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INDUCTIVE DIMENSIONS MODULO SIMPLICIAL COMPLEXES AND ANR-COMPACTA

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Abstract. We introduce and investigate inductive dimensions $\mathcal{K}$-Ind and $\mathcal{L}$-Ind for classes $\mathcal{K}$ of finite simplicial complexes and classes $\mathcal{L}$ of ANR-compacta (if $\mathcal{K}$ consists of the 0-sphere only, then the $\mathcal{K}$-Ind dimension is identical with the classical large inductive dimension Ind). We compare K-Ind to K-Ind introduced by the author [Mat. Vesnik 61 (2009)]. In particular, for every complex K such that $K * K$ is non-contractible, we construct a compact Hausdorff space X with K-Ind X not equal to K-dim X.

Introduction. In [8] we introduced dimension functions $\mathcal{K}$-dim and $\mathcal{L}$-dim for classes $\mathcal{K}$ of finite simplicial complexes and classes $\mathcal{L}$ of ANR-compacta. For the definitions and necessary information see Section 1. The theory of $\mathcal{L}$-dim is a part of extension theory introduced by A. Dranishnikov [2].

Here we introduce and investigate inductive functions $\mathcal{K}$-Ind and $\mathcal{L}$-Ind (Definitions 2.1 and 2.3). For $\mathcal{K}$ and $\mathcal{L}$ consisting of a two-point set $\{0,1\}$ the dimension functions $\mathcal{K}$-Ind and $\mathcal{L}$-Ind coincide with the classical large inductive dimension Ind.

If $\mathcal{L}$ is a class of compact polyhedra and τ is an arbitrary triangulation of the class $\mathcal{L}$ (τ consists of some triangulations of all elements of $\mathcal{L}$), then $\mathcal{L}_\tau\text{-Ind}\, X \le \mathcal{L}\text{-Ind}\, X$ for every normal space X and $\mathcal{L}_\tau\text{-Ind}\, X = \mathcal{L}\text{-Ind}\, X$ for the hereditarily normal space X (Theorem 2.4).

If a hereditarily normal space X is represented as the union of two subspaces X_1 and X_2, then $\mathcal{L}\text{-Ind}\, X \le \mathcal{L}\text{-Ind}\, X_1 + \mathcal{L}\text{-Ind}\, X_2 + 1$ (Theorem 2.8).

For homotopy equivalent classes $\mathcal{L}_1$ and $\mathcal{L}_2$ and an arbitrary hereditarily normal space X we have $\mathcal{L}_1\text{-Ind}\, X = \mathcal{L}_2\text{-Ind}\, X$ (Corollary 3.7). So, when we investigate the $\mathcal{L}$-Ind dimension of hereditarily normal spaces, we can consider only classes $\mathcal{L}$ consisting of compact polyhedra, because by J. West's theorem every ANR-compactum has a homotopy type of some compact polyhedron.

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For every $\mathcal{K}$, $\mathcal{L}$, and X we have $\mathcal{K}\text{-Ind}\,X$, $\mathcal{L}\text{-Ind}\,X \leq \operatorname{Ind} X$ (Theorem 3.12). The equality $\mathcal{K}\text{-Ind}\,X = \operatorname{Ind} X$ holds for every normal space X if and only if $\mathcal{K}$ contains a disconnected complex (Theorem 3.14). The same is true for $\mathcal{L}$-Ind and hereditarily normal spaces X (Theorem 3.15).

We also prove that $\mathcal{K}\text{-dim}\,X \leq \mathcal{K}\text{-Ind}\,X$ for every normal space X (Theorem 3.18) and $\mathcal{K}\text{-dim}\,X = \mathcal{K}\text{-Ind}\,X$ for every metrizable space X (Theorem 3.23).

In Section 5 we construct compact Hausdorff spaces X_n^K with

$$K\text{-dim}\,X_n^K = n < 2n - 1 \leq K\text{-Ind}\,X_n^K \leq 2n,$$

where $n \geq 2$ and K is a complex with $K * K$ non-contractible. To construct X_n^K we apply fully closed mappings and resolutions. In Section 4 we recall necessary information concerning this area.

