COLLOQUIUM MATHEMATICUM

VOL. 120 | 2010 | NO. 2

q-DEFORMED CIRCULARITY FOR AN UNBOUNDED OPERATOR IN HILBERT SPACE

BY

SCHÔICHI ÔTA (Fukuoka)

Abstract. The notion of strong circularity for an unbounded operator is introduced and studied. Moreover, q-deformed circularity as a q-analogue of circularity is characterized in terms of the partially isometric and the positive parts of the polar decomposition.

1. Introduction. The concept of circularity for operators appeared in operator theory and quantum mechanics ([1], [6], [9] and [10]). Mlak and Słociński [10] introduced circularity for bounded operators to investigate a class of operators containing the phase operator from the viewpoint of quantum mechanics. Mlak initiated a systematic study of circular operators and attempted to give a characterization of the canonical commutation relation in the Weyl form in terms of circularity in a series of papers [9]. In [1] and [6] circularity of a bounded operator was studied from the operator theoretical standpoint.

Circularity for a possibly unbounded densely defined operator T in a Hilbert space is introduced as the property that T is unitarily equivalent to $e^{it}T$ for all real numbers t. Every weighted shift in a separable Hilbert space has this property ([16]). In Section 2, motivated by a result for bounded operators ([1]), we observe that, for an irreducible and circular densely defined operator T in a separable Hilbert space, under some condition, there is a strongly continuous one-parameter unitary group that intertwines T and $e^{it}T$ for each t. In Section 3, we give a condition for a closed densely defined operator to have such a one-parameter unitary group in terms of the existence of some self-adjoint operator satisfying a kind of commutation relation with the resolvent.

In connection with a class of q-normal operators, in [13] we investigated unbounded weighted shifts T such that, for some positive real number q with $q \neq 1$, T is unitarily equivalent to $qe^{it}T$ for all real numbers t. An operator with that property is called q-deformed circular and this property

2010 *Mathematics Subject Classification*: Primary 47A05; Secondary 47B25, 81R50.
Key words and phrases: unbounded operator, circularity, q-normal operator, q-deformed circular operator, weighted shift.

DOI: 10.4064/cm120-2-2

may be considered as a q-analogue of circularity. It should be noticed that a q-normal weighted shift has that property. In Section 4, q-deformed circularity is characterized in terms of polar decomposition.

2. Preliminaries. Throughout this paper all operators are assumed to be linear. For an operator T in a Hilbert space $\mathcal{H}$, the domain, range and kernel of T are denoted by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\ker T$, respectively. For operators S and T in $\mathcal{H}$, the relation $S \subseteq T$ means that T extends S, that is,

$$\mathcal{D}(S) \subseteq \mathcal{D}(T) \quad \text{and} \quad S\eta = T\eta \quad \text{for all } \eta \in \mathcal{D}(S).$$

The spectrum of a closed densely defined operator T is denoted by $\sigma(T)$. We write $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}_0$ for the set of complex numbers, the set of real numbers, the set of integers and the set of non-negative integers, respectively. For $\lambda \in \mathbb{C}$, $\Im\lambda$ stands for the imaginary part of λ.

DEFINITION 2.1. A densely defined operator T in a Hilbert space $\mathcal{H}$ is said to be *circular* if T is unitarily equivalent to $e^{it}T$ for all $t \in \mathbb{R}$.

Clearly the spectrum of a circular operator is circularly symmetric about the origin. It is well-known that bounded, bilateral and unilateral weighted shifts are circular. For further information on bounded weighted shifts the reader is referred to [4], [16]. We recall the notion of an unbounded weighted shift ([11]). Let S be a closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If there are an orthonormal basis $\{e_n\}$ $(n \in \mathbb{Z})$ and a sequence $\{w_n\}$ $(w_n \neq 0,\, n \in \mathbb{Z})$ of complex numbers such that

$$\mathcal{D}(S) = \Big\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \Big\}$$

and

$$Se_n = w_n e_{n+1} \quad \text{for all } n \in \mathbb{Z},$$

then S is called a *bilateral (injective) weighted shift* with weights $\{w_n\}$ (with respect to $\{e_n\}$). A unilateral weighted shift is defined analogously by replacing $\mathbb{Z}$ with $\mathbb{N}_0$. Every bilateral or unilateral weighted shift is circular.

