = \pi^*\omega'$ and $\widehat{\omega} = f\eta$ for a function f on M. Then (M, g, D) is an Einstein–Weyl manifold with $r^D(E, F) + r^D(F, E) = \Lambda g(E, F)$ for some function Λ if and only if*

(30)
$$
\begin{aligned}
r^{D'}(\widetilde{X},\widetilde{Y})\circ\pi + r^{D'}(\widetilde{Y},\widetilde{X})\circ\pi - 4g(A^D_X, A^D_Y)&\\
+\left\{\frac{-n+3}{2}\,f^2 + \xi(f)\right\} g(X,Y) &= \Lambda g(X,Y),
\end{aligned}
$$

(31)
$$2g(A^D\xi, A^D\xi) + n\big\{\xi(f) - \tfrac{1}{2}f^2\big\} = \Lambda,$$

(32)
$$-2g((\breve{\delta}A^D)X, \xi) + \frac{n-3}{4}\,(2X(f) + \widetilde{\omega}(X)f) = 0,$$

where X, Y are any horizontal vector fields which are π-related to $\widetilde{X}, \widetilde{Y}$. $\blacksquare$

REMARK. In [3], Calderbank and Pedersen treated a conformal submersion with totally geodesic fibers and $\omega = \frac{n-2}{n-1}\pi^*\omega' + f\eta$. The fibers of a Weyl submersion of the above theorem are Weyl totally geodesic but not necessarily totally geodesic.

Next, we consider a Weyl submersion $\pi : (M, g, D) \to (M', g', D')$ for which ω is horizontal.

LEMMA 7. *Let X be a horizontal vector field and U a vertical vector field. If ω is horizontal, then $d\omega(X, U) = 0$.*

Proof. Since ω is horizontal and $[X, U]$ is vertical, using Lemma 3, we have

$$\begin{aligned} 2d\omega(X,U) &= -U\omega(X) = -(D_U g)(X,B) - g(D_U X, B) - g(X, D_U B) \\ &= -\omega(U)g(X,B) - g(D_X U, B) - g(X, D_B U) \\ &= -g(A^D_X U, B) - g(A^D_B U, X) \\ &= g(U, A^D_X B) + g(U, A^D_B X) = 0. \ \blacksquare \end{aligned}$$

Let $\widehat{r}$ be the Ricci tensor of the induced Riemannian metric $\widehat{g}$ on the fibers. In the case that ω is horizontal, $\widehat{D}_U V = \mathcal{V} D_U V = \mathcal{V}\nabla_U V$, thus $\widehat{D}$ is the Levi-Civita connection of $\widehat{g}$. From Proposition 2 and Lemma 7, we obtain the following theorem.

THEOREM 3. *Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion with Weyl totally geodesic fibers and ω horizontal. Then (M, g, D) is an Einstein–Weyl manifold with $r^D(E,F) + r^D(F,E) = \Lambda g(E,F)$ for some function Λ if and only if*

$$r^{D'}(\widetilde{X}, \widetilde{Y}) \circ \pi + r^{D'}(\widetilde{Y}, \widetilde{X}) \circ \pi - 4g(A^D_X, A^D_Y) = \Lambda g(X,Y), \tag{33}$$

$$2\widehat{r}(U,V) + 2g(A^D U, A^D V) = \Lambda g(U,V), \tag{34}$$

$$\widetilde{\delta} A^D = 0, \tag{35}$$

where X, Y are any horizontal vector fields which are π-related to $\widetilde{X}, \widetilde{Y}$, and U, V are any vertical vector fields. $\blacksquare$

Let r' be the Ricci tensor of the Riemannian metric g'. When $\omega = 0$, from Lemma 6 and Theorem 3 we obtain the following

COROLLARY 3 (cf. [2]). *Let $\pi : (M, g) \to (M', g')$ be a Riemannian submersion with totally geodesic fibers. Then (M, g) is an Einstein manifold with $r(E,F) = \lambda g(E,F)$ for some constant λ if and only if*

$$r'(\widetilde{X}, \widetilde{Y}) \circ \pi - 2g(A^g_X, A^g_Y) = \lambda g(X,Y), \tag{36}$$

$$\widehat{r}(U,V) + g(A^g U, A^g V) = \lambda g(U,V), \tag{37}$$

$$\widetilde{\delta}^g A^g = 0, \tag{38}$$

where X, Y are any horizontal vector fields which are π-related to $\widetilde{X}, \widetilde{Y}$, and U, V are any vertical vector fields. ■

5. Examples

1. *Almost contact metric manifolds.* A Riemannian manifold (M, g) is said to be an *almost contact metric manifold* if there exist a tensor ϕ of type $(1,1)$, a unit vector field ξ and a 1-form η such that

$$\eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where X, Y are arbitrary vector fields on M.