1. Preliminaries

1.1. By a *space* we mean a normal T_1-space. For a space X we denote by $\exp X$ the set of all closed subsets of X (including $\emptyset$).

All mappings are assumed to be continuous. A metrizable compact space is called a compactum. By $\simeq$ we denote homotopy equivalence, and $|S|$ stands for the cardinality of a set S. We denote by $\operatorname{Fin}_s(\exp X)$ the set of all finite sequences $\Phi = (F_1, \dots, F_m)$, $F_j \in \exp X$, i.e.

$$\operatorname{Fin}_s(\exp X) = \bigcup\{(\exp X)^m : m = 1, 2, \dots\}.$$

Recall that an abstract simplicial complex K is said to be *complete* if every face of each simplex from K belongs to K. In what follows, *complexes* are finite abstract complete simplicial complexes. Sometimes we identify a complex K with its geometric realization, i.e. with a *Euclidean* complex $\tilde{K}$ with the same vertex scheme.

In what follows, *polyhedra* are compact polyhedra. Hence every polyhedron is an ANR in the class of all (normal) spaces.

For a complex K we denote by $v(K)$ the set of all its vertices. Let u be a finite family of sets and let $u_0 = \{U \in u : U \neq \emptyset\}$. The *nerve* of the family u is a complex $N(u)$ such that $v(N(u)) = \{a_U : U \in u_0\}$ and a non-empty set $\Delta \subset v(N(u))$ is a simplex of $N(u)$ if and only if $\bigcap\{U : a_U \in \Delta\} \neq \emptyset$.

We now recall several notions and facts. They are well known but important for this article.

1.2. Definition. A pair (X, Y) of spaces has the *Homotopy Extension Property* if, for every closed set $F \subset X$, each mapping $f : (X \times 0) \cup (F \times I) \to Y$ extends over $X \times I$.

1.3. Theorem. (Borsuk's theorem on extension of homotopy; see [13], [14]). *Every pair (X, L), where X is a space and L is an ANR-compactum, has the Homotopy Extension Property.* ■

1.4. Theorem [15]. *Every ANR-compactum is homotopy equivalent to some compact polyhedron.* ■

1.5. Definition. Let X and Y be spaces and let $Z \subset X$. The property that all partial mappings $f : Z \to Y$ extend over X will be denoted by $Y \in AE(X, Z)$. If every mapping $f : Z \to Y$ extends over an open set $U_f \supset Z$, then we write $Y \in ANE(X, Z)$. If $Y \in A(N)E(X, Z)$ for every closed $Z \subset X$, then Y is called an *absolute (neighbourhood) extensor of* X (notation: $Y \in A(N)E(X)$). If $Y \in A(N)E(X)$ for all spaces X, then Y is said to be an *absolute (neighbourhood) extensor* (notation: $Y \in A(N)E$).

The Brouwer–Tietze–Urysohn theorem on extension of functions yields

1.6. Theorem. *If Y is an $A(N)R$-compactum, then $Y \in A(N)E$.* ■

1.7. Lemma (Open enlargement lemma). *Let $\Phi = (F_1, \ldots, F_m) \in \mathrm{Fin}_s(\exp X)$. Then there exists a sequence $u = (U_1, \ldots, U_m)$ of open subsets of X such that $F_j \subset U_j$, $j = 1, \ldots, m$, and $N(\Phi) = N(u)$.* ■

Now we are going to discuss new dim-type functions introduced in [8]. In what follows, K stands for a complex. For each complex K we fix an enumeration of its vertices: $v(K) = (a_1, \ldots, a_m)$.

1.8. Definition. Let K be a complex with $|v(K)| = m$ and let $\Phi = (F_1, \ldots, F_m) \in \mathrm{Fin}_s(\exp X)$. We say that $N(\Phi)$ is *embedded* in K (notation: $N(\Phi) \subset K$) if the correspondence $F_j \to a_j$ generates a simplicial embedding $e : N(\Phi) \to K$.

Put $\mathrm{Exp}_K(X) = \{\Phi \in (\exp X)^m : N(\Phi) \subset K\}$.