For a closed densely defined operator T in $\mathcal{H}$, a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to *reduce* T if the following two conditions are satisfied:

(1) $P_{\mathcal{M}}\mathcal{D}(T) \subseteq \mathcal{D}(T)$.
(2) $T(\mathcal{M} \cap \mathcal{D}(T)) \subseteq \mathcal{M}$ and $T(\mathcal{M}^{\perp} \cap \mathcal{D}(T)) \subseteq \mathcal{M}^{\perp}$.

Here $P_{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ denote the orthogonal projection onto $\mathcal{M}$ and the orthogonal complement of $\mathcal{M}$, respectively. If there is no non-trivial reducing subspace of T, then T is said to be *irreducible*. Let $\mathcal{C}(T)$ be the set of all bounded operators B on $\mathcal{H}$ such that B and B^* commute with T, that is, $BT \subseteq TB$ and $B^*T \subseteq TB^*$. Then $\mathcal{C}(T)$ is a von Neumann algebra on $\mathcal{H}$. If T

is irreducible, then $\mathcal{C}(T)$ is a scalar algebra (see, for example [15]). It should be noticed that every unbounded, bilateral weighted shift is irreducible ([13, Lemma 3.1]) and also all unilateral weighted shifts are irreducible ([8]).

The following lemma is an unbounded analogue of the result by Arveson et al. ([1, Proposition 1.3]) for bounded circular operators. It involves an additional condition caused by the unboundedness of the operator, but the outline of the proof is essentially the same.

LEMMA 2.2. *Let T be an irreducible, closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If T is circular, then there are a family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{H}$ and a mapping $m(\cdot,\cdot):\mathbb{R}\times\mathbb{R}\to\mathbb{T}$ such that*

(i) $U_tT=e^{it}TU_t$ *for all* $t\in\mathbb{R}$.
(ii) $U_sU_t=m(s,t)U_{s+t},\ U_0=I$ (*identity operator*) *for all* $s,t\in\mathbb{R}$.

Here, $\mathbb{T}$ is the multiplicative group of complex numbers with modulus 1.

Moreover, if $t\mapsto U_t$ is measurable, then there exists a strongly continuous one-parameter unitary group $\{V_t\}$ satisfying (i), *that is, $V_tT=e^{it}TV_t$ for all $t\in\mathbb{R}$.*

Proof. Since T is circular, there is a family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators satisfying $U_tT=e^{it}TU_t$ $(t\in\mathbb{R})$. We can assume that $U_0=I$. In fact, if necessary, each U_t is replaced with $U_tU_0{}^{-1}$. For $s,t\in\mathbb{R}$, we have

$$U_{s+t}^{-1}U_sU_tT=U_{s+t}^{-1}e^{i(s+t)}TU_sU_t=TU_{s+t}^{-1}U_sU_t.$$

Hence, the unitary operator $U_{s+t}^{-1}U_sU_t$ commutes with T and T^*. Since T is irreducible, it follows that, for each $s,t\in\mathbb{R}$, there is a constant $m(s,t)$ in $\mathbb{T}$ such that

$$m(s,t)U_{s+t}=U_sU_t.$$

Thus, the map $m(\cdot,\cdot)$ on $\mathbb{R}\times\mathbb{R}$ with values $\mathbb{T}$ satisfies

$$m(s,0)=m(0,s)=1\quad\text{and}\quad m(s+t,u)m(s,t)=m(s,t+u)m(t,u)$$

for $s,t\in\mathbb{R}$.