For an almost contact metric structure (ϕ, ξ, η, g) on M, we put $\Phi(X, Y) = g(X, \phi Y)$. An almost contact metric structure is said to be a *contact metric* if $d\eta = \Phi$.

If the Ricci tensor $r(X, Y)$ of a contact metric manifold (M, ϕ, ξ, η, g) is of the form $r(X, Y) = \beta g(X, Y) + \gamma \eta(X)\eta(Y)$, β and γ being constant, then M is called an *η-Einstein contact metric manifold.*

Now, let (M, ϕ, ξ, η, g) be a contact metric manifold with $\dim M = 2n+1$ and $\omega = f\eta$, where f is a function on M. Let $\pi : (M, \phi, \xi, \eta, g) \to (M', g')$ be a Riemannian submersion with fibers of dimension 1 and η vertical. Let D be a torsion-free affine connection such that $Dg = \omega \otimes g$. Then (M, g, D) is a Weyl manifold. From Theorem 2 we have the following

PROPOSITION 3. *Let (M, ϕ, ξ, η, g) be a contact metric manifold with $\dim M = 2n+1$ and $\omega = f\eta$, where f is a function on M. Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion with Weyl totally geodesic fibers of dimension 1 and η vertical, where $Dg = \omega \otimes g$ and $D'g' = \omega' \otimes g'$ for a 1-form ω'. Assume that $\mathcal{H}$ satisfies the Yang–Mills condition. Then (M, g, D) is an Einstein–Weyl manifold with $r^D(E, F) + r^D(F, E) = \Lambda g(E, F)$ for some function Λ if and only if*

$$2r'(\widetilde{X}, \widetilde{Y}) \circ \pi + \left\{-4 - \frac{2n-1}{2} f^2 + \xi(f)\right\} g(X, Y) = \Lambda g(X, Y),$$
$$4n + 2n\xi(f) = \Lambda, \qquad X(f) = 0,$$

where X, Y are any horizontal vector fields which are π-related to $\widetilde{X}, \widetilde{Y}$.

Proof. Since (M, ϕ, ξ, η, g) is a contact metric manifold, for horizontal vector fields X, Y, we have $A^g_X Y = \frac{1}{2}\mathcal{V}[X, Y] = -d\eta(X, Y)\xi$ and so $A^g_X \xi = -\phi X$ because $\Phi = d\eta$. From $A^D_X \xi = A^g_X \xi - \frac{1}{2} g(\xi, B)X$, we get $g(A^D_X, A^D_Y) = (1 + \frac{1}{4} f^2) g(X, Y)$ and $g(A^D \xi, A^D \xi) = \sum g(A^g_{X_i}\xi, A^g_{X_i}\xi) + \frac{1}{2} n f^2 = 2n + \frac{1}{2} n f^2$. Since the fibers are Weyl totally geodesic and ω is vertical, $T^g = 0$. Since $\mathcal{H}$ satisfies the Yang–Mills condition, we get $\breve{\delta}^g A^g = 0$. From Lemma 6 and $\breve{\delta}^g A^g = 0$, $g((\breve{\delta} A^D)X, \xi) = g((\breve{\delta}^g A^g)X, \xi) - \frac{1}{2}(D_X \omega)(\xi) = -\frac{1}{2} X(f)$. This completes the proof. ■

As a corollary, we have the following

COROLLARY 4 (cf. [9]). *Let* (M,ϕ,ξ,η,g) *be an* η*-Einstein contact metric manifold with* $r(E,F)=\beta g(E,F)+\gamma\eta(E)\eta(F)$ *with* $\dim M=2n+1$ *and* $\omega=f\eta$*, where* f *is a function on* M*. Let* $\pi:(M,g,D)\to(M',g',D')$ *be a Weyl submersion with Weyl totally geodesic fibers of dimension* 1 *and* η *vertical, where* $Dg=\omega\otimes g$ *and* $D'g'=\omega'\otimes g'$ *for a* 1*-form* ω'*. Then* (M,g,D) *is an Einstein–Weyl manifold with* $r^D(E,F)+r^D(F,E)=\Lambda g(E,F)$ *for some function* Λ *if and only if*

$$(39)\qquad 2\beta-\frac{2n-1}{2}f^2+\xi(f)=\Lambda,\qquad 4n+2n\xi(f)=\Lambda,\qquad X(f)=0,$$

where X *is any horizontal vector field.*

In particular, if $\gamma\le 0$ *then* (M,g,D) *admits an Einstein–Weyl structure.*

Proof. For basic vector fields X,Y,Z, we have

$$g(R(X,\xi)Z,Y)=g((\nabla_X A^g)_Y Z,\xi)$$

(cf. [11]). Since M is η-Einstein, we have

$$r(E,F)=\beta g(E,F)+\gamma\eta(E)\eta(F),$$

where β and γ are constant. Hence $g((\breve{\delta}^g A^g)X,\xi)=0$, i.e. $\mathcal{H}$ satisfies the Yang–Mills condition. By using the fundamental equation of a Riemannian submersion, we get $r'(\widetilde{X},\widetilde{Y})\circ\pi=(\beta+2)g(X,Y)$. Proposition 3 yields $2\beta-\frac{1}{2}(2n-1)f^2+\xi(f)=\Lambda$, $4n+2n\xi(f)=\Lambda$ and $X(f)=0$.

