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STANDARD DILATIONS OF q-COMMUTING TUPLES

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Abstract. We study dilations of q-commuting tuples. Bhat, Bhattacharyya and Dey gave the correspondence between the two standard dilations of commuting tuples and here these results are extended to q-commuting tuples. We are able to do this when the q-coefficients q_{ij} are of modulus one. We introduce a "maximal q-commuting subspace" of an n-tuple of operators and a "standard q-commuting dilation". Our main result is that the maximal q-commuting subspace of the standard noncommuting dilation of a q-commuting tuple is the standard q-commuting dilation. We also introduce the q-commuting Fock space as the maximal q-commuting subspace of the full Fock space and give a formula for a projection operator onto this space. This formula helps us in working with the completely positive maps arising in our study. We prove the first version of the Main Theorem (Theorem 21) of the paper for normal tuples by applying some tricky norm estimates and then use it to prove the general version of this theorem. We also study the distribution of a standard tuple associated with the q-commuting Fock space and related operator spaces.

1. Introduction. A generalization of a contraction operator in multivariate operator theory is a contractive n-tuple which is defined as follows:

DEFINITION 1. An n-tuple $\underline{T} = (T_1, \dots, T_n)$ of bounded operators on a Hilbert space $\mathcal{H}$ such that $T_1T_1^* + \dots + T_nT_n^* \le I$ is a *contractive n-tuple*, or a *row contraction*.

Along the lines of [BBD], we will study the dilation of a class of operator tuples defined as follows:

DEFINITION 2. An n-tuple $\underline{T} = (T_1, \dots, T_n)$ is said to be *q-commuting* if $T_jT_i = q_{ij}T_iT_j$ for all $1 \le i, j \le n$, where q_{ij} are nonzero complex numbers. (To avoid trivialities we assume that $q_{ij} = q_{ji}^{-1}$.)

For a q-commuting n-tuple $\underline{T}$ on a finite-dimensional Hilbert space $\mathcal{H}$, say of dimension m, because of the relation

$$\mathrm{Spec}(T_iT_j) \cup \{0\} = \mathrm{Spec}(T_jT_i) \cup \{0\} = \mathrm{Spec}(q_{ij}T_iT_j) \cup \{0\},$$

we see that q_{ij} is either 0 or an mth root of unity. This makes the finite-dimensional case less interesting but for infinite-dimensional Hilbert spaces we do not have such restrictions on the values of q_{ij}.

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Such operator tuples often appear in quantum theory ([Con], [Ma] [Pr]). In Section 2 we introduce a "maximal q-commuting piece" and using this we define a "q-commuting Fock space" when the q-coefficients q_{ij} are of modulus one. (This condition for q-coefficients is in force for almost all results here.) We give another description for this through a particular representation of the permutation group. This q-commuting Fock space is different from the twisted Fock space of M. Bożejko and R. Speicher ([BS1]) or that of P. E. T. Jorgensen ([JSW]). We give a formula for a projection of the full Fock space onto this space. On this Fock space we consider a special tuple of q-commuting operators and show that it is unitarily equivalent to the tuple of shift operators of [BB].

In Section 3 we show that the range of the isometry A defined in (3.1) is contained in the q-commuting Fock space tensored with a Hilbert space when $\underline{T}$ is a pure tuple (this operator was used by Popescu and Arveson in [Po3], [Po4], [Ar2] and for q-commuting case by Bhat and Bhattacharyya in [BB]). Using this we give a condition equivalent to the assertion of the Main Theorem for q-commuting pure tuples. The proof of the particular case of Theorem 19 where $\underline{T}$ is also q-spherical unitary (introduced in Section 3) is more difficult than the version for commuting tuples and we had to choose the terms carefully and proceed so that the q_{ij} of the q-commuting tuples get absorbed or cancel out when we simplify the terms. Also unlike [BBD] we had to use an inequality relating to completely positive maps before getting the result through norm estimates. We have not been able to generalize Section 4 of [BBD]. In the last section we calculate the distribution of $S_i+S_i^*$ with respect to the vacuum expectation for the standard tuple $\underline{S}$ associated with $\Gamma_q(\mathbb{C}^n)$ and study some properties of related operator spaces.

For operator tuples $(T_1,\dots,T_n)$, we need to consider products of the form $\underline{T}^\alpha := T_{\alpha_1}\cdots T_{\alpha_m}$, where $\alpha=(\alpha_1,\dots,\alpha_m)\in\Lambda^m$, $\Lambda:=\{1,\dots,n\}$. Let $\widetilde{\Lambda}$ denote $\bigcup_{m=0}^\infty \Lambda^m$, where Λ^0 is $\{0\}$ by convention, and let $\underline{T}^0$ be the identity operator of the Hilbert space where the T_i's are acting. Let $\mathcal{S}_m$ denote the group of permutations of $\{1,\dots,m\}$. For a q-commuting tuple $\underline{T}=(T_1,\dots,T_n)$, consider the product $T_{x_1}\cdots T_{x_m}$ where $1\le x_i\le n$. If we replace a consecutive pair, say $T_{x_i}T_{x_{i+1}}$, in the above product by $q_{x_{i+1}x_i}T_{x_{i+1}}T_{x_i}$ and do a finite number of such operations with different choices of consecutive pairs in the resulting products, we will get a permutation $\sigma\in\mathcal{S}_m$ such $T_{x_1}\cdots T_{x_m}=kT_{x_{\sigma^{-1}(1)}}\cdots T_{x_{\sigma^{-1}(m)}}$ for some $k\in\mathbb{C}$. To define a q-commuting tuple in Definition 2 we needed the known fact that this k depends only on σ and x_i, and not on the choice of the operations that give rise to the same final product $T_{x_{\sigma^{-1}(1)}}\cdots T_{x_{\sigma^{-1}(m)}}$. This also follows from Proposition 3.

Hereafter, whenever we deal with q-commuting tuples we assume that $|q_{ij}|=1$ for $1\le i,j\le n$. However for Propositions 6, 8 and Corollary 7 we do not need this assumption. Let $\underline{T} = (T_1, \ldots, T_n)$ be a q-commuting tuple and consider the product $T_{x_1} \cdots T_{x_m}$ where $1 \leq x_i \leq n$. Let $\sigma \in \mathcal{S}_m$. As the transpositions $(k, k+1), 1 \leq k \leq m-1$, generate $\mathcal{S}_m$, let $\sigma^{-1} = \tau_1 \cdots \tau_s$ where $\tau_i = (k_i, k_i + 1)$ for each $1 \leq i \leq s$. Let $\widetilde{\sigma}_i = \tau_{i+1} \cdots \tau_s$ for $1 \leq i \leq s-1$ and $\widetilde{\sigma}_s$ be the identity permutation. Define $y_i = x_{\widetilde{\sigma}_i(k_i)}$ and $z_i = x_{\widetilde{\sigma}_i(k_i+1)}$. If we replace $T_{y_s}T_{z_s}$ by $q_{z_s y_s} T_{z_s} T_{y_s}$ corresponding to τ_s, $T_{y_{s-1}} T_{z_{s-1}}$ by $q_{z_{s-1} y_{s-1}} T_{z_{s-1}} T_{y_{s-1}}$ corresponding to τ_{s-1}, and so on, we get $T_{x_1} \cdots T_{x_m} = q_1^\sigma(x) \cdots q_s^\sigma(x) T_{x_{\sigma^{-1}(1)}} \cdots T_{x_{\sigma^{-1}(m)}}$ where $q_i^\sigma(x) = q_{z_i y_i}$. Let $q^\sigma(x) = q_1^\sigma(x) \cdots q_s^\sigma(x)$.

Proposition 3. *Let $\underline{T} = (T_1, \ldots, T_n)$ be a q-commuting tuple and consider the product $T_{x_1} \cdots T_{x_m}$ where $1 \leq x_i \leq n$. Let $\sigma \in \mathcal{S}_m$ and $q^\sigma(x)$ be as defined above. Then*

$$q^\sigma(x) = \prod q_{x_{\sigma^{-1}(k)} x_{\sigma^{-1}(i)}},$$

where the product is over $\{(i,k) : 1 \leq i < k \leq m,\ \sigma^{-1}(i) > \sigma^{-1}(k)\}$. In particular $q^\sigma(x)$ does not depend on the decomposition of σ as a product of transpositions.

Proof. We have

$$q^\sigma(x) = q_1^\sigma(x) \cdots q_s^\sigma(x)$$

where $q_i^\sigma(x) = q_{z_i y_i}$. For $1 \leq i < k \leq m$ let $k' = \sigma^{-1}(k)$ and $i' = \sigma^{-1}(i)$. Define $\sigma = \tau_1 \cdots \tau_s$ and $\widetilde{\sigma}_i$ as above. If $i' > k'$ then there are an odd number of transpositions τ_r for $1 \leq r \leq m$ that interchange the positions of i' and k' in the image of $\widetilde{\sigma}_r$ when we consider the composition $\tau_r \widetilde{\sigma}_r$, while if $i' < k'$ then there are an even number of such transpositions. For the first transposition in τ_r that interchanges i' and k', the corresponding factor in $q^\sigma(x)$, say $q_r^\sigma(x)$, is $q_{x_{k'} x_{i'}}$, for the second such transposition the factor is $q_{x_{i'} x_{k'}}$, for the third it is $q_{x_{k'} x_{i'}}$, and so on. But $(q_{x_{i'} x_{k'}})^{-1} = q_{x_{k'} x_{i'}}$ and so

$$q^\sigma(x) = \prod q_{x_{\sigma^{-1}(k)} x_{\sigma^{-1}(i)}},$$

where the product is over $\{(i,k) : 1 \leq i < k \leq m,\ \sigma^{-1}(i) > \sigma^{-1}(k)\}$. ■

Similar arguments show that if $\sigma \in \mathcal{S}_m$ is such that $(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$, then $q^\sigma(x) = 1$.

Definition 4. Let $\mathcal{H}, \mathcal{L}$ be two Hilbert spaces such that $\mathcal{H}$ is a closed subspace of $\mathcal{L}$ and let $\underline{T}, \underline{R}$ be n-tuples of bounded operators on $\mathcal{H}$, $\mathcal{L}$ respectively. Then $\underline{R}$ is called a *dilation* of $\underline{T}$ if

$$R_i^* u = T_i^* u$$

for all $u \in \mathcal{H}$, $1 \leq i \leq n$. In such a case $\underline{T}$ is called a *piece* of $\underline{R}$. If $\underline{T}$ is a q-commuting tuple (i.e., $T_j T_i = q_{ij} T_i T_j$ for all i, j), then it is called a *q-commuting piece* of $\underline{R}$. A dilation $\underline{R}$ of $\underline{T}$ is said to be a *minimal dilation* if $\overline{\operatorname{span}}\{\underline{R}^\alpha h : \alpha \in \widetilde{\Lambda}, h \in \mathcal{H}\} = \mathcal{L}$. And if $\underline{R}$ is a tuple of n isometries with orthogonal ranges and is a minimal dilation of $\underline{T}$, then it is called the *minimal isometric dilation* or the *standard noncommuting dilation* of $\underline{T}$.

A presentation of the standard noncommuting dilation taken from [Po1] is used here to prove the main theorem. All Hilbert spaces we consider are complex and separable. For a subspace $\mathcal{H}$ of a Hilbert space, $P_{\mathcal{H}}$ will denote the orthogonal projection onto $\mathcal{H}$. The standard noncommuting dilation of an n-tuple of bounded operators is unique up to unitary equivalence (cf. [Po1-4]). An extensive study of the standard noncommuting dilation was carried out by Popescu. He generalized many one-variable results to the multivariable case. It is easy to see that if $\underline{R}$ is a dilation of $\underline{T}$ then

$$\underline{T}^{\alpha}(\underline{T}^{\beta})^{*} = P_{\mathcal{H}}\underline{R}^{\alpha}(\underline{R}^{\beta})^{*}|_{\mathcal{H}}, \tag{1.1}$$

and for any polynomials p, q in n noncommuting variables,

$$p(\underline{T})(q(\underline{T}))^{*} = P_{\mathcal{H}}p(\underline{R})(q(\underline{R}))^{*}|_{\mathcal{H}}.$$

For an n-tuple $\underline{R}$ of bounded operators on a Hilbert space $\mathcal{M}$, consider

$$\mathcal{C}^{q}(\underline{R}) = \{\mathcal{N} : R_i^{*} \text{ leaves } \mathcal{N} \text{ invariant}, R_i^{*}R_j^{*}h = \overline{q}_{ij}R_j^{*}R_i^{*}h, \forall h \in \mathcal{N}, \forall i, j\}.$$

It is a complete lattice, in the sense that arbitrary intersections and closed spans of arbitrary unions of such spaces are again in this collection. So it has a maximal element and we denote it by $\mathcal{M}^{q}(\underline{R})$ (or by $\mathcal{M}^{q}$ when the tuple under consideration is clear).