Next assume $t\mapsto U_t$ is measurable. The map $m(\cdot,\cdot)$ is a Borel multiplier. Since every Borel multiplier on the additive group $\mathbb{R}$ is trivial (see [19, Theorem 10.38]), there is a Borel function φ of $\mathbb{R}$ to $\mathbb{T}$ such that

$$m(s,t)=\frac{\varphi(s+t)}{\varphi(s)\varphi(t)}.$$

Put

$$V_t=\varphi(t)U_t\qquad(t\in\mathbb{R}).$$

Then it follows that

$$V_{s+t}=V_sV_t\quad(s,t\in\mathbb{R}),\qquad V_0=I.$$

Thus $\{V_t\}_{t\in\mathbb{R}}$ is a one-parameter group of unitary operators on the separable Hilbert space $\mathcal{H}$. Since $t \mapsto V_t$ is measurable, it follows from [14, Theorem VIII.9] that $\{V_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group such that $V_tT = e^{it}TV_t$ for all $t \in \mathbb{R}$. ∎

3. Strong circularity. Taking Lemma 2.2 into account, we introduce the following.

DEFINITION 3.1. Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. If there is a strongly continuous one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ such that $U_tT = e^{it}\,TU_t$ for all $t \in \mathbb{R}$, then T is said to be *strongly circular* and $\{U_t\}_{t\in\mathbb{R}}$ is called a *unitary group associated with* T.

EXAMPLE 3.2. As pointed out in [9, III], if S is the creation operator in a separable Hilbert space, that is, the unilateral weighted shift with weights $\{w_n\}$ given by $w_n = \sqrt{n+1}$ ($n \in \mathbb{N}_0$), then S is strongly circular. This observation can be generalized to all weighted shifts. To be more precise, let S be a unilateral or bilateral weighted shift in a separable Hilbert space $\mathcal{H}$. Then S is strongly circular. In fact, let S be a bilateral weighted shift in $\mathcal{H}$ with weights $\{w_n\}$ with respect to $\{e_n\}$. Define a closed densely defined operator by

$$\mathcal{D}(N) = \Big\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |n|^2 < \infty \Big\}$$

and

$$Ne_n = ne_n \qquad (n \in \mathbb{Z}).$$

Then N is self-adjoint. Since $e^{itN}e_n = e^{itn}e_n$ ($n \in \mathbb{Z}$), we have

$$e^{itN}Se_n = e^{it}Se^{itN}e_n \qquad \text{for all } n \in \mathbb{Z}.$$

It follows that S is a strongly circular operator with the associated unitary group $\{e^{itN}\}$. In the analogous way, a unilateral weighted shift S is strongly circular as well.

THEOREM 3.3. *Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then T is strongly circular if and only if there is a self-adjoint operator A in $\mathcal{H}$ such that*

$$(\lambda - A)^{-1}T \subseteq T(\lambda - I - A)^{-1} \tag{3.1}$$

for all $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$.

Proof. Suppose T is strongly circular. Then there is a strongly continuous one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ such that $U_tT = e^{it}TU_t$ for all $t \in \mathbb{R}$. Set

$$V_t = e^{it}U_t \qquad \text{for each } t \in \mathbb{R}.$$

Then $\{V_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$. Let A be the infinitesimal generator of $\{U_t\}_{t\in\mathbb{R}}$. Then it follows from semigroup theory (for example, see [5]) that the infinitesimal generator of $\{V_t\}_{t\in\mathbb{R}}$ is $I + A$. Take $\eta \in \mathcal{D}(T)$. Then, by [14, p. 287], for $\varphi \in \mathcal{D}(T^*)$ and $\lambda \in \mathbb{C}$ with $\Im\lambda < 0$ we have

$$\begin{aligned}\langle(\lambda - A)^{-1}T\eta, \varphi\rangle &= i\int_0^\infty e^{-it\lambda}\langle U_tT\eta, \varphi\rangle\,dt = i\int_0^\infty e^{-it\lambda}\langle V_t\eta, T^*\varphi\rangle\,dt\\ &= \langle(\lambda - I - A)^{-1}\eta, T^*\varphi\rangle.\end{aligned}$$

If $\Im\lambda > 0$, the same equality holds. Since T is closed and $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$, $(\lambda-I-A)^{-1}\eta \in \mathcal{D}(T^{**}) = \mathcal{D}(T)$ and $T(\lambda-I-A)^{-1}\eta = (\lambda-A)^{-1}T\eta$ for $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$. This implies (3.1).