1.9. Definition. Let $\Phi = (F_1, \ldots, F_m) \in \mathrm{Exp}_K(X)$. A sequence $u = (U_1, \ldots, U_m)$ of open subsets of X is called a K-*neighbourhood* of Φ if $F_j \subset U_j$ and the correspondence $U_j \to a_j$ generates a simplicial embedding $N(u) \to K$.

According to Lemma 1.7 each $\Phi \in \mathrm{Exp}_K(X)$ has a K-neighbourhood.

1.10. Definition. A set $P \subset X$ is said to be a K-*partition* of $\Phi \in \mathrm{Exp}_K(X)$ (notation: $P \in \mathrm{Part}(\Phi, K)$) if $P = X \setminus \bigcup u$, where u is a K-neighbourhood of Φ.

1.11. Definition ([8]). A sequence $(K_1 \ldots, K_r)$ of complexes is called *inessential in* X if for every sequence $(\Phi_1, \ldots, \Phi_r)$ such that $\Phi_i \in \mathrm{Exp}_{K_i}(X)$ there exist K_i-partitions P_i of Φ_i with $P_1 \cap \cdots \cap P_r = \emptyset$.

1.12. Definition ([8]). Let $\mathcal{K}$ be a non-empty class of complexes. To every space X one assigns the dimension $\mathcal{K}$-dim$X$, which is an integer ≥ -1 or ∞, defined in the following way:

(1) $\mathcal{K}$-dim $X = -1 \Leftrightarrow X = \emptyset$;
(2) $\mathcal{K}$-dim $X \leq n \geq 0$ if every sequence $(K_1, \ldots, K_{n+1})$, $K_i \in \mathcal{K}$, is inessential in X;

(3) $\mathcal{K}$-dim $X = \infty$ if $\mathcal{K}$-dim $X > n$ for all $n = -1, 0, 1, \dots$.

If the class $\mathcal{K}$ contains only one complex K we write $\mathcal{K} = K$ and $\mathcal{K}$-dim $X = K$-dim X.

Hemmingsen's theorem on partitions ([3, Theorem 3.2.6]) can be reformulated as follows:

1.13. THEOREM. $\{0,1\}$-dim $X = \dim X$. ∎

In what follows, $\mathcal{L}$ stands for a non-empty class of ANR-compacta L. We denote by $X_1 * \dots * X_n \equiv *_{i=1}^n X_i$ the join of the spaces $X_1, \dots, X_n$.

1.14. DEFINITION. To every space X one assigns the dimension $\mathcal{L}$-dim X, which is an integer ≥ -1 or ∞, defined in the following way:

(1) $\mathcal{L}$-dim $X = -1 \Leftrightarrow X = \emptyset$;
(2) $\mathcal{L}$-dim $X \leq n \geq 0$ if $L_1 * \dots * L_{n+1} \in AE(X)$ for any $L_1, \dots, L_{n+1} \in \mathcal{L}$;
(3) $\mathcal{L}$-dim $X = \infty$ if $\mathcal{L}$-dim $X > n$ for all $n \geq -1$.

If the class $\mathcal{L}$ contains only one compactum L we write $\mathcal{L} = L$ and $\mathcal{L}$-dim $X = L$-dim X.

1.15. REMARK. In [8, Definition 3.9], $\mathcal{L}$-dim was defined in a slightly different but equivalent way (see [8, Corollary 3.13]).

Since $S^n = (S^0)^{*(n+1)}$, from a characterization of the Lebesgue dimension by means of mappings to spheres we get

1.16. THEOREM. *For every space X, S^0-dim $X = \dim X$.* ∎

Let $\mathcal{L}$ be a non-empty class of polyhedra. For each $L \in \mathcal{L}$ we fix a triangulation $t = t(L)$ of L. The pair (L, t) is a simplicial complex which is denoted by L_t. The family $\tau = \{t(L) : L \in \mathcal{L}\}$ is said to be a *triangulation* of the class $\mathcal{L}$. Let $\mathcal{L}_\tau = \{L_t : t \in \tau\}$.