If $\gamma\le 0$, we set $f^2=\frac{-4}{2n-1}\gamma$ (= constant). From (3) and Proposition 2, we obtain $r^D(\xi,\xi)=\beta+\gamma+n\xi(f)$ and $r^D(\xi,\xi)=2n+n\xi(f)$. Thus $\beta+\gamma=2n$ and so we obtain (39). Therefore (M,g,D) admits an Einstein–Weyl structure. ∎

2. *Warped products.* Let (M',g') and $(\widehat{F},\widehat{g}_0)$ be Riemannian manifolds of dimension n and m respectively. Let $M=M'\times_{f^2}\widehat{F}$ be their warped product with metric $g=g'+f^2\widehat{g}_0$, where f^2 is a positive function on M'. Let ∇, ∇' be the Levi-Civita connections of g,g' respectively. Then $\pi:M\to M'$ is a Riemannian submersion whose fiber at $x'\in M'$ is $(\widehat{F},f(x')^2\widehat{g}_0)$. It is known that $A^g=0$, $T^g_U V=g(U,V)(-f^{-1}\nabla f)$ and $N^g=\sum_{j=1}^m T^g_{V_j}V_j=-mf^{-1}\nabla f$ is a basic vector field which is π-related to $-mf^{-1}\nabla' f$, where ∇f is the gradient of f for g (cf. [2]). We set $B=2f^{-1}\nabla f$ and $B'=2f^{-1}\nabla' f$. Then B is a basic vector field which is π-related to B'. Let $\omega(X)=g(X,B)$ and $\omega'(\widetilde{X})=g'(\widetilde{X},B')$. We define torsion-free affine connections D, D' on M,M' by $Dg=\omega\otimes g$ and $D'g'=\omega'\otimes g'$. From $\omega(X)=\omega'(\widetilde{X})\circ\pi$ for a basic vector field X which is π-related to $\widetilde{X}$, it follows that $\mathcal{H}D_X Y$ is a basic vector field which is π-related to $D'_{\widetilde{X}}\widetilde{Y}$ for basic vector fields X,Y. Therefore $\pi:(M,g,D)\to(M',g',D')$ is a Weyl submersion with ω horizontal. Since

$T_U^D V = T_U^g V + \frac{1}{2} g(U,V)B = g(U,V)(-f^{-1}\nabla f + f^{-1}\nabla f) = 0$, the fibers are Weyl totally geodesic. Since $A^g = 0$ and ω is horizontal, $A^D = 0$. As $\widehat{D}_U V = \mathcal{V} D_U V = \mathcal{V}\nabla_U V$, $\widehat{D}$ is the Levi-Civita connection of $\widehat{g} = f(x')^2 \widehat{g}_0$. Therefore, from Theorem 3 we obtain

PROPOSITION 4. *Let $M = M' \times_{f^2} \widehat{F}$ be the warped product of (M', g') and $(\widehat{F}, \widehat{g}_0)$ with metric $g = g' + f^2\widehat{g}_0$, where f^2 is a positive function on M'. Set $B = 2f^{-1}\nabla f$, $B' = 2f^{-1}\nabla' f$, $\omega(X) = g(X,B)$ and $\omega'(\widetilde{X}) = g'(\widetilde{X}, B')$. Define torsion-free affine connections D and D' by $Dg = \omega \otimes g$ and $D'g' = \omega' \otimes g'$. Then $\pi : (M, g, D) \to (M', g', D')$ is a Weyl submersion with Weyl totally geodesic fibers and $A^D = 0$. Therefore (M, g, D) admits an Einstein–Weyl structure with $r^D(E,F) + r^D(F,E) = \Lambda g(E,F)$ for some function Λ if and only if $(\widehat{F}, \widehat{g}_0)$ is Einstein with $\widehat{r}_0 = \widehat{\lambda}\widehat{g}_0$, $2\widehat{r}(U,V) = \Lambda g(U,V)$, i.e. $2\widehat{\lambda}/f^2 = \Lambda$, and*

$$r^{D'}(\widetilde{X}, \widetilde{Y}) \circ \pi + r^{D'}(\widetilde{Y}, \widetilde{X}) \circ \pi = \Lambda g(X,Y),$$

where X, Y are any horizontal vector fields which are π-related to $\widetilde{X}, \widetilde{Y}$, and U, V are any vertical vector fields. ∎