DEFINITION 5. Let $\underline{R}$ be an n-tuple of operators on a Hilbert space $\mathcal{M}$. The q-commuting piece $\underline{R}^{q} = (R_1^{q}, \dots, R_n^{q})$ obtained by compressing $\underline{R}$ to the maximal element $\mathcal{M}^{q}(\underline{R})$ of $\mathcal{C}^{q}(\underline{R})$ is called the *maximal q-commuting piece* of $\underline{R}$. The maximal q-commuting piece is said to be *trivial* if $\mathcal{M}^{q}(\underline{R})$ is the zero space.

The following result gives a description of maximal q-commuting pieces.

PROPOSITION 6. *Let $\underline{R} = (R_1, \dots, R_n)$ be an n-tuple of bounded operators on a Hilbert space $\mathcal{M}$, $\mathcal{K}_{ij} = \overline{\operatorname{span}}\{\underline{R}^{\alpha}(q_{ij}R_iR_j - R_jR_i)h : h \in \mathcal{M}, \alpha \in \widetilde{\Lambda}\}$ for all $1 \le i, j \le n$, and $\mathcal{K} = \overline{\operatorname{span}}\{\bigcup_{i,j=1}^{n}\mathcal{K}_{ij}\}$. Then $\mathcal{M}^{q}(\underline{R}) = \mathcal{K}^{\perp}$ and $\mathcal{M}^{q}(\underline{R}) = \{h \in \mathcal{M} : (\overline{q}_{ij}R_j^{*}R_i^{*} - R_i^{*}R_j^{*})(\underline{R}^{\alpha})^{*}h = 0, \forall 1 \le i, j \le n, \alpha \in \widetilde{\Lambda}\}$.*

The above proposition can be easily proved using arguments similar to the proof of Proposition 4 of [BBD].

COROLLARY 7. *Suppose $\underline{R}$, $\underline{T}$ are n-tuples of operators on two Hilbert spaces $\mathcal{L}, \mathcal{M}$. Then the maximal q-commuting piece of $(R_1 \oplus T_1, \dots, R_n \oplus T_n)$ acting on $\mathcal{L} \oplus \mathcal{M}$ is $(R_1^{q} \oplus T_1^{q}, \dots, R_n^{q} \oplus T_n^{q})$ acting on $\mathcal{L}^{q} \oplus \mathcal{M}^{q}$, and the maximal q-commuting piece of $(R_1 \otimes I, \dots, R_n \otimes I)$ acting on $\mathcal{L} \otimes \mathcal{M}$ is $(R_1^{q} \otimes I, \dots, R_n^{q} \otimes I)$ acting on $\mathcal{L}^{q} \otimes \mathcal{M}$.*

Proof. Clear from Proposition 6. ∎

PROPOSITION 8. *Let $\underline{T}, \underline{R}$ be n-tuples of bounded operators on $\mathcal{H}$, $\mathcal{L}$ with $\mathcal{H} \subseteq \mathcal{L}$ such that $\underline{R}$ is a dilation of $\underline{T}$. Then $\mathcal{H}^q(\underline{T}) = \mathcal{L}^q(\underline{R}) \cap \mathcal{H}$ and $\underline{R}^q$ is a dilation of $\underline{T}^q$.*

Proof. This can be proved using arguments similar to the proof of Proposition 7 of [BBD]. ∎

2. A q-commuting Fock space. In this section we introduce a q-commuting Fock space and give two descriptions of it. For any Hilbert space $\mathcal{K}$, we have the full Fock space over $\mathcal{K}$,

$$\Gamma(\mathcal{K}) = \mathbb{C} \oplus \mathcal{K} \oplus \mathcal{K}^{\otimes^2} \oplus \cdots \oplus \mathcal{K}^{\otimes^m} \oplus \cdots .$$

We denote the vacuum vector $1 \oplus 0 \oplus \cdots$ by ω. For fixed $n \geq 2$, let $\mathbb{C}^n$ be the n-dimensional complex Euclidian space with the usual inner product, and let $\Gamma(\mathbb{C}^n)$ be the full Fock space over $\mathbb{C}^n$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$. For $\alpha \in \widetilde{\Lambda}$, $e^\alpha := e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m} \in \Gamma(\mathbb{C}^n)$ and $e^0 := \omega$. Then define the (left) *creation operators* V_i on $\Gamma(\mathbb{C}^n)$ by

$$V_i x = e_i \otimes x \quad \text{ for } 1 \leq i \leq n \text{ and } x \in \Gamma(\mathbb{C}^n)$$

(here $e_i \otimes \omega$ is interpreted as e_i). It is obvious that the tuple $\underline{V} = (V_1, \ldots, V_n)$ consists of isometries with orthogonal ranges and $\sum V_i V_i^* = I - I_0$, where I_0 is the projection onto the vacuum space. Define the *q-commuting Fock space* $\Gamma_q(\mathbb{C}^n)$ as the subspace $(\Gamma(\mathbb{C}^n))^q(\underline{V})$ of the full Fock space. Let $\underline{S} = (S_1, \ldots, S_n)$ be the tuple of operators on $\Gamma_q(\mathbb{C}^n)$ where S_i is the compression of V_i to $\Gamma_q(\mathbb{C}^n)$:

$$S_i = P_{\Gamma_q(\mathbb{C}^n)} V_i|_{\Gamma_q(\mathbb{C}^n)}.$$

Clearly each V_i^* leaves $\Gamma_q(\mathbb{C}^n)$ invariant. Observe that the vacuum vector is in $\Gamma_q(\mathbb{C}^n)$. It is easy to see that $\sum S_i S_i^* = I^q - I_0^q$ (where I^q, I_0^q are the identity and the projection onto the vacuum space respectively in $\Gamma_q(\mathbb{C}^n)$). So $\underline{V}$ and $\underline{S}$ are contractive tuples, $S_j S_i = q_{ij} S_i S_j$ for all $1 \leq i, j \leq n$, and $S_i^* x = V_i^* x$ for $x \in \Gamma_q(\mathbb{C}^n)$.

Define $U_\sigma^{m,q}$ on $(\mathbb{C}^n)^{\otimes^m}$ by

$$U_\sigma^{m,q}(e_{x_1} \otimes \cdots \otimes e_{x_m}) = q^\sigma(x) e_{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma^{-1}(m)}} \tag{2.1}$$

on the standard basis vectors and extend it linearly to $(\mathbb{C}^n)^{\otimes^m}$. As $|q_{ij}| = 1$ for $1 \leq i, j \leq n$, U_σ^m is unitary and it extends uniquely to a unitary operator on $(\mathbb{C}^n)^{\otimes^m}$. Let

$$(\mathbb{C}^n)^{\textcircled{q}^m} = \{u \in (\mathbb{C}^n)^{\otimes^m} : U_\sigma^{m,q} u = u \ \forall \sigma \in \mathcal{S}_m\}$$

and $(\mathbb{C}^n)^{\textcircled{q}^0} = \mathbb{C}$. The dimension of $(\mathbb{C}^n)^{\textcircled{q}^m}$ is the number of ways in which m identical objects can be distributed in n buckets. From standard combinatorics it follows that

$$\dim\,(\mathbb{C}^n)^{\circledS m} = \binom{n+m-1}{m}.$$

LEMMA 9. *The map from $\mathcal{S}_m$ to $B((\mathbb{C}^n)^{\otimes^m})$ defined by $\sigma \mapsto U_\sigma^{m,q}$ is a unitary representation of the permutation group $\mathcal{S}_m$.*

Proof. Let $\otimes_{i=1}^m e_{x_i}, \otimes_{i=1}^m e_{y_i} \in (\mathbb{C}^n)^{\otimes^m}, 1 \le x_i, y_i \le n$. Suppose there exist $\sigma \in \mathcal{S}_m$ such that $\otimes_{i=1}^m e_{y_i} = \otimes_{i=1}^m e_{x_{\sigma^{-1}(i)}}$. Then $\langle U_\sigma^{m,q}(\otimes_{i=1}^m e_{x_i}), \otimes_{i=1}^m e_{y_i}\rangle$ $= q^\sigma(x)$ and $\langle \otimes_{i=1}^m e_{x_i}, U_{\sigma^{-1}}^{m,q}(\otimes_{i=1}^m e_{y_i})\rangle = \overline{q^{(\sigma^{-1})}(y)}$. Also

$$q^{(\sigma^{-1})}(y) = \prod q_{y_{\sigma(k)} y_{\sigma(i)}} = \prod q_{x_k x_i}$$

where the products are over $\{(i,k) : 1 \le i < k \le m,\ \sigma(i) > \sigma(k)\}$. If we substitute $k = \sigma^{-1}(i')$ and $i = \sigma^{-1}(k')$ in the last term we get

$$q^{\sigma^{-1}}(y) = \prod q_{x_{\sigma^{-1}(i')} x_{\sigma^{-1}(k')}} = \Big(\prod q_{x_{\sigma^{-1}(k')} x_{\sigma^{-1}(i')}}\Big)^{-1} = (q^\sigma(x))^{-1}$$

where the products are over $\{(i',k') : 1 \le i' < k' \le m,\, \sigma^{-1}(i') > \sigma^{-1}(k')\}$. So

$$q^\sigma(x) = (q^{\sigma^{-1}}(y))^{-1} = \overline{q^{\sigma^{-1}}(y)}.$$

The last equality holds as $|q_{ij}| = 1$. This implies

$$\langle U_\sigma^{m,q}(\otimes_{i=1}^m e_{x_i}), \otimes_{i=1}^m e_{y_i}\rangle = \langle \otimes_{i=1}^m e_{x_i}, U_{\sigma^{-1}}^{m,q}(\otimes_{i=1}^m e_{y_i})\rangle.$$

If there does not exist any $\sigma \in \mathcal{S}_m$ such that $\otimes_{i=1}^m e_{y_i} = \otimes_{i=1}^m e_{x_{\sigma^{-1}(i)}}$ then

$$\langle U_{\sigma'}^{m,q}(\otimes_{i=1}^m e_{x_i}), \otimes_{i=1}^m e_{y_i}\rangle = 0 = \langle \otimes_{i=1}^m e_{x_i}, U_{(\sigma')^{-1}}^{m,q}(\otimes_{i=1}^m e_{y_i})\rangle$$

for all $\sigma' \in \mathcal{S}_m$. So for all $\sigma \in \mathcal{S}_m$, $(U_\sigma^{m,q})^* = U_{\sigma^{-1}}^{m,q}$ on the basis elements of $(\mathbb{C}^n)^{\otimes^m}$, and hence on the whole of $(\mathbb{C}^n)^{\otimes^m}$.

Next let $\sigma = \sigma_1\sigma_2$ for some $\sigma_1, \sigma_2 \in \mathcal{S}_m$. We show that $U_\sigma^{m,q} = U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}$. Let $e_x = e_{x_1} \otimes \cdots \otimes e_{x_m}$ where $x_j \in \{1,\dots,n\}$ for $1 \le j \le m$. Let $\sigma_1^{-1} = \tau_1 \cdots \tau_r$ and $\sigma_2^{-1} = \tau_{r+1} \cdots \tau_s$ where the τ_i are transpositions of the form (k_i, k_i+1). Then

$$\begin{aligned} U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}(e_{x_1} \otimes \cdots \otimes e_{x_m}) &= U_{\sigma_1}^{m,q}(q^{\sigma_2}(x) e_{x_{\sigma_2^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma_2^{-1}(m)}}) \\ &= q^{\sigma_1}(z) q^{\sigma_2}(x) e_{x_{\sigma_2^{-1}\sigma_1^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma_2^{-1}\sigma_1^{-1}(m)}} \end{aligned}$$

where $e_z = e_{z_1} \otimes \cdots \otimes e_{z_m}$, i.e. $z_i = x_{\sigma_2^{-1}(i)}$. But as $\sigma = \tau_1 \dots \tau_r \tau_{r+1} \dots \tau_s$ it is easy to see that $q^\sigma(x) = q^{\sigma_1}(z) q^{\sigma_2}(x)$. So

$$\begin{aligned} U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}(e_{x_1} \otimes \cdots \otimes e_{x_m}) &= q^\sigma(x) e_{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma^{-1}(m)}} \\ &= U_\sigma^{m,q}(e_{x_1} \otimes \cdots \otimes e_{x_m}), \end{aligned}$$

and hence $U_{\sigma_1\sigma_2}^{m,q} = U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}$. ∎

In the next lemma and theorem we derive a formula for the projection operator onto the q-commuting Fock space.