1.17. THEOREM ([8]). *Let $\mathcal{L}$ be a non-empty class of polyhedra and let τ be a triangulation of $\mathcal{L}$. Then $\mathcal{L}_\tau$-dim $X = \mathcal{L}$-dim X for every space X.* ∎

1.18. DEFINITION. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be non-empty classes of ANR-compacta. We say that $\mathcal{L}_1$ is *dominated* by $\mathcal{L}_2$ (notation: $\mathcal{L}_1 \leq_h \mathcal{L}_2$) if every $L_1 \in \mathcal{L}_1$ is homotopically dominated by some $L_2 \in \mathcal{L}_2$. The class $\mathcal{L}_1$ is *homotopy equivalent* to $\mathcal{L}_2$ (notation: $\mathcal{L}_1 \simeq \mathcal{L}_2$) if both $\mathcal{L}_1 \leq_h \mathcal{L}_2$ and $\mathcal{L}_2 \leq_h \mathcal{L}_1$.

1.19. PROPOSITION ([8]). *If $\mathcal{L}_1 \simeq \mathcal{L}_2$, then $\mathcal{L}_1$-dim $X = \mathcal{L}_2$-dim X for every space X.* ∎

Theorem 1.4 and Proposition 1.19 yield

1.20. THEOREM. *For every non-empty class $\mathcal{R}$ of ANR-compacta there exists a class $\mathcal{L} = \mathcal{L}(\mathcal{R})$ of polyhedra such that $\mathcal{R}$-dim $X = \mathcal{L}$-dim X for every space X.* ∎

So, when we investigate dimension functions of type $\mathcal{L}$-dim, we can consider only classes $\mathcal{L}$ consisting of compact polyhedra. In the remainder of this section, L stands for a compact polyhedron and $\mathcal{L}$ for a non-empty class of compact polyhedra.

1.21. DEFINITION. Let F be a closed subset of a space X. A mapping $f : F \to L$ is called a *partial mapping* of X to L (notation: $f \in PC(X, L)$).

1.22. DEFINITION. Every mapping $f \in PC(X, L)$ extends over an open set $U \supset F = \operatorname{dom} f$. Such a set U is said to be an *L-neighbourhood* of f. Its complement $P = X \setminus U$ is called an *L-partition* of f (notation: $P \in \operatorname{Part}(f, L)$).

1.23. DEFINITION. A sequence $(f_1, \ldots, f_r)$, $f_i \in PC(X, L_i)$, is said to be *inessential in* X if there exist partitions $P_i \in \operatorname{Part}(f_i, L_i)$ such that $P_1 \cap \cdots \cap P_r = \emptyset$.

Theorem 1.3 implies

1.24. LEMMA. *Let X be a hereditarily normal space, $f_1, f_2 \in PC(X, L)$, $\operatorname{dom} f_1 = \operatorname{dom} f_2$, and $f_1 \simeq f_2$. Then* $\operatorname{Part}(f_1, L) = \operatorname{Part}(f_2, L)$. ■

The following statement is well known.

1.25. LEMMA. *Let X be a space, $u = (U_1, \ldots, U_m)$ be an open covering of X, and $F \subset X$ be a closed subset. Assume $(\varphi_1, \ldots, \varphi_m)$ is a partition of unity on F subordinated to the covering $u|F$. Then the functions φ_j, $j = 1, \ldots, m$, can be extended over X to functions ψ_j so that $(\psi_1, \ldots, \psi_m)$ is a partition of unity on X subordinated to the covering u.* ■

In what follows we identify a complex K with its geometric realization $\tilde{K}$. So K is both a complex and a polyhedron.

1.26. DEFINITION. Let $u = (U_1, \ldots, U_m)$ be an open covering of a space X. A mapping $f : X \to N(u)$ is said to be *u-barycentric* if $f(x) = (\varphi_1(x), \ldots, \varphi_m(x))$, where $(\varphi_1, \ldots, \varphi_m)$ is some partition of unity subordinated to the covering u, and $\varphi_j(x)$ is the barycentric coordinate of $f(x)$ corresponding to the vertex $a_j \equiv U_j \in v(N(u))$.

If $e : N(u) \to K$ is a simplicial embedding, then the composition $e \circ f : X \to K$ is also called a u-barycentric mapping.