Conversely, suppose A is a self-adjoint operator satisfying (3.1). Put

$$U_t = e^{itA} \quad\text{and}\quad V_t = e^{it}e^{itA}$$

for $t \in \mathbb{R}$. For $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$, $\eta \in \mathcal{D}(T)$ and $\varphi \in \mathcal{D}(T^*)$, we have

$$\langle(\lambda - A)^{-1}T\eta, \varphi\rangle = \int \frac{1}{\lambda-\nu}\,d\langle E^A_\nu T\eta, \varphi\rangle$$

and

$$\langle T(\lambda - I - A)^{-1}\eta, \varphi\rangle = \int \frac{1}{\lambda-\nu}\,d\langle E^{I+A}_\nu \eta, T^*\varphi\rangle.$$

Here $E^A(\cdot)$ and $E^{I+A}(\cdot)$ are the spectral measures of A and $I + A$, respectively. In view of [18, Lemma 5.2], it follows that

$$E^A(\mathcal{M})T \subseteq TE^{I+A}(\mathcal{M})$$

for all Borel sets $\mathcal{M}$. Therefore, $U_tT \subseteq e^{it}TU_t$ for all $t \in \mathbb{R}$. Since each U_t is unitary, $U_t\mathcal{D}(T) = \mathcal{D}(T)$. Hence, $U_tT = e^{it}\,TU_t$ for all $t \in \mathbb{R}$. ∎

For densely defined operators S and T, $[S, T]$ stands for their commutator, that is, $\mathcal{D}([S, T]) = \mathcal{D}(ST) \cap \mathcal{D}(TS)$ and $[S, T]\eta = ST\eta - TS\eta$ for $\eta \in \mathcal{D}([S, T])$.

PROPOSITION 3.4. *Let T be a strongly circular operator in a Hilbert space $\mathcal{H}$ and let A be the infinitesimal generator of a unitary group $\{U_t\}_{t\in\mathbb{R}}$ associated with T. If $\eta \in \mathcal{D}(AT) \cap \mathcal{D}(A)$, then*

$$\eta \in \mathcal{D}([A, T]) \quad\textit{and}\quad [A, T]\eta = T\eta.$$

Proof. Take $\eta \in \mathcal{D}(AT) \cap \mathcal{D}(A)$. By Definition 3.1, we have

$$T\Big(\frac{U_t - I}{t}\Big)\eta = \frac{e^{-it}U_tT\eta - T\eta}{t}.$$

Since $\mathcal{D}(T)$ is invariant for each U_t and also $\eta \in \mathcal{D}(A)$,

$$\Big(\frac{U_t - I}{t}\Big)\eta \in \mathcal{D}(T) \quad\text{and}\quad \Big(\frac{U_t - I}{t}\Big)\eta \to iA\eta \quad\text{as } t \to 0.$$

On the other hand, since $T\eta \in \mathcal{D}(A)$, it follows that

$$\lim_{t\to 0} \frac{e^{-it}\, U_t T\eta - T\eta}{t} = -iT\eta + iAT\eta.$$

Since T is closed,

$$A\eta \in \mathcal{D}(T) \quad \text{and} \quad iTA\eta = -iT\eta + iAT\eta.$$

This implies the proposition. ∎

4. q-Deformed circularity. Let T be a densely defined operator in a Hilbert space $\mathcal{H}$. If there is a positive real number q with $q \neq 1$ such that T is unitarily equivalent to qT, then we say that T *has property Q*. If a nontrivial T has property Q, it must be unbounded and its spectrum contains zero [13]. We should remark that Makarov and Tsekanovskii ([7]) introduced the concept of a μ-scale invariance, which is just the same as property Q, to investigate Friedrichs and Krein–von Neumann self-adjoint extensions.