LEMMA 10. *Define P_m on $(\mathbb{C}^n)^{\otimes^m}$ by*

$$P_m = \frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m} U_\sigma^{m,q}. \tag{2.2}$$

Then P_m is the projection of $(\mathbb{C}^n)^{\otimes^m}$ onto $(\mathbb{C}^n)^{@^m}$.

Proof. First we see that

$$P_m^* = \frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m}(U_\sigma^{m,q})^* = \frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m} U_{\sigma^{-1}}^{m,q} = P_m.$$

For $\sigma'\in\mathcal{S}_m$ we have

$$P_m U_{\sigma'}^{m,q} = \frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m} U_{\sigma\sigma'}^{m,q} = \frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m} U_\sigma^{m,q} = P_m. \tag{2.3}$$

Similarly $U_{\sigma'}^{m,q}P_m = P_m$. So $P_m^2 = P_m$ and hence P_m is a projection. ∎

THEOREM 11. $\bigoplus_{m=0}^{\infty}(\mathbb{C}^n)^{@^m} = \Gamma_q(\mathbb{C}^n)$.

Proof. Let $Q = \bigoplus_{m=0}^{\infty} P_m$ be a projection of $\Gamma(\mathbb{C}^n)$ onto $\bigoplus_{m=0}^{\infty}(\mathbb{C}^n)^{@^m}$ where P_m is defined in Lemma 10. Next we show that $\bigoplus_{m=0}^{\infty}(\mathbb{C}^n)^{@^m}$ is invariant under V_i^*. Let $\otimes_{j=1}^m e_{x_j}\in(\mathbb{C}^n)^{\otimes^m}, 1\le x_j\le n$. Then $V_i^*\{P_m(\otimes_{j=1}^m e_{x_j})\}$ is zero if no x_j is equal to i. Otherwise $V_i^*\{P_m(\otimes_{j=1}^m e_{x_j})\}$ is a nonzero element of $\bigoplus_{m=0}^{\infty}(\mathbb{C}^n)^{@^{(m-1)}}$ because of the following: Suppose $x_j = i$ if and only if $j\in\{r_1,\dots,r_p\}$, and let $\mathcal{A}_k$ be the set of all $\sigma\in S_m$ such that σ^{-1} sends 1 to r_k, $1\le k\le p$. Then $\mathcal{A}_k$ consists of all compositions $\tau\varrho$ where $\tau=(1,r_k)$ and ϱ keeps r_k fixed and permutes the other $m-1$ symbols. Let $x=(x_1,\dots,x_m)$ and $y=(x_{\tau^{-1}(1)},\dots,x_{\tau^{-1}(m)})$. As the V_i are isometries with orthogonal ranges,

$$\begin{aligned}
V_i^*\{P_m(\otimes_{j=1}^m e_{x_j})\} &= V_i^*\Big\{\frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m}U_\sigma^{m,q}(\otimes_{j=1}^m e_{x_j})\Big\}\\
&= \frac{1}{m!}\sum_{k=1}^{p} V_i^*\Big\{\sum_{\tau\varrho\in\mathcal{A}_k} U_\tau^{m,q}U_\varrho^{m,q}(\otimes_{j=1}^m e_{x_j})\Big\}\\
&= \frac{1}{m!}\sum_{k=1}^{p} q_{x_1 i}q^{\varrho}(y)V_i^*\Big\{\sum_{\tau\varrho\in\mathcal{A}_k} e_{x_{\varrho^{-1}(r_k)}}\otimes e_{x_{\varrho^{-1}(2)}}\otimes\cdots\\
&\qquad \otimes e_{x_{\varrho^{-1}(r_{k-1})}}\otimes e_{x_{\varrho^{-1}(1)}}\otimes e_{x_{\varrho^{-1}(r_{k+1})}}\otimes\cdots\otimes e_{x_{\varrho^{-1}(m)}}\Big\}
\end{aligned}$$

$$= \frac{1}{m!}\sum_{k=1}^{p} q_{x_1 i} q^{\varrho}(y) V_i^* \Big\{ \sum_{\tau\varrho\in\mathcal{A}_k} e_i \otimes e_{x_{\varrho^{-1}(2)}} \otimes \cdots$$

$$\otimes e_{x_{\varrho^{-1}(r_{k-1})}} \otimes e_{x_{\varrho^{-1}(1)}} \otimes e_{x_{\varrho^{-1}(r_{k+1})}} \otimes \cdots \otimes e_{x_{\varrho^{-1}(m)}} \Big\}$$

$$= \sum_{k=1}^{p} \frac{q_{x_1 i}}{m!} \Big\{ q^{\varrho}(y) \sum_{\varrho\in\mathcal{S}_{m-1}} e_{x_{\varrho^{-1}(2)}} \otimes \cdots \otimes e_{x_{\varrho^{-1}(r_{k-1})}}$$

$$\otimes e_{x_{\varrho^{-1}(1)}} \otimes e_{x_{\varrho^{-1}(r_{k+1})}} \otimes \cdots \otimes e_{x_{\varrho^{-1}(m)}} \Big\}$$

$$= \sum_{k=1}^{p} a_k(x) P_{m-1}(e_{x_1} \otimes \cdots \otimes \hat{e}_{x_{i_k}} \otimes \cdots \otimes e_{x_m})$$

where $a_k(x)$ are constants and the hat denotes omission of the corresponding term. This shows that $\bigoplus_{m=0}^{\infty}(\mathbb{C}^n)^{\text{\textcircled{q}}m}$ is invariant under V_i^*.

Taking $R_i = QV_iQ$ we show that $\underline{R}$ is q-commuting. Define $U_{(1,2)}^{q} = \bigoplus_{m=0}^{\infty} U_{(1,2)}^{m,q}$ where $U_{(1,2)}^{0,q} = I$ and $U_{(1,2)}^{1,q} = I$. Let $\otimes_{i=1}^{k} e_{\alpha_i} \in (\mathbb{C}^n)^{\otimes^k}, 1 \leq \alpha_i \leq n$. Using Lemma 10 we get

$$\begin{aligned} R_iR_j\underline{R}^{\alpha}\omega &= QV_iV_j\underline{V}^{\alpha}\omega = QU_{(1,2)}^{q} V_iV_j(\otimes_{i=1}^{k} e_{\alpha_i}) \\ &= QU_{(1,2)}^{q}\{e_i\otimes e_j\otimes(\otimes_{i=1}^{k} e_{\alpha_i})\} = Qq_{ji}e_j\otimes e_i\otimes(\otimes_{i=1}^{k} e_{\alpha_i}) \\ &= q_{ji}QV_jV_i\underline{V}^{\alpha}\omega = q_{ji}R_jR_i\underline{R}^{\alpha}\omega, \end{aligned}$$

and clearly $\bigoplus_{m=0}^{\infty}(\mathbb{C}^n)^{\text{\textcircled{q}}m} = \overline{\text{span}}\{\underline{R}^{\alpha}\omega : \alpha \in \widetilde{\Lambda}\}$. So $(R_1, \ldots, R_n)$ is a q-commuting piece of $\underline{V}$.

To show maximality we make use of Proposition 6. Suppose $x \in \Gamma(\mathbb{C}^n)$ and $\langle x, \underline{V}^{\alpha}(q_{ij}V_iV_j - V_jV_i)y\rangle = 0$ for all $\alpha \in \widetilde{\Lambda}$, $1 \leq i, j \leq n$ and $y \in \Gamma(\mathbb{C}^n)$. We wish to show that $x \in \Gamma_q(\mathbb{C}^n)$. Suppose x_m is the m-particle component of x, i.e., $x = \oplus_{m\geq 0} x_m$ with $x_m \in (\mathbb{C}^n)^{\otimes^m}$. For $m \geq 2$ and any $\sigma \in \mathcal{S}_m$ we need to show that the unitary $U_{\sigma}^{m,q} : (\mathbb{C}^n)^{\otimes^m} \to (\mathbb{C}^n)^{\otimes^m}$ defined by (2.1) leaves x_m fixed. Since $\mathcal{S}_m$ is generated by $\{(1,2), (2,3), \ldots, (m-1, m)\}$ it is enough to verify $U_{\sigma}^{m,q}(x_m) = x_m$ for σ of the form $(i, i+1)$. So fix m and i with $m \geq 2$ and $1 \leq i \leq m-1$. We have

$$\langle \oplus_p x_p, \underline{V}^{\alpha}(q_{kl}V_kV_l - V_lV_k)\underline{V}^{\beta}\omega\rangle = 0 \tag{2.4}$$

for every $\beta \in \widetilde{\Lambda}$, $1 \leq k, l \leq n$. This implies that

$$\langle x_m, e^{\alpha} \otimes (q_{kl}e_k \otimes e_l - e_l \otimes e_k) \otimes e^{\beta}\rangle = 0$$

for any $\alpha \in \Lambda^{i-1}$, $\beta \in \Lambda^{m-i-1}$. So if

$$x_m = \sum a(s,t,\alpha,\beta)e^{\alpha} \otimes e_s \otimes e_t \otimes e^{\beta}$$

where the sum is over $\alpha \in \Lambda^{i-1}, \beta \in \Lambda^{m-i-1}$ and $1 \leq s, t \leq n$, and $a(s,t,\alpha,\beta)$ are constants, then for fixed α and β it follows from (2.4) that

$\overline{q}_{kl}a(k,l,\alpha,\beta) = a(l,k,\alpha,\beta)$ or $q_{lk}a(k,l,\alpha,\beta) = a(l,k,\alpha,\beta)$. Hence for $\sigma = (i,i+1)$,

$$\begin{aligned}&U_\sigma^{m,q}(a(k,l,\alpha,\beta)e^\alpha\otimes e_k\otimes e_l\otimes e^\beta + a(l,k,\alpha,\beta)e^\alpha\otimes e_l\otimes e_k\otimes e^\beta)\\&\quad = q_{lk}a(k,l,\alpha,\beta)e^\alpha\otimes e_l\otimes e_k\otimes e^\beta + q_{kl}a(l,k,\alpha,\beta)e^\alpha\otimes e_k\otimes e_l\otimes e^\beta\\&\quad = a(l,k,\alpha,\beta)e^\alpha\otimes e_l\otimes e_k\otimes e^\beta + a(k,l,\alpha,\beta)e^\alpha\otimes e_k\otimes e_l\otimes e^\beta.\end{aligned}$$

This clearly implies $U_\sigma^{m,q}(x_m) = x_m$. ∎

COROLLARY 12. *For* $u \in (\mathbb{C}^n)^{\otimes^k}, v \in (\mathbb{C}^n)^{\otimes^l}, w \in (\mathbb{C}^n)^{\otimes^m}$,

$$P_{k+l+m}\{P_{k+l}(u\otimes v)\otimes w\} = P_{k+l+m}\{u\otimes P_{l+m}(v\otimes w)\}.$$

Proof. If we identify $\mathcal{S}_{k+l}$ and $\mathcal{S}_{l+m}$ with the subgroups of $\mathcal{S}_{k+l+m}$ such that $\sigma\in\mathcal{S}_{k+l}$ fixes the last m elements of $\{1,\dots,k+l+m\}$ and $\sigma\in\mathcal{S}_{l+m}$ fixes the first k elements of $\{1,\dots,k+l+m\}$, the assertion follows easily using (2.3). ∎

When $q_{ij} = 1$ for all i,j, we denote $(\mathbb{C}^n)^{ⓠ^m}$ by $(\mathbb{C}^n)^{ⓢ^m}$ and the q-commuting Fock space $\Gamma_q(\mathbb{C}^n)$ by $\Gamma_s(\mathbb{C}^n)$, and call it the *symmetric Fock space* (or the *boson Fock space*) (cf. [BBD]). The map $U^{m,q} : \mathcal{S}_m \to B(\mathbb{C}^n)^{\otimes^m}$ given by

$$U^{m,q}(\sigma) = U_\sigma^{m,q}$$

gives a representation of $\mathcal{S}_m$ on $B(\mathbb{C}^n)^{\otimes^m}$. Denote $U_\sigma^{m,q}$ by $U_\sigma^{m,s}$ if $q_{ij}=1$ for all i,j. It is easy to see that for all $q=(q_{ij})_{n\times n}$ with $|q_{ij}|=1$, the representations are unitarily equivalent. So there exists a unitary $W^{m,q} : (\mathbb{C}^n)^{\otimes^m}\to(\mathbb{C}^n)^{\otimes^m}$ such that

$$W^{m,q}U_\sigma^{m,s} = U_\sigma^{m,q}W^{m,q}. \tag{2.5}$$

This $W^{m,q}$ is not unique as for $k\in\mathbb{C}$ with $|k|=1$, the operator $kW^{m,q}$ is also a unitary satisfying (2.5). We will give one such $W^{m,q}$ explicitly.