1.27. PROPOSITION. *If $u = (U_1, \ldots, U_m)$ is an open covering of a space X, then there exists a u-barycentric mapping $f : X \to N(u)$.* ■

1.28. LEMMA. *Let $\Phi = (F_1, \ldots, F_m) \in \operatorname{Exp}_K(X)$ and let $F = F_1 \cup \cdots \cup F_m$. Assume that u is a K-neighbourhood of Φ such that $U = \bigcup u$ is normal. Then the set $P = X \setminus U$ is a K-partition of any partial mapping $f : F \to K$ which is $(u|F)$-barycentric.*

Proof. Since f is $(u|F)$-barycentric, $f(x) = (\varphi_1(x), \dots, \varphi_m(x))$, where $(\varphi_1, \dots, \varphi_m)$ is a partition of unity on F subordinated to the covering $u|F = (U_1 \cap f, \dots, U_m \cap F)$. From Lemma 1.25 and normality of U it follows that the functions $\varphi_1, \dots, \varphi_m$ extend to functions $\psi_j : U \to I$, $j = 1, \dots, m$, so that $(\psi_1, \dots, \psi_m)$ is a partition of unity on U subordinated to the covering u of U. Then the mapping $g : U \to K$ defined as $g(x) = (\psi_1(x), \dots, \psi_m(x))$ is an extension of f. Consequently, $P = X \setminus U \in \mathrm{Part}(f, K)$. ∎

1.29. Definition. Let K be a complex with vertices $a_1, \dots, a_m$, $\Phi = (F_1, \dots, F_m) \in \mathrm{Fin}_s(\exp X)$, and $F = F_1 \cup \dots \cup F_m$. The sequence Φ is *f-generated* by K, where $f : F \to K$ is a mapping, if there exists a closed covering $(\Gamma_1, \dots, \Gamma_m)$ of K such that $\Gamma_j \subset Oa_j \equiv \mathrm{St}(a_j, K)$ and $F_j = f^{-1}(\Gamma_j)$.

1.30. Lemma. *Let $f \in PC(X, K)$ with $F = \mathrm{dom}\, f$. If $P \in \mathrm{Part}(f, K)$, then $P \in \mathrm{Part}(\Phi, K)$ for any sequence $\Phi = (F_1, \dots, F_m)$ which is f-generated by K.*

Proof. By Definition 1.29 there exists a closed covering $(\Gamma_1, \dots, \Gamma_m)$ of K such that $\Gamma_j \subset Oa_j$ and $F_j = f^{-1}(\Gamma_j)$. Since $P \in \mathrm{Part}(f, K)$, f extends to a mapping $g : X \setminus P \to K$. Put $U_j = g^{-1}(Oa_j)$, $j = 1, \dots, m$. Then

$$F_j = f^{-1}(\Gamma_j) \subset g^{-1}(\Gamma_j) \subset g^{-1}(Oa_j) = U_j.$$

Hence $u = (U_1, \dots, U_m)$ is a K-neighbourhood of Φ. Moreover, u is a covering of $X \setminus P$, because $(Oa_1, \dots, Oa_m)$ is a covering of K. Thus $P \in \mathrm{Part}(\Phi, K)$. ∎

1.31. Theorem. *Let X be a space and let $\mathcal{K}$ be a class of complexes. Then $\mathcal{K}\text{-}\dim X \le n$ if and only if every sequence $(f_1, \dots, f_{n+1})$ with $f_i \in PC(X, K_i)$ and $K_i \in \mathcal{K}$ is inessential.*

Proof. Necessity. Let $\mathcal{K}\text{-}\dim X \le n$ and let $f_i \in PC(X, K_i)$, $K_i \in \mathcal{K}$, $i = 1, \dots, n+1$. Let $v(K_i) = (a^i_1, \dots, a^i_{m_i})$ and $\mathrm{dom}\, f_i = F^i$. There exist closed sets $\Gamma^i_j \subset K_i$ such that

- $\Gamma^i_j \subset Oa^i_j \equiv \mathrm{St}(a^i_j, K_i)$;
- $\gamma_i = (\Gamma^i_1, \dots, \Gamma^i_{m_i})$ is a covering of K_i.