3. *Locally conformal cosymplectic manifolds.* An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be *locally conformal cosymplectic* if the Nijenhuis tensor N_ϕ is zero and if there exists a closed 1-form θ on M such that $d\eta = \eta \wedge \theta$ and $d\Phi = -2\Phi \wedge \theta$, where

$$N_\phi(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y].$$

Let (M, ϕ, ξ, η, g) and $(M', \phi', \xi', \eta', g')$ be almost contact metric manifolds. A Riemannian submersion $\pi : (M, \phi, \xi, \eta, g) \to (M', \phi', \xi', \eta', g')$ is called an *almost contact metric submersion* if π is an almost contact mapping, i.e. $\phi' \circ \pi_* = \pi_* \circ \phi$. An almost contact metric submersion between locally conformal cosymplectic manifolds is called *locally conformal cosymplectic* (cf. [4], [7]).

Let $\pi : (M, \phi, \xi, \eta, g) \to (M', \phi', \xi', \eta', g')$ be a locally conformal cosymplectic submersion. Let ω, ω' be the Lee forms of (M, ϕ, ξ, η, g), $(M', \phi', \xi', \eta', g')$ respectively. For the Lee form $\widetilde{\omega}$ in the sense of Chinea, Marrero and Rocha [4], our Lee form ω is $\omega = -2\widetilde{\omega}$. Then the Lee vector field B on M is horizontal and the integrability tensor A^g is zero, moreover $\omega(X) = \omega'(\widetilde{X}) \circ \pi$ for any basic vector field X on M which is π-related to $\widetilde{X}$ on M' (cf. [4], [7]). Let D and D' be torsion-free affine connections such that $Dg = \omega \otimes g$ and $D'g' = \omega' \otimes g'$. From $\omega(X) = \omega'(\widetilde{X}) \circ \pi$, it follows that $\mathcal{H} D_X Y$ is a basic vector field which is π-related to $D'_{\widetilde{X}}\widetilde{Y}$, for any basic vector fields X, Y. Therefore $\pi : (M, g, D) \to (M', g', D')$ is a Weyl submersion. Since $A^g = 0$ and B is horizontal, $A^D = 0$. Thus, from Theorem 1, if

(M, g, D) is Weyl flat, i.e. $R^D = 0$, then (M', g', D') is also Weyl flat. Hence we obtain

PROPOSITION 5. *Let* $\pi : (M, \phi, \xi, \eta, g) \to (M', \phi', \xi', \eta', g')$ *be a locally conformal cosymplectic submersion and* ω, ω' *be the Lee forms of* (M, ϕ, ξ, η, g), $(M', \phi', \xi', \eta', g')$ *respectively. Let* D *and* D' *be torsion-free affine connections such that* $Dg = \omega \otimes g$ *and* $D'g' = \omega' \otimes g'$. *Then* $\pi : (M, g, D) \to (M', g', D')$ *is a Weyl submersion with* ω *horizontal and* $A^D = 0$. *If* (M, g, D) *is Weyl flat, i.e.* $R^D = 0$, *then* (M', g', D') *is also Weyl flat.* ■

4. *Locally conformal Kähler manifolds.* Let M be an almost Hermitian manifold with metric g, Levi-Civita connection ∇ and almost complex structure J. The Kähler form Ω is given by $\Omega(X, Y) = g(X, JY)$. An almost Hermitian manifold (M, J, g) is said to be *locally conformal Kähler* if $N_J = 0$, ω is closed and $d\Omega = \omega \wedge \Omega$, where

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

and ω is the Lee form.

Let (M, J, g) and (M', J', g') be almost Hermitian manifolds. A Riemannian submersion $\pi : (M, J, g) \to (M', J', g')$ is called *almost Hermitian* if $\pi_* \circ J = J' \circ \pi_*$.

An almost Hermitian submersion $\pi : (M, J, g) \to (M', J', g')$ is called *locally conformal Kähler* if (M, J, g) is a *locally conformal Kähler* manifold (cf. [6]).

Let $\pi : (M, J, g) \to (M', J', g')$ be a locally conformal Kähler submersion. Let ω, ω' be the Lee forms of $(M, J, g), (M', J', g')$ respectively. Then $\omega(X) = \omega'(\widetilde{X}) \circ \pi$ for any basic vector field X on M π-related to $\widetilde{X}$ on M', and $\mathcal{H}B$ is a basic vector field π-related to B' (cf. [6]). Let D and D' be torsion-free affine connections such that $Dg = \omega \otimes g$ and $D'g' = \omega' \otimes g'$. Then $\mathcal{H}D_XY$ is a basic vector field which is π-related to $D'_{\widetilde{X}}\widetilde{Y}$. Therefore $\pi : (M, g, D) \to (M', g', D')$ is a Weyl submersion.