For $m\in\mathbb{N}$, $y_i\in\Lambda$ define $W^{m,q}$ over $(\mathbb{C}^n)^{\otimes^m}$ as

$$W^{m,q}(e_{y_1}\otimes\cdots\otimes e_{y_m}) = q^{\tau^{-1}}(x)e_{y_1}\otimes\cdots\otimes e_{y_m}$$

where $x=(x_1,\dots,x_m)$ is the tuple $(y_1,\dots,y_m)$ rearranged in nondecreasing order and $\tau\in\mathcal{S}_m$ is such that $y_i = x_{\tau(i)}$. From Proposition 3 it is clear that $q^{\tau^{-1}}(x)$ does not depend upon the choice of τ and

$$\begin{aligned}W^{m,q}U_\sigma^{m,s}(e_{y_1}\otimes\cdots\otimes e_{y_m}) &= W^{m,q}(e_{y_{\sigma^{-1}(1)}}\otimes\cdots\otimes e_{y_{\sigma^{-1}(m)}})\\&= q^{(\sigma^{-1}\tau)^{-1}}(x)e_{y_{\sigma^{-1}(1)}}\otimes\cdots\otimes e_{y_{\sigma^{-1}(m)}}\\&= q^{\tau^{-1}\sigma}(x)e_{y_{\sigma^{-1}(1)}}\otimes\ldots\otimes e_{y_{\sigma^{-1}(m)}}\end{aligned}$$

$$
\begin{aligned}
&= q^{\sigma}(x_{\tau(1)}, \ldots, x_{\tau(m)}) q^{\tau^{-1}}(x) e_{y_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{y_{\sigma^{-1}(m)}} \\
&= U_{\sigma}^{m,q} q^{\tau^{-1}}(x) e_{y_1} \otimes \cdots \otimes e_{y_m} \\
&= U_{\sigma}^{m,q} W^{m,q}(e_{y_1} \otimes \cdots \otimes e_{y_m}).
\end{aligned}
$$

So, $W^{m,q}U_{\sigma}^{m,s} = U_{\sigma}^{m,q}W^{m,q}$. Denoting the unitary operator $\bigoplus_{m=0}^{\infty} W^{m,q}$ on $\Gamma(\mathbb{C}^n)$ by W^q where $W^{0,q} = I$, we get

$$W^q P_{\Gamma_S(\mathbb{C}^n)} = P_{\Gamma_q(\mathbb{C}^n)} W^q$$

and for q and q' we get the intertwining unitary $W^{q'}(W^q)^*$ such that

$$W^{q'}(W^q)^* P_{\Gamma_q(\mathbb{C}^n)} = P_{\Gamma_{q'}(\mathbb{C}^n)} W^{q'}(W^q)^*.$$

Under the Schur product, $\mathcal{Q} = \{q = (q_{ij})_{n\times n} : |q_{ij}| = 1\}$ forms a group.

PROPOSITION 13. *The map from $\mathcal{Q}$ to $B((\mathbb{C}^n)^{\otimes^m})$ given by $q \mapsto W^{m,q}$ is a unitary representation of $\mathcal{Q}$.*

Proof. From the definition of $W^{m,q}$ we get

$$W^{m,q.q'} = W^{m,q}W^{m,q'} \quad \text{and} \quad (W^{m,q})^{-1} = W^{m,q^{-1}}$$

for $q, q' \in \mathcal{Q}$ and $q^{-1} = (q_{ij}^{-1})_{n\times n}$. When q is the identity element of $\mathcal{Q}$, all entries q_{ij} are 1 and hence $W^{m,q}$ is the identity matrix. Hence the assertion holds. ∎

Define

$$(\mathbb{C}^n)^{@^m} = \{u \in (\mathbb{C}^n)^{\otimes^m} : U_{\sigma}^{m,s}(u) = \operatorname{sign}(\sigma)u \ \forall \sigma \in \mathcal{S}_m\}.$$

Then define the *antisymmetric Fock space* or the *fermion Fock space* $\Gamma_a(\mathbb{C}^n)$ as

$$\Gamma_a(\mathbb{C}^n) = \bigoplus_{m=0}^{\infty} (\mathbb{C}^n)^{@^m}.$$

We observed before that the symmetric Fock space is the q-commuting Fock space where $q_{ij} = 1$. But the antisymmetric Fock space is not equal to any $\Gamma_q(\mathbb{C}^n)$. However, consider the case when $q = (q_{ij})_{n\times n}$ is such that $q_{ij} = -1$ for $1 \le i \ne j \le n$. Then the antisymmetric Fock space $\Gamma_a(\mathbb{C}^n)$ is a proper subset of $\Gamma_q(\mathbb{C}^n)$ because clearly $(\mathbb{C}^n)^{@^m}$ is the set of all $u \in (\mathbb{C}^n)^{@^m}$ which are orthogonal to those $P_m e^{\beta}$ for which there exist $s, t \in \{1, \ldots, m\}$, $s \ne t$, such that $\beta_s = \beta_t$ (P_m is given by (2.2)).

Next we give another realization of the standard tuple $\underline{S}$. Let $\mathcal{P}$ be the vector space of all polynomials in q-commuting variables $z_1, \ldots, z_n$, that is, $z_j z_i = q_{ij} z_i z_j$. Any multi-index $\underline{k}$ is an ordered n-tuple of non-negative integers $(k_1, \ldots, k_n)$. We write $|\underline{k}| = k_1 + \cdots + k_n$. The multi-index with 0 in all positions except the ith which is 1, is denoted by $\underline{e}_i$. For any nonzero multi-index $\underline{k}$ the monomial $z_1^{k_1} \cdots z_n^{k_n}$ will be denoted by $\underline{z}^{\underline{k}}$; for $\underline{k} = (0, \ldots, 0)$, let $\underline{z}^{\underline{k}}$ be the complex number 1. Let us equip $\mathcal{P}$ with the following inner product. Declare $\underline{z}^{\underline{k}}$ and $\underline{z}^{\underline{l}}$ orthogonal if $\underline{k} \neq \underline{l}$ as ordered multi-indices. Let

$$\|\underline{z}^{\underline{k}}\|^2 = \frac{k_1! \cdots k_n!}{|k|!}.$$

Note that this inner product also appears in [BB, Definition (1.1)] in the general case. Now define $\mathcal{H}'$ to be the closure of $\mathcal{P}$ with respect to this inner product. Define $\underline{S}' = (S'_1, \ldots, S'_n)$ where for $f \in \mathcal{P}$,

$$S'_i f(z_1, \ldots, z_n) := z_i f(z_1, \ldots, z_n)$$

and S_i is linearly extended to $\mathcal{H}'$. In the case of our standard q-commuting n-tuple $\underline{S}$ of operators on $\Gamma_q(\mathbb{C}^n)$, when $\underline{k} = (k_1, \ldots, k_n)$ let $\underline{S}^{\underline{k}} = S_1^{k_1} \ldots S_n^{k_n}$ and when $\underline{k} = (0, \ldots, 0)$ let $\underline{S}^{\underline{k}} = 1$.

Using (2.2) and the fact that the V_i's are isometries with orthogonal ranges, for $\underline{k} = (k_1, \ldots, k_n)$ with $|\underline{k}| = m$ we get

$$\|\underline{S}^{\underline{k}}\omega\| = \langle P_m \underline{V}^{\underline{k}}\omega, \underline{V}^{\underline{k}}\omega\rangle = \left\langle \frac{1}{|\underline{k}|!} \sum_{\sigma \in \mathcal{S}_m} U_\sigma^{m,q} \underline{V}^{\underline{k}}\omega, \underline{V}^{\underline{k}}\omega \right\rangle = \frac{k_1! \cdots k_n!}{|\underline{k}|!}.$$

If we denote $\underline{V}^{\underline{k}}\omega$ by $e_{x_1} \otimes \cdots \otimes e_{x_m}$, $1 \leq x_i \leq n$, then to get the last term of the above equation we used the fact that there are $k_1! \cdots k_n!$ permutations $\sigma \in \mathcal{S}_m$ such that

$$e_{x_1} \otimes \cdots \otimes e_{x_m} = e_{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma^{-1}(m)}}.$$

Next we show that the above tuples $\underline{S}'$ and $\underline{S}$ are unitarily equivalent.

Proposition 14. *Let $\underline{S}' = (S'_1, \ldots, S'_n)$ be the operator tuple on $\mathcal{H}'$ as introduced above and let $\underline{S} = (S_1, \ldots, S_n)$ be the standard q-commuting tuple of operators on $\Gamma_q(\mathbb{C}^n)$. Then there exists a unitary $U : \mathcal{H}' \to \mathcal{H}$ such that $US'_i = S_i U$ for $1 \leq i \leq n$.*

Proof. Define $U : \mathcal{P} \to \Gamma_q(\mathbb{C}^n)$ as

$$U\Big(\sum_{|\underline{k}| \leq s} b_{\underline{k}} \underline{z}^{\underline{k}}\Big) = \sum_{|\underline{k}| \leq s} b_{\underline{k}} \underline{S}^{\underline{k}}\omega$$

for any constants $b_{\underline{k}}$. As $\|\underline{z}^{\underline{k}}\| = \|\underline{S}^{\underline{k}}\omega\|$ we have

$$\Big\| \sum_{|\underline{k}| \leq s} b_{\underline{k}} \underline{z}^{\underline{k}} \Big\|^2 = \sum_{|\underline{k}| \leq s} |b_{\underline{k}}|^2 \|\underline{z}^{\underline{k}}\|^2 = \sum_{|\underline{k}| \leq s} |b_{\underline{k}}|^2 \|\underline{S}^{\underline{k}}\omega\|^2 = \Big\| \sum_{|\underline{k}| \leq s} b_{\underline{k}} \underline{S}^{\underline{k}}\omega \Big\|^2.$$

So we can extend U linearly to $\mathcal{H}'$ and it is a unitary. Moreover,

$$\begin{aligned}US_i'\Big(\sum_{|\underline{k}|\le s} b_{\underline{k}}\underline{z}^{\underline{k}}\Big) &= U\Big(z_i\sum_{|\underline{k}|\le s} b_{\underline{k}}\underline{z}^{\underline{k}}\Big) = q_{1i}^{k_1}\cdots q_{i-1,i}^{k_{i-1}}U\Big(\sum_{|\underline{k}|\le s} b_{\underline{k}}\underline{z}^{\underline{k}+\underline{e}_i}\Big)\\ &= q_{1i}^{k_1}\cdots q_{i-1,i}^{k_{i-1}}\sum_{|\underline{k}|\le s} b_{\underline{k}}\underline{S}^{\underline{k}+\underline{e}_i}\omega = S_i\Big(\sum_{|\underline{k}|\le s} b_{\underline{k}}\underline{S}^{\underline{k}}\omega\Big)\\ &= S_iU\Big(\sum_{|\underline{k}|\le s} b_{\underline{k}}\underline{z}^{\underline{k}}\Big),\end{aligned}$$

i.e., $US_i' = S_iU$ for $1\le i\le n$. ∎

For any complex number z, define the *z-commutator* of two operators A, B as

$$[A,B]_z = AB - zBA.$$

As $\underline{S}'$ and $\underline{S}$ are unitarily equivalent and the same properties have been proved for $\underline{S}'$ in [BB], we have

Lemma 15.