Put $F^i_j = f_i^{-1}(\Gamma^i_j)$, $\Phi_i = (F^i_1, \dots, F^i_{m_i})$, and $O^i_j = f_i^{-1}(Oa^i_j)$. Then $\Phi_i \in \mathrm{Exp}_{K_i}(X)$ and $F^i = F^i_1 \cup \dots \cup F^i_{m_i} = O^i_1 \cup \dots \cup O^i_{m_i}$. As $\mathcal{K}\text{-}\dim X \le n$, there exist K_i-neighbourhoods $u_i = (U^i_1, \dots, U^i_{m_i})$ of Φ_i such that $P_1 \cap \dots \cap P_{n+1} = \emptyset$, where $P_i = X \setminus \bigcup u_i$. By Lemma 1.7 and the Urysohn lemma we can enlarge partitions P_i to zero-sets P'_i with $P'_1 \cap \dots \cap P'_{n+1} = \emptyset$. So we may assume that $U^i = \bigcup u_i$ are F_σ-sets and hence normal subspaces of X. We can also assume that

$$U^i_j \cap F^i \subset O^i_j. \tag{1.1}$$

In fact, if (1.1) is not satisfied, we can define new sets ${}^1U^i_j = (U^i_j \setminus F^i) \cup (U^i_j \cap O^i_j)$. Then the sequences $u^1_i = ({}^1U^i_1, \dots, {}^1U^i_{m_i})$ are K_i-neighbourhoods of Φ_i with $\bigcup u^1_i = \bigcup u_i$.

Assuming (1.1) take some $(u_i|F^i)$-barycentric mappings $f^1_i : F^i \to K_i$. Since $O^i_j = f_i^{-1}(Oa^i_j)$, condition (1.1) implies that

$$f_i(x) \in Oa^i_j \;\Rightarrow\; f^1_i(x) \in Oa^i_j. \tag{1.2}$$

By a result of R. Cauty [1] condition (1.2) yields $f^1_i \simeq f_i$. Then Lemma 1.24 implies that $\mathrm{Part}(f^1_i, K_i) = \mathrm{Part}(f_i, K_i)$. On the other hand, $P_i \in \mathrm{Part}(f^1_i, K_i)$ in view of Lemma 1.28. Consequently, $P_i \in \mathrm{Part}(f_i, K_i)$ and the sequence $(f_1, \dots, f_{n+1})$ is inessential.

Sufficiency. Let $\Phi_i = (F^i_1, \dots, F^i_{m_i}) \in \mathrm{Exp}_{K_i}(X)$, $F^i = F^i_1 \cup \dots \cup F^i_{m_i}$, $v(K_i) = (a^i_1, \dots, a^i_{m_i})$, $i = 1, \dots, n+1$. According to Lemma 1.7 there exist sequences $\omega_i = (O^i_1, \dots, O^i_{m_i})$ of open subsets of F^i such that $F^i_j \subset O^i_j$ and $N(\omega_i) = N(\Phi_i)$.

By the usual procedure we construct partitions of unity $(\varphi^i_1, \dots, \varphi^i_{m_i})$ subordinated to the coverings ω_i so that

$$x \in F^i_j \;\Rightarrow\; \varphi^i_j(x) \ge 1/m_i. \tag{1.3}$$

The functions $(\varphi^i_1, \dots, \varphi^i_{m_i})$ generate ω_i-barycentric mappings

$$f_i : F^i \to K_i, \quad i = 1, \dots, n+1.$$

For $z \in K_i$, let $\mu^i_j(z)$, $j = 1, \dots, m_i$, be the barycentric coordinates of z in K_i. Put

$$\Gamma^i_j = \{z \in K_i : \mu^i_j(z) \ge 1/m_i\}, \quad j = 1, \dots, m_i;\ i = 1, \dots, n+1. \tag{1.4}$$

Clearly

$$\Gamma^i_j \subset Oa^i_j = \{z \in K_i : \mu^i_j(z) > 0\}, \tag{1.5}$$

$$\gamma_i = (\Gamma^i_1, \dots, \Gamma^i_{m_i}) \text{ is a covering of } K_i. \tag{1.6}$$

Since $\varphi^i_j(x) = \mu^i_j(f_i(x))$, (1.3) and (1.4) yield

$$F^i_j \subset f_i^{-1}(\Gamma^i_j). \tag{1.7}$$

Put ${}^1F^i_j = f_i^{-1}(\Gamma^i_j)$ and $\Phi^1_i = ({}^1F^i_1, \dots, {}^1F^i_{m_i})$. From (1.4), (1.6), and (1.7) it follows that the sequence Φ^1_i is f_i-generated by K_i. Consequently,

$$\mathrm{Part}(f_i, K_i) \subset \mathrm{Part}(\Phi^1_i, K_i) \tag{1.8}$$

according to Lemma 1.30.