We assume that ω is horizontal. Then $A^g = 0$ (cf. [6]) and so $A^D = 0$. Thus we get

PROPOSITION 6. *Let* $\pi : (M, J, g) \to (M', J', g')$ *be a locally conformal Kähler submersion and* ω, ω' *be the Lee forms of* (M, J, g), (M', J', g') *respectively. Let* D *and* D' *be torsion-free affine connections such that* $Dg = \omega \otimes g$ *and* $D'g' = \omega' \otimes g'$. *Assume that* ω *is horizontal. Then* $\pi : (M, g, D) \to (M', g', D')$ *is a Weyl submersion with* $A^D = 0$. ■

6. Canonical variations. Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion. Recall that the canonical variation g_t of the Riemannian metric g on M is defined for $t > 0$ by setting $g_t|\mathcal{V} = tg|\mathcal{V}$, $g_t|\mathcal{H} = g|\mathcal{H}$ and $g_t(\mathcal{V}, \mathcal{H}) = 0$ (cf. [2]).

Let D and D^t be torsion-free affine connections such that $Dg = \omega \otimes g$ and $D^t g_t = \omega \otimes g_t$. Since $\pi : (M, g, D) \to (M', g', D')$ is a Weyl submersion, so is $\pi : (M, g_t, D^t) \to (M', g', D')$. Let T^{D^t} and A^{D^t} be the fundamental tensors of the Weyl submersion $\pi : (M, g_t, D^t) \to (M', g', D')$. Let B and B_t be the dual vector fields of ω with respect to g and g_t respectively. Then $\mathcal{V}B = t\mathcal{V}B_t$ and $\mathcal{H}B = \mathcal{H}B_t$.

LEMMA 8. *If X, Y are horizontal and U, V are vertical, then*

(a) $$A^{D^t}_X Y = A^D_X Y + \tfrac{1}{2}(1/t - 1)g(X, Y)\mathcal{V}B,$$

(b) $$A^{D^t}_X U = tA^D_X U + \tfrac{1}{2}(t - 1)\omega(U)X,$$

(c) $$T^{D^t}_U V = tT^D_U V,$$

(d) $$T^{D^t}_U X = T^D_U X.$$

Proof. Since $D^t g_t = \omega \otimes g_t$, we have

$$D^t_E F = \nabla^t_E F - \tfrac{1}{2}\omega(E)F - \tfrac{1}{2}\omega(F)E + \tfrac{1}{2}g_t(E, F)B_t, \tag{40}$$

where ∇^t is the Levi-Civita connection of g_t. Let T^t and A^t be the fundamental tensors of a Riemannian submersion $\pi : (M, g_t) \to (M', g')$. For Riemannian submersions $\pi : (M, g) \to (M', g')$ and $\pi : (M, g_t) \to (M', g')$, we have $A^t_X Y = A^g_X Y$, $A^t_X U = tA^g_X U$, $T^t_U V = tT^g_U V$ and $T^t_U X = T^g_U X$ (cf. [2]). Thus we obtain

$$\begin{aligned}
A^{D^t}_X Y &= \mathcal{V}D^t_X Y = \mathcal{V}\nabla^t_X Y + \tfrac{1}{2}g_t(X, Y)\mathcal{V}B_t \\
&= A^t_X Y + \frac{1}{2t}\,g(X, Y)\mathcal{V}B = A^g_X Y + \frac{1}{2t}\,g(X, Y)\mathcal{V}B \\
&= A^D_X Y + \frac{1}{2}\left(\frac{1}{t} - 1\right)g(X, Y)\mathcal{V}B, \\
A^{D^t}_X U &= A^t_X U - \tfrac{1}{2}\omega(U)X \\
&= tA^g_X U - \tfrac{1}{2}\omega(U)X = tA^D_X U + \tfrac{1}{2}(t - 1)\omega(U)X, \\
T^{D^t}_U V &= T^t_U V + \tfrac{1}{2}tg(U, V)\mathcal{H}B = tT^g_U V + \tfrac{1}{2}tg(U, V)\mathcal{H}B = tT^D_U V, \\
T^{D^t}_U X &= T^t_U X - \tfrac{1}{2}\omega(X)U = T^g_U X - \tfrac{1}{2}\omega(X)U = T^D_U X. \ \blacksquare
\end{aligned}$$

Now we consider a Weyl submersion $\pi : (M, g, D) \to (M', g', D')$ with Weyl totally geodesic fibers of dimension 1 and ω vertical, where $Dg = \omega \otimes g$. Since ω is vertical, D' is the Levi-Civita connection of g'. We set $(\breve{\delta}_t A^{D^t})X = -\sum_{i=1}^n (D^t_{X_i} A^{D^t})_{X_i} X$.