(1) *Each monomial $\underline{S}^{\underline{k}}\omega$ is an eigenvector for $\sum S_i^*S_i - I$, so the latter operator is diagonal on the standard basis. In fact,*

$$\sum_{i=1}^n S_i^*S_i(\underline{S}^{\underline{k}}\omega) = \Big(\sum_{i=1}^n \frac{\|\underline{S}^{\underline{k}+\underline{e}_i}\omega\|^2}{\|\underline{S}^{\underline{k}}\omega\|^2}\Big)\underline{S}^{\underline{k}}\omega.$$

Also $\sum S_i^*S_i - I$ is compact.

(2) *The commutator $[S_i^*, S_i]$ is as follows:*

$$[S_i^*, S_i]\underline{S}^{\underline{k}}\omega = \Big(\frac{\|\underline{S}^{\underline{k}+\underline{e}_i}\omega\|^2}{\|\underline{S}^{\underline{k}}\omega\|^2} - \frac{\|\underline{S}^{\underline{k}}\omega\|^2}{\|\underline{S}^{\underline{k}-\underline{e}_i}\omega\|^2}\Big)\underline{S}^{\underline{k}}\omega \qquad \text{when } k_i\ne 0.$$

If $k_i = 0$, then

$$[S_i^*, S_i]\underline{S}^{\underline{k}}\omega = S_i^*S_i\underline{S}^{\underline{k}}\omega = \frac{\|\underline{S}^{\underline{k}+\underline{e}_i}\omega\|^2}{\|\underline{S}^{\underline{k}}\omega\|^2}\,\underline{S}^{\underline{k}}\omega.$$

(3) *$[S_i^*, S_j]_{q_{ij}}$ is compact for all $1\le i,j\le n$.*

3. Dilation of q-commuting tuples and the main theorem

Definition 16. Let $\underline{T} = (T_1,\ldots,T_n)$ be a contractive tuple on a Hilbert space $\mathcal{H}$. The operator $\Delta_{\underline{T}} = [I - (T_1T_1^* + \cdots + T_nT_n^*)]^{1/2}$ is called the *defect operator* of $\underline{T}$ and the subspace $\overline{\Delta_{\underline{T}}(\mathcal{H})}$ is called the *defect space* of $\underline{T}$. The tuple $\underline{T}$ is said to be *pure* if $\sum_{\alpha\in\Lambda^m}\underline{T}^\alpha(\underline{T}^\alpha)^*$ converges to zero in the strong operator topology as m tends to infinity.

When $\sum T_iT_i^* = I$, we have $\sum_{\alpha\in\Lambda^m}\underline{T}^\alpha(\underline{T}^\alpha)^* = I$ for all m and hence $\underline{T}$ is not pure. Let $\underline{T}$ be a pure tuple on $\mathcal{H}$. Set $\widetilde{\mathcal{H}} = \Gamma(\mathbb{C}^n)\otimes\overline{\Delta_{\underline{T}}(\mathcal{H})}$, and define an operator $A : \mathcal{H} \to \widetilde{\mathcal{H}}$ by

$$Ah = \sum_{\alpha} e^{\alpha} \otimes \Delta_{\underline{T}} (\underline{T}^{\alpha})^* h, \tag{3.1}$$

where the sum is taken over all $\alpha \in \widetilde{\Lambda}$ (this operator was used by Popescu and Arveson in [Po3, Po4, Ar2] and in the q-commuting case by Bhat and Bhattacharyya in [BB]). Then A is an isometry and $\underline{T}^{\alpha} = A^*(\underline{V}^{\alpha} \otimes I)A$ for all $\alpha \in \widetilde{\Lambda}$ (see [Po4]). Also the tuple $\widetilde{\underline{V}} = (V_1 \otimes I, \dots, V_n \otimes I)$ of operators on $\widetilde{\mathcal{H}}$ is a realization of the minimal noncommuting dilation of $\underline{T}$.

Lemma 17. *Suppose $\underline{T} = (T_1, \dots, T_n)$ is a pure q-commuting tuple on a Hilbert space $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K}$ such that $(S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}})$ is a dilation of $\underline{T}$ and* $\dim(\mathcal{K}) = \operatorname{rank}(\Delta_{\underline{T}})$.

Proof. Let A be the operator introduced in (3.1). Let $\mathcal{B}^m$ denote the set of all $\alpha \in \Lambda^m$ such that $\alpha_1 \le \dots \le \alpha_m$. Then for $f \in \mathcal{H}$,

$$A(h) = \sum_{m=0}^{\infty} \sum_{\sigma,\alpha} e_{\alpha_{\sigma^{-1}(1)}} \otimes \dots \otimes e_{\alpha_{\sigma^{-1}(m)}} \otimes \Delta_{\underline{T}} (T_{\alpha_{\sigma^{-1}(1)}} \cdots T_{\alpha_{\sigma^{-1}(m)}})^* h$$

where the second summation is over $\sigma \in \mathcal{S}_m$ and $\alpha \in \mathcal{B}^m$. Further

$$\begin{aligned} A(h) &= \sum_{m=0}^{\infty} \sum_{\sigma,\alpha} e_{\alpha_{\sigma^{-1}(1)}} \otimes \dots \otimes e_{\alpha_{\sigma^{-1}(m)}} \otimes \overline{(q^{\sigma}(\alpha))^{-1}}\, \Delta_{\underline{T}} (T_{\alpha_1} \cdots T_{\alpha_m})^* h \\ &= \sum_{m=0}^{\infty} \sum_{\sigma,\alpha} q^{\sigma}(\alpha) e_{\alpha_{\sigma^{-1}(1)}} \otimes \dots \otimes e_{\alpha_{\sigma^{-1}(m)}} \otimes \Delta_{\underline{T}} (T_{\alpha_1} \cdots T_{\alpha_m})^* h \\ &= \sum_{m=0}^{\infty} \sum_{\alpha \in \mathcal{B}^m} (m!) P_m e_{\alpha_1} \otimes \dots \otimes e_{\alpha_m} \otimes \Delta_{\underline{T}} (T_{\alpha_1} \cdots T_{\alpha_m})^* h. \end{aligned}$$

So the range of A is contained in $\widetilde{\mathcal{H}}_q = \Gamma_q(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$. This with the above stated properties of A implies that $\underline{S} \otimes I_{\mathcal{K}}$ is a dilation of $\underline{T}$ for some space $\mathcal{K}$ with $\dim(\mathcal{K}) = \operatorname{rank}(\Delta_{\underline{T}})$. ∎

In other words, now $\mathcal{H}$ can be considered as a subspace of $\widetilde{\mathcal{H}}_q$. Moreover, $\widetilde{\underline{S}} = (S_1 \otimes I, \dots, S_n \otimes I)$, as a tuple of operators in $\widetilde{\mathcal{H}}_q$, is the standard q-commuting dilation of $(T_1, \dots, T_n)$. More abstractly we can get a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ can be isometrically embedded in $\Gamma_q(\mathbb{C}^n) \otimes \mathcal{K}$ and $(S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}})$ is a dilation of $\underline{T}$ and $\overline{\operatorname{span}}\{(\underline{S}^{\alpha} \otimes I_{\mathcal{K}})h : h \in \mathcal{H}, \alpha \in \widetilde{\Lambda}\} = \Gamma_q(\mathbb{C}^n) \otimes \mathcal{K}$. There is a unique such dilation up to unitary equivalence and $\dim(\mathcal{K}) = \operatorname{rank}(\Delta_{\underline{T}})$.

Let $C^*(\underline{V})$ and $C^*(\underline{S})$ be the unital C^*-algebras generated by tuples $\underline{V}$ and $\underline{S}$ (defined in the introduction) on the Fock spaces $\Gamma(\mathbb{C}^n)$ and $\Gamma_q(\mathbb{C}^n)$ respectively. For any $\alpha, \beta \in \widetilde{\Lambda}$, $\underline{V}^{\alpha}(I - \sum V_i V_i^*)(\underline{V}^{\beta})^*$ is the rank one operator

$x \mapsto \langle e^\beta, x\rangle e^\alpha$, and so $C^*(\underline{V})$ contains all compact operators. Similarly we see that $C^*(\underline{S})$ also contains all compact operators on $\Gamma_q(\mathbb{C}^n)$. As $V_i^*V_j = \delta_{ij}I$, it is easy to see that $C^*(\underline{V}) = \overline{\operatorname{span}}\{\underline{V}^\alpha(\underline{V}^\beta)^* : \alpha,\beta \in \widetilde{\Lambda}\}$. As the q_{ij}-commutators $[S_i^*, S_j]_{q_{ij}}$ are compact for all i,j, we can also get $C^*(\underline{S}) = \overline{\operatorname{span}}\{\underline{S}^\alpha(\underline{S}^\beta)^* : \alpha,\beta \in \widetilde{\Lambda}\}$.

Consider a contractive tuple $\underline{T}$ on a Hilbert space $\mathcal{H}$. For $0 < r < 1$ the tuple $r\underline{T} = (rT_1, \dots, rT_n)$ is clearly pure. So by (3.1) we have an isometry $A_r : \mathcal{H} \to \Gamma(\mathbb{C}^n) \otimes \overline{\Delta_r(\mathcal{H})}$ defined by

$$A_r h = \sum_\alpha e^\alpha \otimes \Delta_r((r\underline{T})^\alpha)^* h, \qquad h \in \mathcal{H},$$

where $\Delta_r = (I - r^2 \sum T_i T_i^*)^{1/2}$. So for every $0 < r < 1$ we have a completely positive map $\psi_r : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H})$ defined by $\psi_r(X) = A_r^*(X \otimes I)A_r, X \in C^*(\underline{V})$. By taking the limit as $r \nearrow 1$ (see [Po1-4] for details), we get a unital completely positive map ψ from $C^*(\underline{V})$ to $\mathcal{B}(\mathcal{H})$ (Popescu's Poisson transform) satisfying

$$\psi(\underline{V}^\alpha(\underline{V}^\beta)^*) = \underline{T}^\alpha(\underline{T}^\beta)^* \quad \text{for } \alpha,\beta \in \widetilde{\Lambda}.$$

As $C^*(\underline{V}) = \overline{\operatorname{span}}\{\underline{V}^\alpha(\underline{V}^\beta)^* : \alpha,\beta \in \widetilde{\Lambda}\}$, ψ is the unique such completely positive map. Let the minimal Stinespring dilation of ψ be a unital $*$-homomorphism $\pi : C^*(\underline{V}) \to \mathcal{B}(\widetilde{\mathcal{H}})$ where $\widetilde{\mathcal{H}}$ is a Hilbert space containing $\mathcal{H}$, and

$$\psi(X) = P_{\mathcal{H}}\pi(X)|_{\mathcal{H}} \quad \forall X \in C^*(\underline{V}),$$

and $\overline{\operatorname{span}}\{\pi(X)h : X \in C^*(\underline{V}),\, h \in \mathcal{H}\} = \widetilde{\mathcal{H}}$. Let $\underline{\widetilde{V}} = (\widetilde{V}_1, \dots, \widetilde{V}_n)$ where $\widetilde{V}_i = \pi(V_i)$ and so $\underline{\widetilde{V}}$ is the unique standard noncommuting dilation of $\underline{T}$ and clearly $(\widetilde{V}_i)^*$ leaves $\mathcal{H}$ invariant. If $\underline{T}$ is q-commuting, by considering $C^*(\underline{S})$ instead of $C^*(\underline{V})$, and restricting the range of A_r to $\Gamma_q(\mathbb{C}^n) \otimes \Delta_{\underline{T}}(\mathcal{H})$, and taking limits as $r \nearrow 1$ as before we get the unique unital completely positive map $\phi : C^*(\underline{S}) \to \mathcal{B}(\mathcal{H})$ (see also [BB]) satisfying

$$\phi(\underline{S}^\alpha(\underline{S}^\beta)^*) = \underline{T}^\alpha(\underline{T}^\beta)^*, \qquad \alpha,\beta \in \widetilde{\Lambda}. \tag{3.2}$$

DEFINITION 18. Let $\underline{T}$ be a q-commuting tuple. Then we have a unique unital completely positive map $\phi : C^*(\underline{S}) \to \mathcal{B}(\mathcal{H})$ satisfying (3.2). Consider the minimal Stinespring dilation of ϕ, so there is a Hilbert space $\mathcal{H}_1$ containing $\mathcal{H}$ and a unital $*$-homomorphism $\pi_1 : C^*(\underline{S}) \to \mathcal{B}(\mathcal{H}_1)$ such that

$$\phi(X) = P_{\mathcal{H}}\pi_1(X)|_{\mathcal{H}} \quad \forall X \in C^*(\underline{S}),$$

and $\overline{\operatorname{span}}\{\pi_1(X)h : X \in C^*(\underline{S}),\, h \in \mathcal{H}\} = \mathcal{H}_1$. Let $\widetilde{S}_i = \pi_1(S_i)$ and $\underline{\widetilde{S}} = (\widetilde{S}_1, \dots, \widetilde{S}_n)$. Then $\underline{\widetilde{S}}$ is called the *standard q-commuting dilation* of $\underline{T}$.