Since $(f_1, \dots, f_{n+1})$ is inessential, there exist partitions $P_i \in \mathrm{Part}(f_i, K_i)$ such that $P_1 \cap \dots \cap P_{n+1} = \emptyset$. Then $(\Phi^1_1, \dots, \Phi^1_{n+1})$ is inessential by (1.8). Hence $(\Phi_1, \dots, \Phi_{n+1})$ is inessential, because $\mathrm{Part}(\Phi^1_i, K_i) \subset \mathrm{Part}(\Phi_i, K_i)$ in view of (1.7). Thus $\mathcal{K}\text{-}\dim X \le n$. ∎

1.32. Proposition. *If $\mathcal{L}$-dim $X \leq n$ and F is a closed subspace of X, then $\mathcal{L}$-dim $F \leq n$.* ■

Since ANR-compacta are ANE's for normal spaces, we have

1.33. Proposition. *If F is a closed subspace of a space X such that $\mathcal{L}$-dim $X \leq n$ and $\mathcal{L}$-dim $E \leq n$ for any closed subset $E \subset X$ with $E \cap F = \emptyset$, then $\mathcal{L}$-dim $X \leq n$.* ■

1.34. Proposition ([8]). *If a space X is the union of its closed subspaces $X_1, X_2, \dots$ with $\mathcal{L}$-dim $X_i \leq n$, $i \in \mathbb{N}$, then $\mathcal{L}$-dim $X \leq n$.* ■

1.35. Theorem ([8]).

(i) $\mathcal{L}$-dim $X \leq \dim X$ *for every* $\mathcal{L}$;
(ii) $\mathcal{L}$-dim $X = \dim X$ *if and only if* $\mathcal{L}$ *contains a disconnected space.* ■

1.36. Theorem ([8]). *If a hereditarily normal space X is the union of subspaces X_1 and X_2 such that $\mathcal{L}$-dim $X_1 \leq m$ and $\mathcal{L}$-dim $X_2 \leq n$, then $\mathcal{L}$-dim $X \leq m + n + 1$.* ■

1.37. Theorem ([8]). *If X is a metrizable space with L-dim $X \leq n$, then $X = X_1 \cup \dots \cup X_{n+1}$, where L-dim $X_i \leq 0$, $i = 1, \dots, n+1$.* ■

1.38. Theorem ([8]). *If X is the limit space of an inverse system $\{X_\alpha, \pi^\alpha_\beta, A\}$ of compact Hausdorff spaces X_α with $\mathcal{L}$-dim $X_\alpha \leq n$, then $\mathcal{L}$-dim $X \leq n$.* ■

1.39. Theorem ([9]). *If $L * L$ is not contractible, then for every $n \geq 0$ there is m such that L-dim $I^m = n$.* ■

1.40. Proposition ([11]). *Let X be a hereditarily normal space and let A be an arbitrary subspace of X. Then for every mapping $f : A \to L$ there exist an open subspace $U \subset X$ and a mapping $f_1 : U \to L$ such that $A \subset U$ and $f \simeq f_1|_A$.* ■

2. Inductive dimensions and some of their properties

2.1. Definition. To every space X one assigns the dimension $\mathcal{K}$-Ind X, which is an integer $n \geq -1$ or ∞, defined in the following way:

(1) $\mathcal{K}$-Ind $X = -1 \Leftrightarrow X = \emptyset$;
(2) $\mathcal{K}$-Ind $X \leq n \geq 0$ if for every $\Phi \in \mathrm{Exp}_K(X)$, $K \in \mathcal{K}$, there exists a K-partition P of Φ such that $\mathcal{K}$-Ind $P \leq n - 1$;
(3) $\mathcal{K}$-Ind $X = \infty$ if $\mathcal{K}$-Ind $X > n$ for $n = -1, 0, 1, \dots$.