EXAMPLE 4.1. Let T be a closed densely defined operator in $\mathcal{H}$. If

$$TT^* = qT^*T \qquad (q > 0,\, q \neq 1),$$

then T is called a *q-normal operator*. It should be noticed that elements satisfying this relation in a formal algebraic sense appear in various circumstances in quantum group theory (see [3] and the references cited therein). A nontrivial q-normal operator T is always unbounded and has large spectrum in the sense of the planar Lebesgue measure. In particular, every q-normal operator T is unitarily equivalent to qT. Thus the class of operators possessing property Q contains all q-normal operators.

DEFINITION 4.2 ([13]). Let T be a densely defined operator in $\mathcal{H}$. If there is a positive real number q with $q \neq 1$ such that T is unitarily equivalent to $qe^{it}T$ for all $t \in \mathbb{R}$, then T is called a *deformed circular operator* with *deformation parameter q*, or simply a *q-deformed circular operator*.

Clearly, a q-deformed circular operator has property Q.

EXAMPLE 4.3 (q-deformed circular weighted shifts). If a bilateral weighted shift has property Q, then it is q-deformed circular. Hence, a q-normal bilateral weighted shift is q-deformed circular. Moreover the spectrum of a q-deformed circular weighted shift is the whole complex plane. Further information on q-deformed circular weighted shifts can be found in [13].

PROPOSITION 4.4. *Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Then T is q-deformed circular if and only if T is circular and has property Q.*

Proof. Suppose T is q-deformed circular. Then there is a family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{H}$ such that

$$U_tT = qe^{it}TU_t \quad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

It is clear that T has property Q. Put

$$V_t = U_tU_0^{-1} \quad \text{for all } t \in \mathbb{R}. \tag{4.2}$$

By (4.1),

$$V_tT = U_tU_0^{-1}T = q^{-1}U_tTU_0^{-1} = e^{it}TU_tU_0^{-1} = e^{it}TV_t.$$

Thus T is circular. The converse is easily proved by a simple calculation. ∎

THEOREM 4.5. *Let T be a closed densely defined operator in $\mathcal{H}$ with the polar decomposition $T = U|T|$. Then T is q-deformed circular if and only if the following statements hold:*

(i) *There is a unitary operator U_0 on $\mathcal{H}$ that commutes with U and satisfies*

$$U_0|T| = q|T|U_0. \tag{4.3}$$

(ii) *There is a family $\{V_t\}_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{H}$ such that*

$$V_tU = e^{it}UV_t \quad \textit{and} \quad V_t|T| = |T|V_t \tag{4.4}$$

for all $t \in \mathbb{R}$.

In particular, in that case U is circular and $|T|$ has property Q.

Proof. Suppose T is q-deformed circular and let $\{U_t\}_{t\in\mathbb{R}}$ be a family of unitary operators satisfying (4.1). Since $U_0T = qTU_0$, we have

$$(U_0|T|U_0{}^*)^2 = U_0T^*TU_0{}^* = q^2T^*T.$$

Since the square root is unique, we obtain (4.3). Moreover, we have

$$UU_0|T| = qU|T|U_0 = qTU_0 = U_0T = U_0U|T|,$$

and so $UU_0 = U_0U$ on $\overline{\mathcal{R}(|T|)}$. By (4.3), U_0 leaves $\ker|T|$ ($\equiv \ker T$) invariant. Since $U = 0$ on $\ker T$, $UU_0 = 0$ on $\ker|T|$. Hence, $UU_0 = U_0U = 0$ on $\ker|T|$. Thus, U and U_0 commute.