The standard q-commuting dilation is also unique up to unitary equivalence as the minimal Stinespring dilation is unique up to unitary equivalence.

THEOREM 19. *Let $\underline{T}$ be a pure tuple on a Hilbert space $\mathcal{H}$.*

(1) *Then the maximal q-commuting piece $\underline{\widetilde{V}}^q$ of the standard noncommuting dilation $\underline{\widetilde{V}}$ of $\underline{T}$ is a realization of the standard q-commuting dilation of $\underline{T}^q$ if and only if $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^q(\underline{T}))}$. Moreover, if $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^q(\underline{T}))}$ then* $\operatorname{rank}(\Delta_{\underline{T}}) = \operatorname{rank}(\Delta_{\underline{T}^q}) = \operatorname{rank}(\Delta_{\underline{\widetilde{V}}}) = \operatorname{rank}(\Delta_{\underline{\widetilde{V}}^q})$.

(2) *Let $\underline{\widetilde{V}}$ be the standard noncommuting dilation of $\underline{T}$. If* $\operatorname{rank}(\Delta_{\underline{T}})$ *and* $\operatorname{rank}(\Delta_{\underline{T}^q})$ *are finite and equal then $\underline{\widetilde{V}}^q$ is a realization of the standard q-commuting dilation of $\underline{T}^q$.*

Proof. The proof is similar to the proofs of Theorem 10 and Remark 11 of [BBD]. ■

If the ranks of both $\Delta_{\underline{T}}$ and $\Delta_{\underline{T}^q}$ are infinite then we cannot ensure that $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^q(\underline{T}))}$ and hence cannot ensure the converse of the second part of the last theorem, as seen by the following example. For any $n \geq 2$ consider the Hilbert space $\mathcal{H}_0 = \Gamma_q(\mathbb{C}^n) \otimes \mathcal{M}$ where $\mathcal{M}$ is of infinite dimension, and let $\underline{R} = (S_1 \otimes I, \ldots, S_n \otimes I)$ be a q-commuting pure n-tuple. In fact, one can take $\underline{R}$ to be any q-commuting pure n-tuple on some Hilbert space $\mathcal{H}_0$ with $\overline{\Delta_{\underline{R}}(\mathcal{H}_0)}$ of infinite dimension. Suppose $P_k = (p^k_{ij})_{n\times n}$ for $1 \leq k \leq n$ are $n \times n$ matrices with complex entries such that

$$p^k_{ij} = \begin{cases} t_k & \text{if } i = k,\ j = k+1, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for } 1 \leq k < n,$$

$$p^n_{ij} = \begin{cases} t_n & \text{if } i = n,\ j = 1, \\ 0 & \text{otherwise}, \end{cases}$$

where t_k's are complex numbers satisfying $0 < |t_k| < 1$. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{C}^n$. Set $\underline{T} = (T_1, \ldots, T_n)$ where T_k for $1 \leq k \leq n$ are operators on $\mathcal{H}$ defined by

$$T_k = \begin{bmatrix} R_k & \\ & P_k \end{bmatrix}.$$

So $\underline{T}$ is a pure tuple, the maximal q-commuting piece of $\underline{T}$ is $\underline{R}$, and $\mathcal{H}^q(\underline{T}) = \mathcal{H}_0$ (by Corollary 7). Here $\operatorname{rank}(\Delta_{\underline{T}^q}) = \operatorname{rank}(\Delta_{\underline{T}}) = \infty$ but $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{R}}(\mathcal{H}_0)} \oplus \mathbb{C}^n$. But the converse of Theorem 17(2) holds when the rank of $\Delta_{\underline{T}}$ is finite.

Consider the case when $\underline{T}$ is a q-commuting tuple on a Hilbert space $\mathcal{H}$ satisfying $\sum T_i T_i^* = I$. As $C^*(\underline{S})$ contains the ideal of all compact operators, by standard C^*-algebra theory we have a direct sum decomposition of π_1 as follows. Set $\mathcal{H}_1 = \mathcal{H}_{1C} \oplus \mathcal{H}_{1N}$ where $\mathcal{H}_{1C} = \overline{\operatorname{span}}\{\pi_1(X)h : h \in \mathcal{H},\ X \in C^*(\underline{S}) \text{ and } X \text{ is compact}\}$ and $\mathcal{H}_{1N}$ is the orthogonal complement of it.

Clearly $\mathcal{H}_{1C}$ is a reducing subspace for π_1. Therefore $\pi_1 = \pi_{1C} \oplus \pi_{1N}$ where $\pi_{1C}(X) = P_{\mathcal{H}_{1C}} \pi_1(X) P_{\mathcal{H}_{1C}}$ and $\pi_{1N}(X) = P_{\mathcal{H}_{1N}} \pi_1(X) P_{\mathcal{H}_{1N}}$. Also $\pi_{1C}(X)$ is just the identity representation with some multiplicity. In fact $\mathcal{H}_{1C}$ can be written as $\mathcal{H}_{1C} = \Gamma_q(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$ (see Theorem 4.5 of [BB]) and $\pi_{1N}(X) = 0$ for compact X. But $\Delta_{\underline{T}}(\mathcal{H}) = 0$ and the commutators $[S_i^*, S_i]$ are compact. So $\underline{W} = (W_1, \dots, W_n)$, $W_i = \pi_{1N}(S_i)$, is a tuple of normal operators. It follows that the standard q-commuting dilation of $\underline{T}$ is a tuple of normal operators.

Definition 20. A q-commuting n-tuple $\underline{T} = (T_1, \dots, T_n)$ of operators on a Hilbert space $\mathcal{H}$ is called a *q-spherical unitary* if each T_i is normal and $T_1 T_1^* + \dots + T_n T_n^* = I$.

If $\mathcal{H}$ is a finite-dimensional Hilbert space and $\underline{T}$ is a q-commuting tuple on $\mathcal{H}$ satisfying $\sum T_i T_i^* = I$, then $\underline{T}$ is a q-spherical unitary because in this case each T_i is subnormal and all finite-dimensional subnormal operators are normal (see [Ha]).

Theorem 21 (Main Theorem). *Let $\underline{T}$ be a q-commuting contractive tuple on a Hilbert space $\mathcal{H}$. Then the maximal q-commuting piece of the standard noncommuting dilation of $\underline{T}$ is a realization of the standard q-commuting dilation of $\underline{T}$.*

Proof. Let $\widetilde{\underline{S}}$ denote the standard q-commuting dilation of $\underline{T}$ on a Hilbert space $\mathcal{H}_1$ and we follow the notations as at the beginning of this section. As $\underline{S}$ is also a contractive tuple, we have a unique unital completely positive map $\eta : C^*(\underline{V}) \to C^*(\underline{S})$ satisfying

$$\eta(\underline{V}^{\alpha}(\underline{V}^{\beta})^*) = \underline{S}^{\alpha}(\underline{S}^{\beta})^*, \qquad \alpha, \beta \in \widetilde{\Lambda}.$$

It is easy to see that $\psi = \phi \circ \eta$. Let the unital $*$-homomorphism $\pi_2 : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H}_2)$, for some Hilbert space $\mathcal{H}_2$ containing $\mathcal{H}_1$, be the minimal Stinespring dilation of the map $\pi_1 \circ \eta : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H}_1)$ such that $\pi_1 \circ \eta(X) = P_{\mathcal{H}_1} \pi_2(X)|_{\mathcal{H}_1}$ for $X \in C^*(\underline{V})$, and

$$\overline{\operatorname{span}}\{\pi_2(X)h : X \in C^*(\underline{V}),\, h \in \mathcal{H}_1\} = \mathcal{H}_2.$$

We get the following commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{B}(\mathcal{H}_2) \arrow[d] \\
 & & \mathcal{B}(\mathcal{H}_1) \arrow[d] \\
C^*(\underline{V}) \arrow[r, "\eta"'] \arrow[uurr, "\pi_2"] & C^*(\underline{S}) \arrow[r, "\phi"'] \arrow[ur, "\pi_1"] & \mathcal{B}(\mathcal{H})
\end{tikzcd}$$

where the vertical arrows are compression maps, the horizontal arrows are unital completely positive maps and the diagonal arrows are unital $*$-homomorphisms. Let $\underline{\widehat{V}} = (\widehat{V}_1, \dots, \widehat{V}_n)$ where $\widehat{V}_i = \pi_2(V_i)$. We now show that $\underline{\widehat{V}}$ is the standard noncommuting dilation of $\underline{T}$. We will have this result if we can show that π_2 is a minimal dilation of $\psi = \phi \circ \eta$, as the minimal Stinespring dilation is unique up to unitary equivalence. For this we first show that $\underline{\widetilde{S}} = (\pi_1(S_1), \dots, \pi_1(S_n))$ is the maximal q-commuting piece of $\underline{\widehat{V}}$.

First we consider a particular case when $\underline{T}$ is a q-spherical unitary on a Hilbert space $\mathcal{H}$. In this case we prove that the standard q-commuting dilation and the maximal q-commuting piece of the standard noncommuting dilation of $\underline{T}$ is $\underline{T}$ itself. We have $\phi(\underline{S}^\alpha(I - \sum S_i S_i^*)(\underline{S}^\beta)^*) = \underline{T}^\alpha(I - \sum T_i T_i^*)(\underline{T}^\beta)^* = 0$ for any $\alpha, \beta \in \widetilde{\Lambda}$. This forces $\phi(X) = 0$ for any compact operator X in $C^*(\underline{S})$. Now as the q_{ij}-commutators $[S_i^*, S_j]_{q_{ij}}$ are all compact we see that ϕ is a unital $*$-homomorphism. So the minimal Stinespring dilation of ϕ is ϕ itself and the standard q-commuting dilation of $\underline{T}$ is $\underline{T}$ itself. Next we show that the maximal q-commuting piece of the standard noncommuting dilation of $\underline{T}$ is $\underline{T}$. The presentation of the standard noncommuting dilation which we use is taken from [Po1]. The dilation space $\widetilde{\mathcal{H}}$ can be decomposed as $\widetilde{\mathcal{H}} = \mathcal{H} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathcal{D})$ where $\mathcal{D}$ is the closure of the range of the operator

$$D : \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n\,\text{copies}} \to \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n\,\text{copies}}$$

where D is the positive square root of

$$D^2 = [\delta_{ij} I - T_i^* T_j]_{n \times n}.$$

For convenience, at some places we identify $\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n\,\text{copies}}$ with $\mathbb{C}^n \otimes \mathcal{H}$ so that $(h_1, \dots, h_n) = \sum_{i=1}^n e_i \otimes h_i$. Then

$$D(h_1, \dots, h_n) = D\Big(\sum_{i=1}^n e_i \otimes h_i\Big) = \sum_{i=1}^n e_i \otimes \Big(h_i - \sum_{j=1}^n T_i^* T_j h_j\Big) \tag{3.3}$$

and the standard noncommuting dilation $\widetilde{V}_i$ is

$$\widetilde{V}_i\Big(h \oplus \sum_{\alpha \in \widetilde{\Lambda}} e^\alpha \otimes d_\alpha\Big) = T_i h \oplus D(e_i \otimes h) \oplus e_i \otimes \Big(\sum_{\alpha \in \widetilde{\Lambda}} e^\alpha \otimes d_\alpha\Big) \tag{3.4}$$

for $h \in \mathcal{H}$, $d_\alpha \in \mathcal{D}$ for $\alpha \in \widetilde{\Lambda}$, and $1 \le i \le n$ ($\mathbb{C}^n \omega \otimes \mathcal{D}$ has been identified with $\mathcal{D}$). We have

$$T_i T_i^* = T_i^* T_i \quad \text{and} \quad T_j T_i = q_{ij} T_i T_j \quad \forall 1 \le i, j \le n.$$

Also by the Fuglede–Putnam theorem ([Ha], [Pu])

$$T_j^* T_i = \overline{q}_{ij} T_i T_j^* = q_{ji} T_i T_j^* \quad \text{and} \quad T_j^* T_i^* = q_{ij} T_i^* T_j^* \quad \forall 1 \le i, j \le n.$$

As $\sum T_iT_i^* = I$, by direct computation D^2 is seen to be a projection. So, $D = D^2$. Note that $q_{ij}\overline{q}_{ij} = 1$, i.e., $\overline{q}_{ij} = q_{ji}$. Then we get

$$D(h_1,\dots,h_n) = \sum_{i,j=1}^{n} e_i \otimes T_j(T_j^*h_i - \overline{q}_{ji}T_i^*h_j) \tag{3.5}$$
$$= \sum_{i,j=1}^{n} e_i \otimes T_j(h_{ij})$$

where $h_{ij} = T_j^*h_i - \overline{q}_{ji}T_i^*h_j = T_j^*h_i - q_{ij}T_i^*h_j$ for $1 \le i,j \le n$. Note that $h_{ii} = 0$ and $h_{ji} = -\overline{q}_{ij}h_{ij}$.