THEOREM 4. *Let $\pi : (M, g, D) \to (M', g', D')$ be a Weyl submersion with Weyl totally geodesic fibers of dimension* 1 *and* $\dim M = n + 1$. *Let ξ be a unit vertical vector field and η its dual* 1*-form with respect to g. Assume that*

$\omega = f\eta$, where f is a function on M. Assume that (M', g') is an Einstein manifold with $r'(\widetilde{X}, \widetilde{Y}) = \lambda' g'(\widetilde{X}, \widetilde{Y})$ whose scalar curvature is positive and (M, g, D) is an Einstein–Weyl manifold with $r^D(E, F) + r^D(F, E) = \Lambda g(E, F)$ and $A^g \neq 0$.

If there exists a positive $t \neq 1$ such that (M, g_t, D^t) is an Einstein–Weyl manifold, then

$$X(f) = 0 \quad \text{and} \quad 0 < 2\xi(f) + f^2 \neq \frac{2}{n-1}\lambda',$$

where X is any horizontal vector field.

If f is constant, then (M, g_t, D^t) admits an Einstein–Weyl structure for

$$t = \frac{(n-1)f^2}{4\lambda' - (n-1)f^2}.$$

Proof. Since (M, g, D) is an Einstein–Weyl manifold, from Theorem 2, we have

$$2r'(\widetilde{X}, \widetilde{Y}) \circ \pi - 4g(A^D_X, A^D_Y) + \left\{\frac{-n+3}{2} f^2 + \xi(f)\right\} g(X, Y) = \Lambda g(X, Y), \tag{41}$$

$$2g(A^D\xi, A^D\xi) + n\{\xi(f) - \tfrac{1}{2} f^2\} = \Lambda, \tag{42}$$

$$-2g((\breve{\delta} A^D)X, \xi) + \frac{n-3}{2} X(f) = 0. \tag{43}$$

Since the fibers of a Weyl submersion $\pi : (M, g, D) \to (M', g', D')$ are Weyl totally geodesic, the fibers of the Weyl submersion $\pi : (M, g_t, D^t) \to (M', g', D')$ are also Weyl totally geodesic because $T^{D^t}_U V = tT^D_U V$. Since (M', g') is an Einstein manifold with $r'(\widetilde{X}, \widetilde{Y}) = \lambda' g'(\widetilde{X}, \widetilde{Y})$, from Proposition 2, we obtain

$$r^{D^t}(X, Y) + r^{D^t}(Y, X) = 2\lambda' g'(\widetilde{X}, \widetilde{Y}) \circ \pi - 4g_t(A^{D^t}_X, A^{D^t}_Y) + \left\{\frac{-n+4}{2} |B_t|^2 + \frac{1}{t} g_t(D^t_\xi B_t, \xi)\right\} g_t(X, Y), \tag{44}$$

$$2r^{D^t}(\xi, \xi) = 2g_t(A^{D^t}\xi, A^{D^t}\xi) + ng_t(D^t_\xi B_t, \xi), \tag{45}$$

$$r^{D^t}(X, \xi) + r^{D^t}(\xi, X) = -2g_t((\breve{\delta}_t A^{D^t})X, \xi) + (n-2)g_t(D^t_X B_t, \xi) + (n-1)d\omega(\xi, X). \tag{46}$$

From $B = f\xi$, we have $g_t(D^t_\xi B_t, \xi) = \xi(f) - f^2/2$ and $|B_t|^2 = t^{-1}f^2$. Using Lemma 8, we get $g_t(A^{D^t}\xi, A^{D^t}\xi) = t^2 g(A^D\xi, A^D\xi) + \frac{1}{4}(1-t^2)nf^2$ and

$$g_t(A^{D^t}_X, A^{D^t}_Y) = tg(A^D_X, A^D_Y) + \frac{1-t^2}{4t} f^2 g(X, Y).$$

From (41) and (44), we have

$$r^{D^t}(X,Y)+r^{D^t}(Y,X)=2\lambda' g'(\widetilde{X},\widetilde{Y})\circ\pi - t\Big\{2\lambda' g'(\widetilde{X},\widetilde{Y})\circ\pi + \Big(\frac{-n+3}{2}f^2+\xi(f)-\Lambda\Big)g(X,Y)\Big\} + \Big\{\frac{2(t^2-1)-n+3}{2t}f^2+\frac{\xi(f)}{t}\Big\}g(X,Y). \tag{47}$$

From (42) and (45), we have

$$2r^{D^t}(\xi,\xi)=t^2\{\Lambda-n\xi(f)\}+n\xi(f). \tag{48}$$

Since ω is vertical, we have $\mathcal{H}D^t_XY=\mathcal{H}D_XY$ and $\mathcal{V}D^t_XU=\mathcal{V}D_XU$, where X,Y are any horizontal vector fields and U is a vertical vector field. Using Lemma 8, we obtain $g_t((\breve{\delta}_tA^{D^t})X,\xi)=tg((\breve{\delta}A^D)X,\xi)+\frac{1}{2}(t-1)X(f)$.