We next show (ii). As in the proof of Proposition 4.4, define V_t by (4.2), so that $V_tT = e^{it}TV_t$ for all $t \in \mathbb{R}$. It follows that

$$T^*TV_t = T^*e^{-it}V_tT = V_tT^*T.$$

Therefore, $|T|V_t = V_t|T|$ for all $t \in \mathbb{R}$. Furthermore, this equality implies that $\ker|T|$ is invariant under each V_t. Hence,

$$V_tU = 0 = e^{it}UV_t \quad \text{on } \ker|T|.$$

On the other hand, we have

$$V_tU|T| = e^{it}TV_t = e^{it}U|T|V_t = e^{it}UV_t|T|.$$

It follows that $V_tU = e^{it}UV_t$ for all $t \in \mathbb{R}$.

Conversely, assume that (i) and (ii) hold. Then, for each $t \in \mathbb{R}$,

$$\begin{aligned}(V_tU_0)T &= V_tU_0U|T| = V_tUU_0|T| && \text{(since } U_0 \text{ and } U \text{ commute)}\\ &= qV_tU|T|U_0 && \text{(by (4.3))}\\ &= qe^{it}\,U|T|V_tU_0 && \text{(by (4.4))}\\ &= qe^{it}T(V_tU_0).\end{aligned}$$

This implies that T is q-deformed circular. ■

Acknowledgements. This work was done while the author was visiting Leipzig University. It is pleasure to thank Professor K. Schmüdgen for stimulating discussions. He also thanks Professor F. H. Szafraniec for useful comments and pointing out [7]. This research was partially supported by Grant-in-Aid for Scientific Research No. 20540178.

REFERENCES

[1] W. Arveson, D. W. Hadwin, T. W. Hoover and E. E. Kymala, *Circular operators*, Indiana Univ. Math. J. 33 (1984), 583–595.

[2] J. Blank, P. Exner and M. Havlíček, *Hilbert Space Operators in Quantum Physics*, AIP Press, New York, 1994.

[3] M. Chaichian and A. Demichev, *Introduction to Quantum Groups*, World Sci., Singapore, 1996.

[4] J. B. Conway, *The Theory of Subnormal Operators*, Amer. Math. Soc., Providence, RI, 1991.

[5] A.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, Berlin, 1999.

[6] R. Gellar, *Circularly symmetric normal and subnormal operators*, J. Anal. Math. 32 (1977), 93–117.

[7] K. A. Makarov and E. Tsekanovskii, *On μ-scale invariant operators*, Methods Funct. Anal. Topology 13 (2007), 181–186.

[8] W. Mlak, *The Schrödinger type couples related to weighted shifts*, Univ. Iagell. Acta Math. 27 (1988), 297–301.

[9] —, *Notes on circular operators I–IV*, ibid. 29 (1992), 145–175.

[10] W. Mlak and M. Słociński, *Quantum phase and circular operators*, ibid. 29 (1992), 133–144.

[11] S. Ôta, *Some classes of q-deformed operators*, J. Operator Theory 48 (2002), 151–186.

[12] —, *On q-deformed hyponormal operators*, Math. Nachr. 248–249 (2003), 144–150, .

[13] —, *q-deformed circular operators*, Integral Equations Operator Theory 54 (2006), 555–569.

[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. I, Academic Press, New York, 1973.

[15] K. Schmüdgen, *Unbounded Operator Algebras and Representation Theory*, Akademie-Verlag, Berlin, 1988.

[16] A. L. Shields, *Weighted shift operators and analytic function theory*, in: Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc., Providence, RI, 1974, 49–128.

[17] J. Stochel and F. H. Szafraniec, *Circular invariance of the Weyl form of the canonical commutation relation*, in: Mathematical Physics and Stochastic Analysis, World Sci., Singapore, 2000, 379–383.

[18] M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, Amer. Math. Soc. Colloq. Publ. 15, Providence, RI, 1932.

[19] V. S. Varadarajan, *Geometry of Quantum Theory*, Vol. II, Van Nostrand, New York, 1970.

[20] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, Berlin, 1980.

Schôichi Ôta
Department of Content and Creative Design
Kyushu University
4-9-1 Shiobaru, Fukuoka 815-8540, Japan
E-mail: ota@design.kyushu-u.ac.jp

Received 26 August 2009 (5263)