As clearly $\mathcal{H} \subseteq \widetilde{\mathcal{H}}^q(\underline{V})$, let $y \in \mathcal{H}^\perp \cap \widetilde{\mathcal{H}}^q(\underline{\widetilde{V}})$. We wish to show that $y = 0$. Decompose y as $y = 0 \oplus \sum_{\alpha\in\widetilde{\Lambda}} e^\alpha \otimes y_\alpha$ with $y_\alpha \in \mathcal{D}$. We assume $y \neq 0$ and arrive at a contradiction. If for some α, $y_\alpha \neq 0$, then $\langle \omega \otimes y_\alpha, (\underline{\widetilde{V}}^\alpha)^*y\rangle = \langle e^\alpha \otimes y_\alpha, y\rangle = \langle y_\alpha, y_\alpha\rangle \neq 0$. Since $(\underline{\widetilde{V}}^\alpha)^*y \in \widetilde{\mathcal{H}}^q(\underline{\widetilde{V}})$, we can assume $\|y_0\| = 1$. Setting $\widetilde{y}_m = \sum_{\alpha\in\Lambda^m} e^\alpha \otimes y_\alpha$, we get $y = 0 \oplus (\oplus_{m\ge 0}\widetilde{y}_m)$. Since D is a projection, its range is closed, and as $y_0 \in \mathcal{D}$, there exist some $(h_1,\dots,h_n)$ such that $y_0 = D(h_1,\dots,h_n)$. Let $\widetilde{x}_0 = \widetilde{y}_0 = y_0$ and $\widetilde{x}_1 = \sum_{i,j=1}^n e_i \otimes D(e_j \otimes h_{ij})$. Further denoting $\prod_{1\le r<s\le m} q_{i_ri_s}$ by p_m, for $m \ge 1$ let

$$\widetilde{x}_m = \sum_{i_1,\dots,i_{m-1},i,j=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes e_i$$
$$\otimes D\Big(e_j \otimes p_{m-1}\Big(\prod_{k=1}^{m-1} q_{i_ki}q_{i_kj}\Big)T_{i_1}^* \cdots T_{i_{m-1}}^* h_{ij}\Big).$$

So $\widetilde{x}_m \in (\mathbb{C}^n)^{\otimes m} \otimes \mathcal{D}$ for all $m \in \mathbb{N}$. As $\underline{T}$ is a q-commuting n-tuple and D is a projection, we have

$$\sum_{1\le i<j\le n} (q_{ij}\widetilde{V}_i\widetilde{V}_j - \widetilde{V}_j\widetilde{V}_i)q_{ji}h_{ij}$$
$$= \sum_{1\le i<j\le n} (q_{ij}T_iT_j - T_jT_i)q_{ji}h_{ij} + \sum_{1\le i<j\le n} D(e_i \otimes T_jh_{ij} - q_{ji}e_j \otimes T_ih_{ij})$$
$$+ \sum_{1\le i<j\le n} (e_i \otimes D(e_j \otimes h_{ij}) - q_{ji}e_j \otimes D(e_i \otimes h_{ij}))$$
$$= 0 + D\Big(\sum_{i,j=1}^{n} e_i \otimes T_jh_{ij}\Big) + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij})$$
$$= D^2(h_1,\dots,h_n) + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij}) = \widetilde{x}_0 + \widetilde{x}_1.$$

So by Proposition 6, $\langle y, \widetilde{x}_0 + \widetilde{x}_1 \rangle = 0$. Next let $m \geq 2$. Then

$$\sum_{i_1,\dots,i_{m-1}=1}^{n} \widetilde{V}_{i_1} \dots \widetilde{V}_{i_{m-1}} \Big\{ \sum_{i,j=1}^{n} (q_{ij} \widetilde{V}_i \widetilde{V}_j - \widetilde{V}_j \widetilde{V}_i) p_{m-1} \Big(\prod_{k=1}^{m-2} q_{i_k j} \Big) \cdot (T_i^* T_{i_1}^* \dots T_{i_{m-2}}^* h_{i_{m-1} j}) \Big\}$$

$$= \sum_{i_1,\dots,i_{m-1}=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-1}} \otimes \Big[\sum_{i,j=1}^{n} D\Big(p_{m-1} \Big(\prod_{k=1}^{m-2} q_{i_k j} \Big) (q_{ij} e_i \otimes T_j T_i^* \cdot T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1} j} - e_j \otimes T_i T_i^* T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1} j}) \Big)$$

$$+ \sum_{i,j=1}^{n} p_{m-1} \Big(\prod_{k=1}^{m-2} q_{i_k j} \Big) \Big\{ q_{ij} e_i \otimes D(e_j \otimes T_i^* T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1} j}) - e_j \otimes D(e_i \otimes T_i^* T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1} j}) \Big\} \Big]$$

$$= - \sum_{i_1,\dots,i_{m-1}=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-1}} \otimes \Big\{ \sum_{j=1}^{n} p_{m-1} \Big(\prod_{k=1}^{m-2} q_{i_k j} \Big) D(e_j \otimes T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1} j}) \Big\}$$

$$+ \sum_{i_1,\dots,i_{m-1}=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-1}} \otimes \Big\{ \sum_{i,j=1}^{n} e_i \otimes D\Big(e_j \otimes q_{ij} p_{m-1} \Big(\prod_{k=1}^{m-2} q_{i_k j} \Big) (T_i^* T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1} j}) \Big)$$

$$- \sum_{i,j=1}^{n} e_i \otimes D\Big(e_j \otimes p_{m-1} \Big(\prod_{k=1}^{m-2} q_{i_k i} \Big) (T_j^* T_{i_1}^* \cdots T_{i_{m-2}}^* h_{i_{m-1} i}) \Big) \Big\}$$

(in the term above, i and j have been interchanged in the last summation)

$$= - \sum_{i_1,\dots,i_{m-2},i=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-2}} \otimes e_i \otimes \Big\{ \sum_{j=1}^{n} p_{m-2} q_{i_r i_s} \Big(\prod_{k=1}^{m-2} q_{i_k i} q_{i_k j} \Big) D(e_j \otimes T_{i_1}^* \cdots T_{i_{m-2}}^* h_{ij}) \Big\}$$

$$+ \sum_{i_1,\dots,i_{m-1}=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-1}} \otimes \sum_{i,j=1}^{n} e_i$$

$$\otimes D\Big(e_j\Big\{\otimes p_{m-1}q_{ij}\Big(\prod_{k=1}^{m-2} q_{i_kj}\Big)(T_i^*T_{i_1}^*\cdots T_{i_{m-2}}^*T_j^*h_{i_{m-1}}$$
$$-q_{i_{m-1}j}T_i^*T_{i_1}^*\cdots T_{i_{m-2}}^*T_{i_{m-1}}^*h_j)$$
$$-p_{m-1}\Big(\prod_{k=1}^{m-2} q_{i_ki}\Big)(T_j^*T_{i_1}^*\cdots T_{i_{m-2}}^*T_i^*h_{i_{m-1}}-q_{i_{m-1}i}T_j^*T_{i_1}^*\cdots T_{i_{m-2}}^*T_{i_{m-1}}^*h_i)\Big)\Big\}$$

(in the term above, i_{m-1} has been replaced by i in the first summation)

$$\begin{aligned}
&= -\sum_{i_1,\ldots,i_{m-2},i,j=1}^{n} e_{i_1}\otimes\cdots\otimes e_{i_{m-2}}\otimes e_i\\
&\qquad\otimes p_{m-2}\Big(\prod_{k=1}^{m-2} q_{i_ki}q_{i_kj}\Big)D(e_j\otimes T_{i_1}^*\cdots T_{i_{m-2}}^*h_{ij})\\
&\qquad+\sum_{i_1,\ldots,i_{m-1},i,j=1}^{n} e_{i_1}\otimes\cdots\otimes e_{i_{m-1}}\otimes e_i\\
&\qquad\otimes p_{m-1}\Big(\prod_{k=1}^{m-1} q_{i_ki}q_{i_kj}\Big)D(e_j\otimes T_{i_1}^*\cdots T_{i_{m-1}}^*h_{ij})\\
&= -\widetilde{x}_{m-1}+\widetilde{x}_m.
\end{aligned}$$

Hence by Proposition 6, $\langle y,\widetilde{x}_{m-1}-\widetilde{x}_m\rangle = 0$. Further for all $m\in\mathbb{N}$, $\|\widetilde{x}_m\|^2$ equals

$$\begin{aligned}
&\Big\langle \sum_{i_1,\ldots,i_{m-1},i,j=1}^{n} e_{i_1}\otimes\cdots\otimes e_{i_{m-1}}\otimes e_i\\
&\qquad\otimes D\Big(e_j\otimes p_{m-1}\Big(\prod_{k=1}^{m-1} q_{i_ki}q_{i_kj}\Big)T_{i_1}^*\cdots T_{i_{m-1}}^*h_{ij}\Big),\\
&\qquad\sum_{i'_1,\ldots,i'_{m-1},i',j'=1}^{n} e_{i'_1}\otimes\cdots\otimes e_{i'_{m-1}}\otimes e_{i'}\\
&\qquad\otimes D\Big(e_{j'}\otimes p_{m-1}\Big(\prod_{k'=1}^{m-1} q_{i'_{k'}i'}q_{i'_{k'}j'}\Big)T_{i'_1}^*\cdots T_{i'_{m-1}}^*h_{i'j'}\Big)\Big\rangle\\
&= \sum_{i_1,\ldots,i_{m-1},i=1}^{n}\Big\langle\sum_{j=1}^{n} D\Big(e_j\otimes p_{m-1}\Big(\prod_{k=1}^{m-1} q_{i_ki}q_{i_kj}\Big)T_{i_1}^*\cdots T_{i_{m-1}}^*h_{ij}\Big),\\
&\qquad\sum_{j'=1}^{n} D\Big(e_{j'}\otimes p_{m-1}\Big(\prod_{k'=1}^{m-1} q_{i_{k'}i}q_{i_{k'}j'}\Big)T_{i_1}^*\cdots T_{i_{m-1}}^*h_{ij'}\Big)\Big\rangle\\
&= \sum_{i_1,\ldots,i_{m-1},i=1}^{n}\Big\langle D\Big(\sum_{j=1}^{n} e_j\otimes p_{m-1}\Big(\prod_{k=1}^{m-1} q_{i_ki}q_{i_kj}\Big)T_{i_1}^*\cdots T_{i_{m-1}}^*h_{ij}\Big),
\end{aligned}$$

$$\sum_{j'=1}^{n} e_{j'} \otimes p_{m-1}\Big(\prod_{k'=1}^{m-1} q_{i_{k'}i}q_{i_{k'}j'}\Big)T_{i_1}^* \cdots T_{i_{m-1}}^* h_{ij'}\Big\rangle$$

$$= \sum_{i_1,\ldots,i_{m-1},i=1}^{n} \Big\langle p_{m-1}\Big\{ \sum_{j,l=1}^{n} \Big(\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j}\Big)(e_j \otimes T_l(T_l^* T_{i_1}^* \cdots T_{i_{m-1}}^* h_{ij}$$

$$- q_{jl} T_j^* T_{i_1}^* \ldots T_{i_{m-1}}^* h_{il}))\Big\}, \sum_{j'=1}^{n} p_{m-1}\Big(\prod_{k'=1}^{m-1} q_{i_{k'}i}q_{i_{k'}j'}\Big) e_{j'} \otimes T_{i_1}^* \cdots T_{i_{m-1}}^* h_{ij'}\Big\rangle$$

$$= \sum_{i_1,\ldots,i_{m-1},i,j=1}^{n} \Big\langle p_{m-1}\Big(\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j}\Big) \sum_{l=1}^{n} T_l(T_l^* T_{i_1}^* \cdots T_{i_{m-1}}^* h_{ij}$$

$$- q_{jl} T_j^* T_{i_1}^* \cdots T_{i_{m-1}}^* h_{il}), p_{m-1}\Big(\prod_{k'=1}^{m-1} q_{i_{k'}i}q_{i_{k'}j}\Big) T_{i_1}^* \cdots T_{i_{m-1}}^* h_{ij}\Big\rangle$$

$$= \sum_{i,j=1}^{n} \langle h_{ij}, h_{ij}\rangle - \sum_{i_1,\ldots,i_{m-1},i,j,l=1}^{n} \langle T_{i_{m-1}} \ldots T_{i_1} T_j^* T_l T_{i_1}^* \cdots T_{i_{m-1}}^* h_{il}, h_{ij}\rangle.$$