Thus, from (43) and (46), we have

$$r^{D^t}(X,\xi)+r^{D^t}(\xi,X)=\tfrac{1}{2}(n-1)(1-t)X(f). \tag{49}$$

From Lemma 3, we have $A^D_X\xi=A^g_X\xi-\frac{1}{2}fX$. Thus $g(A^D_X,A^D_Y)=g(A^g_X,A^g_Y)+\frac{1}{4}f^2g(X,Y)$ and $g(A^D\xi,A^D\xi)=g(A^g\xi,A^g\xi)+\frac{1}{4}nf^2$. Equations (41), (42) imply $g(A^g_X,A^g_Y)=\frac{1}{4}(2\lambda'-\frac{1}{2}(n-1)f^2+\xi(f)-\Lambda)g(X,Y)$ and $g(A^g\xi,A^g\xi)=\frac{1}{2}(\Lambda-n\xi(f))$. Since $A^g\neq 0$, we obtain $4\lambda'-(n-1)(2\xi(f)+f^2)>0$.

Let (M,g_t,D_t) be an Einstein–Weyl manifold with $r^{D^t}(E,F)+r^{D^t}(F,E)=\Lambda_tg_t(E,F)$. From (47) and (48), we have

$$t\Lambda_t=-t^2\Big(2\lambda'-\Lambda+\frac{-n+1}{2}f^2+\xi(f)\Big)+2\lambda't+\frac{-n+1}{2}f^2+\xi(f) \tag{50}$$

and

$$t\Lambda_t=t^2\{\Lambda-n\xi(f)\}+n\xi(f). \tag{51}$$

Using (50) and (51) we obtain

$$\Big\{\lambda'-\frac{n-1}{4}(2\xi(f)+f^2)\Big\}t^2-\lambda't+\frac{n-1}{4}(2\xi(f)+f^2)=0. \tag{52}$$

One solution is $t=1$, and the other

$$t=\frac{(n-1)(2\xi(f)+f^2)}{4\lambda'-(n-1)(2\xi(f)+f^2)}$$

is positive and $\neq 1$ if and only if $0<2\xi(f)+f^2\neq\frac{2}{n-1}\lambda'$.

Next, we assume that f is constant. From (47)–(49), for

$$t=\frac{(n-1)f^2}{4\lambda'-(n-1)f^2},$$

we have $r^{D^t}(E,F) + r^{D^t}(F,E) = t\Lambda g_t(E,F)$, where E,F are any vector fields on M. Thus (M, g_t, D_t) admits an Einstein–Weyl structure. ∎

As a corollary, we have the following

COROLLARY 5. *Let (M,ϕ,ξ,η,g) be an η-Einstein contact metric manifold with $r(E,F) = \beta g(E,F) + \gamma\eta(E)\eta(F)$, $\dim M = 2n+1$ and $\omega = f\eta$, where f is a function on M. Let $\pi : (M,g,D) \to (M',g',D')$ be a Weyl submersion with Weyl totally geodesic fibers of dimension 1 and η vertical, where $Dg = \omega\otimes g$ and $D'g' = \omega'\otimes g'$ for a 1-form ω'.*

If $\gamma < 0$ and we set $f^2 = \frac{-4}{2n-1}\gamma$, then (M, g_t, D^t) admits an Einstein–Weyl structure for $t = \frac{-4}{8(n+1)}\gamma$.

Proof. From Corollary 4, (M,g,D) admits an Einstein–Weyl structure. Since $r'(\widetilde{X},\widetilde{Y})\circ\pi = (\beta+2)g(X,Y)$ and $\beta+\gamma = 2n$, we have

$$t = \frac{(2n-1)f^2}{4(\beta+2)-(2n-1)f^2} = \frac{-4}{8(n+1)}\,\gamma.$$

Therefore, from Theorem 4, (M, g_t, D^t) admits an Einstein–Weyl structure for $t = \frac{-4}{8(n+1)}\gamma$. ∎

Next, let $\pi : (M,g,D) \to (M',g',D')$ be a Weyl submersion with Weyl totally geodesic fibers and ω horizontal. We study the canonical variation of the metric of the total space. Let D, D' and D^t be the torsion-free affine connections such that $Dg = \omega\otimes g$, $D'g' = \omega'\otimes g'$ and $D^t g_t = \omega\otimes g_t$. Since ω is horizontal, $\widehat{D}$ is the Levi-Civita connection of the induced Riemannian metric $\widehat{g}$ of the fiber.

THEOREM 5. *Let $\pi : (M,g,D) \to (M',g',D')$ be a Weyl submersion with Weyl totally geodesic fibers over an Einstein–Weyl manifold (M',g',D') with $r^{D'}(\widetilde{X},\widetilde{Y}) + r^{D'}(\widetilde{Y},\widetilde{X}) = \Lambda' g'(\widetilde{X},\widetilde{Y})$ and $A^D \neq 0$. Suppose ω is horizontal and Λ' is constant. Assume that the fibers $(\widehat{F},\widehat{g})$ are Einstein manifolds with $\widehat{r}(U,V) = \widehat{\lambda}\widehat{g}(U,V)$ and (M,g,D) is an Einstein–Weyl manifold with $r^D(E,F)+r^D(F,E) = \Lambda g(E,F)$. Then there exists a positive $t \neq 1$ such that (M, g_t, D^t) is also an Einstein–Weyl manifold if and only if $0 < 4\widehat{\lambda} \neq \Lambda'$.*

Proof. By Theorem 3, we have (33)–(35).

Since the fibers of the Weyl submersion $\pi : (M,g,D) \to (M',g',D')$ are Weyl totally geodesic, so are the fibers of the Weyl submersion $\pi : (M,g_t,D^t) \to (M',g',D')$. From Proposition 2, we have

$$r^{D^t}(X,Y) + r^{D^t}(Y,X) = r^{D'}(\widetilde{X},\widetilde{Y})\circ\pi + r^{D'}(\widetilde{Y},\widetilde{X})\circ\pi - 4g_t(A^{D^t}_X, A^{D^t}_Y), \tag{53}$$

$$r^{D^t}(U,V) + r^{D^t}(V,U) = r^{\widehat{D}^t}(U,V) + r^{\widehat{D}^t}(V,U) + 2g_t(A^{D^t}U, A^{D^t}V), \tag{54}$$

$$r^{D^t}(X,U) + r^{D^t}(U,X) = -2g_t((\breve{\delta}_t A^{D^t})X,U). \tag{55}$$

Since ω is horizontal, from Lemma 8 we have $A^{D^t}_X Y = A^D_X Y$, $A^{D^t}_X U = tA^D_X U$, $T^{D^t}_U V = tT^D_U V$, and $T^{D^t}_U X = T^D_U X$. Thus $g_t(A^{D^t}U, A^{D^t}V) = t^2 g(A^D U, A^D V)$ and $g_t(A^{D^t}_X, A^{D^t}_Y) = tg(A^D_X, A^D_Y)$. Since $\mathcal{V}D^t_X Y = \mathcal{V}D_X Y$, $\mathcal{H}D^t_X Y = \mathcal{H}D_X Y$ and $\mathcal{V}D^t_X U = \mathcal{V}D_X U$, we have $\breve{\delta}_t A^{D^t} = \breve{\delta} A^D$.

Thus we obtain

$$r^{D^t}(X,Y)+r^{D^t}(Y,X) = r^{D'}(\widetilde{X},\widetilde{Y})\circ\pi + r^{D'}(\widetilde{Y},\widetilde{X})\circ\pi - 4tg(A^D_X, A^D_Y), \tag{56}$$

$$r^{D^t}(U,V) + r^{D^t}(V,U) = r^{\widehat{D}^t}(U,V) + r^{\widehat{D}^t}(V,U) + 2t^2 g(A^D U, A^D V), \tag{57}$$

$$r^{D^t}(X,U) + r^{D^t}(U,X) = -2tg((\breve{\delta}A^D)X, U) = 0. \tag{58}$$

Since ω is horizontal, $r^{\widehat{D}^t}(U,V) = \widehat{r}(U,V)$. From (33) and (34), since $A^D \neq 0$, we obtain $\Lambda' > 2\widehat{\lambda}$. Then (M, g_t, D^t) is an Einstein–Weyl manifold with $r^{D^t}(E,F) + r^{D^t}(F,E) = \Lambda_t g_t(E,F)$ if and only if there exists a positive $t \neq 1$ such that $\Lambda_t = \Lambda' - t(\Lambda' - \Lambda)$ and $t\Lambda_t = 2\widehat{\lambda} + t^2(\Lambda - 2\widehat{\lambda})$. That is, t satisfies

$$(2\widehat{\lambda} - \Lambda')t^2 + \Lambda' t - 2\widehat{\lambda} = 0. \tag{59}$$

One solution of the quadratic equation is $t = 1$, and the other $t = 2\widehat{\lambda}/(\Lambda' - 2\widehat{\lambda})$ is positive and $\neq 1$ if and only if $0 < 4\widehat{\lambda} \neq \Lambda'$. ∎

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