Define $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $\tau(X) = \sum_{i=1}^n T_i X T_i^*$ for $X \in \mathcal{B}(\mathcal{H})$, and $\widetilde{\tau}^m : M_n(\mathcal{B}(\mathcal{H})) \to M_n(\mathcal{B}(\mathcal{H}))$ by $\widetilde{\tau}^m(X) = (\tau^m(X_{ij}))_{n\times n}$ for $X = (X_{ij})_{n\times n} \in M_n(\mathcal{B}(\mathcal{H}))$. As τ is a contractive completely positive map, so is $\widetilde{\tau}^m$.

Hence we have $\widetilde{\tau}^m(D) \le I$ and

$$\begin{aligned}
\|\widetilde{x}_m\|^2 &= \sum_{r=1}^{n} \langle \widetilde{\tau}^{m-1}(D)(h_{r1},\ldots,h_{rn}),(h_{r1},\ldots,h_{rn})\rangle \\
&\le \sum_{r=1}^{n} \langle (h_{r1},\ldots,h_{rn}),(h_{r1},\ldots,h_{rn})\rangle \\
&= \sum_{r,i=1}^{n} \langle h_{ri}, h_{ri}\rangle = \sum_{i,r=1}^{n} \langle T_i^* h_r - \overline{q}_{ir} T_r^* h_i, T_i^* h_r - \overline{q}_{ir} T_r^* h_i\rangle \\
&= \sum_{i,r=1}^{n} \{\langle T_i^* T_i h_r - T_r^* T_i h_i, h_r\rangle - \langle T_i^* T_r h_r - T_r^* T_r h_i, h_i\rangle\} \\
&= \sum_{r=1}^{n} \Big\langle h_r - \sum_{i=1}^{n} T_r^* T_i h_i, h_r\Big\rangle - \sum_{i=1}^{n} \Big\langle \sum_{r=1}^{n} T_i^* T_r h_r - h_i, h_i\Big\rangle \\
&= 2\sum_{r=1}^{n} \Big\langle h_r - \sum_{i=1}^{n} T_r^* T_i h_i, h_r\Big\rangle = 2\langle D(h_1,\ldots,h_n),(h_1,\ldots,h_n)\rangle \\
&= 2\|\widetilde{x}_0\|^2 = 2.
\end{aligned}$$

As $\langle y, \widetilde{x}_0+\widetilde{x}_1\rangle = 0$ and $\langle y, \widetilde{x}_{m-1}-\widetilde{x}_m\rangle = 0$ for $m+1 \in \mathbb{N}$, we get $\langle y, \widetilde{x}_0+\widetilde{x}_m\rangle = 0$ for $m \in \mathbb{N}$. So $1 = \langle \widetilde{y}_0, \widetilde{y}_0\rangle = \langle \widetilde{y}_0, \widetilde{x}_0\rangle = -\langle \widetilde{y}_m, \widetilde{x}_m\rangle$. By the Cauchy–Schwarz inequality, $1 \le \|\widetilde{y}_m\|\,\|\widetilde{x}_m\|$, which implies $1/\sqrt{2} \le \|\widetilde{y}_m\|$ for $m \in \mathbb{N}$. This is a contradiction as $y = 0 \oplus (\oplus_{m\ge 0}\widetilde{y}_m)$ is in the Hilbert space $\widetilde{\mathcal{H}}$. This proves the particular case.

Using arguments similar to those for Theorem 13 of [BBD], the proof of the general case (that is, when T_i is not necessarily normal) and the proof of "$\hat{\underline{V}}$ is the standard noncommuting dilation of $\underline{T}$" both follow. ∎

4. Distribution of $S_i + S_i^*$ and related operator spaces. Let $\mathcal{R}$ be the von Neumann algebra generated by $G_i = S_i + S_i^*$ for all $1 \le i \le n$ where

$$S_i = P_{\Gamma_q(\mathbb{C}^n)} V_i|_{\Gamma_q(\mathbb{C}^n)}$$

as in Section 2. We are interested in calculating the moments of $S_i + S_i^*$ with respect to the vacuum state and inferring about the distribution. The vacuum expectation is given by $\epsilon(T) = \langle \omega, T\omega\rangle$ where $T \in \mathcal{R}$. So,

$$\epsilon((S_i+S_i^*)^n) = \langle \omega, (S_i+S_i^*)^n\omega\rangle = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ C_{n/2} = \dfrac{1}{n/2+1}\dbinom{n}{n/2} & \text{otherwise,} \end{cases}$$

where C_n is the Catalan number (cf. [Com]). The above follows on observing that for A_k's equal to S_i or S_i^* the scalar product $\langle \omega, A_nA_{n-1}\cdots A_1\omega\rangle$ is 1 if n is even and if for each k the number of S_i's in $A_kA_{k-1}\cdots A_1$ is greater than or equal to the number of S_i^*'s. In the remaining cases $\langle \omega, A_nA_{n-1}\cdots A_1\omega\rangle = 0$. So the expectation turns out to be the number of Catalan paths. This shows that $S_i + S_i^*$ has semicircular distribution (cf. [Vo]). Further this vacuum expectation is not tracial on $\mathcal{R}$ for $n \ge 2$ as

$$\begin{aligned}
\epsilon(G_2G_2G_1G_1) &= \langle \omega, (S_2+S_2^*)(S_2+S_2^*)(S_1+S_1^*)(S_1+S_1^*)\omega\rangle \\
&= \langle \omega, (S_2^*S_2^*S_1S_1 + S_2^*S_2S_1^*S_1)\omega\rangle = 1, \\
\epsilon(G_2G_1G_1G_2) &= \langle \omega, (S_2+S_2^*)(S_1+S_1^*)(S_1+S_1^*)(S_2+S_2^*)\omega\rangle \\
&= \langle \omega, (S_2^*S_1^*S_1S_2 + S_2^*S_1S_1^*S_2)\omega\rangle = 1/2.
\end{aligned}$$

We now investigate the operator space generated by the G_i's, using notions of the theory of operator spaces introduced by Effros and Ruan [ER]. Here we follow the ideas of [BS2] and [HP]. For some Hilbert space $\widetilde{\mathcal{H}}$ and $a_i \in B(\widetilde{\mathcal{H}}), 1 \le i \le n$, define

$$\|(a_1,\dots,a_n)\|_{\max} = \max\Big(\Big\|\sum_{i=1}^n a_ia_i^*\Big\|^{1/2}, \Big\|\sum_{i=1}^n a_i^*a_i\Big\|^{1/2}\Big).$$

Denote the operator space

$$\left\{\begin{pmatrix} r_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ r_n & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} r_1 & \cdots & r_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} : r_1, \ldots, r_n \in \mathbb{C}\right\} \subset M_n \oplus M_n$$

by E_n. Let $\{e_{ij} : 1 \leq i, j \leq n\}$ denote the standard basis of M_n and $\delta_i = e_{i1} \oplus e_{1i}$. Then

$$\Big\|\sum_{i=1}^{n} a_i \otimes \delta_i\Big\|_{B(\widetilde{\mathcal{H}})\otimes M_n} = \|(a_1, \ldots, a_n)\|_{\max}.$$

THEOREM 22. *The operator space generated by G_i, $1 \leq i \leq n$, is completely isomorphic to E_n.*

Proof. It is enough to show that for $a_i \in B(\widetilde{\mathcal{H}}), 1 \leq i \leq n$, we have

$$\|(a_1, \ldots, a_n)\|_{\max} \leq \Big\|\sum_{i=1}^{n} a_i \otimes G_i\Big\|_{\widetilde{\mathcal{H}}\otimes \Gamma_q(\mathbb{C}^n)} \leq 2\|(a_1, \ldots, a_n)\|_{\max}.$$

Note that

$$\begin{aligned}\Big\|\sum_{i=1}^{n} a_i \otimes S_i^*\Big\|_{\widetilde{\mathcal{H}}\otimes \Gamma_q(\mathbb{C}^n)} &= \Big\|\sum_{i=1}^{n} (a_i \otimes 1)(1 \otimes S_i^*)\Big\|_{\widetilde{\mathcal{H}}\otimes \Gamma_q(\mathbb{C}^n)} \\ &\leq \Big\|\sum_{i=1}^{n} a_i a_i^*\Big\|_{\widetilde{\mathcal{H}}}^{1/2} \Big\|\sum_{i=1}^{n} S_i S_i^*\Big\|_{\Gamma_q(\mathbb{C}^n)}^{1/2} \leq \Big\|\sum_{i=1}^{n} a_i a_i^*\Big\|_{\widetilde{\mathcal{H}}}^{1/2}.\end{aligned}$$

Similarly

$$\Big\|\sum_{i=1}^{n} a_i \otimes S_i\Big\|_{\widetilde{\mathcal{H}}\otimes \Gamma_q(\mathbb{C}^n)} = \Big\|\sum_{i=1}^{n} (1 \otimes S_i)(a_i \otimes 1)\Big\|_{\widetilde{\mathcal{H}}\otimes \Gamma_q(\mathbb{C}^n)} \leq \Big\|\sum_{i=1}^{n} a_i^* a_i\Big\|_{\widetilde{\mathcal{H}}}^{1/2}.$$

So

$$\Big\|\sum_{i=1}^{n} a_i \otimes G_i\Big\|_{\widetilde{\mathcal{H}}\otimes \Gamma_q(\mathbb{C}^n)} \leq 2\|(a_1, \ldots, a_n)\|_{\max}.$$

Let $\mathcal{S}$ denote the set of all states on $B(\widetilde{\mathcal{H}})$. Since $\epsilon(G_i G_j) = \langle \omega, S_i^* S_j \omega\rangle = \delta_{ij}$ we get

$$\begin{aligned}\Big\|\sum_{i=1}^{n} a_i \otimes G_i\Big\|_{\widetilde{\mathcal{H}}\otimes \Gamma_q(\mathbb{C}^n)}^2 &\geq \sup_{\tau\in\mathcal{S}} (\tau \otimes \epsilon)\Big[\Big(\sum_{i=1}^{n} a_i \otimes G_i\Big)^* \sum_{j=1}^{n} a_j \otimes G_j\Big] \\ &= \sup_{\tau\in\mathcal{S}} \tau\Big(\sum_{i=1}^{n} a_i^* a_i\Big) = \Big\|\sum_{i=1}^{n} a_i^* a_i\Big\|.\end{aligned}$$

Similar arguments give

$$\Big\|\sum_{i=1}^{n} a_i \otimes G_i\Big\|^2_{\widetilde{\mathcal{H}}\otimes\Gamma_q(\mathbb{C}^n)} \geq \Big\|\sum_{i=1}^{n} a_i a_i^*\Big\|. \blacksquare$$

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