If the class $\mathcal{K}$ contains only one complex K we write $\mathcal{K}$-Ind $X = K$-Ind X.

This dimension function is a generalization of the large inductive dimension in view of

2.2. Proposition. $\{0, 1\}$-Ind $X = \mathrm{Ind}\, X$. ■

2.3. Definition. To every space X one assigns the dimension $\mathcal{L}\text{-Ind}\,X$, which is an integer $n \geq -1$ or ∞, defined in the following way:

(1) $\mathcal{L}\text{-Ind}\,X = -1 \Leftrightarrow X = \emptyset$;
(2) $\mathcal{L}\text{-Ind}\,X \leq n \geq 0$ if for every $f \in PC(X,L)$, $L \in \mathcal{L}$, there exists a partition $P \in \text{Part}(f,L)$ such that $\mathcal{L}\text{-Ind}\,P \leq n-1$;
(3) $\mathcal{L}\text{-Ind}\,X = \infty$ if $\mathcal{L}\text{-Ind}\,X > n$ for $n = -1, 0, 1, \ldots$.

If the class $\mathcal{L}$ contains only one ANR-compactum L we write $\mathcal{L}\text{-Ind}\,X = L\text{-Ind}\,X$.

2.4. Theorem. *If X is a hereditarily normal space and τ is an arbitrary triangulation of a class $\mathcal{L}$ of polyhedra, then* $\mathcal{L}\text{-Ind}\,X = \mathcal{L}_\tau\text{-Ind}\,X$.

Proof. Denote the class $\mathcal{L}_\tau$ by $\mathcal{K} = \mathcal{K}(\mathcal{L})$ and its members L_t by $K = K(L)$. We have to prove the inequalities

$$\mathcal{K}\text{-Ind}\,X \leq \mathcal{L}\text{-Ind}\,X, \tag{2.1}$$

$$\mathcal{L}\text{-Ind}\,X \leq \mathcal{K}\text{-Ind}\,X. \tag{2.2}$$

To prove (2.1) we apply induction on $\mathcal{L}\text{-Ind}\,X$. Let $\mathcal{L}\text{-Ind}\,X = n$ and let $\varPhi = (F_1, \ldots, F_m) \in \text{Exp}_K(X)$, $K = K(L)$, $L \in \mathcal{L}$. Let $v(K) = (a_1, \ldots, a_m)$. As in the proof of Theorem 1.31 (Sufficiency) we construct a mapping $f : F = F_1 \cup \cdots F_m \to K \overset{\text{top}}{=} L$ and a sequence $\varPhi_1 = (F_1^1, \ldots, F_m^1)$ such that $F_j \subset F_j^1$ and $\varPhi_1$ is f-generated by K. Since $\mathcal{L}\text{-Ind}\,X = n$ there exists a partition $P \in \text{Part}(f,K)$ with $\mathcal{L}\text{-Ind}\,P \leq n-1$. By the inductive assumption we have $\mathcal{K}\text{-Ind}\,P \leq n-1$. But, by (Lemma 1.30), $P \in \text{Part}(\varPhi_1, K) \subset \text{Part}(\varPhi, K)$. Thus $\mathcal{K}\text{-Ind}\,X \leq n$.

We prove (2.2) by induction on $\mathcal{K}\text{-Ind}\,X$. Let $\mathcal{K}\text{-Ind}\,X = n$ and let $f \in PC(X, L(K)) = PC(X,K)$. Using the argument of the proof of Theorem 1.31 (Necessity) we construct a sequence $\varPhi = (F_1, \ldots, F_m)$ so that $\operatorname{dom} f \equiv F = F_1 \cup \cdots \cup F_m$ and $\varPhi$ is f-generated by K. Then we take a K-neighbourhood u of $\varPhi$ with $\mathcal{K}\text{-Ind}\,P \leq n-1$, where $P = X \setminus \bigcup u$, and construct a $(u|F)$-barycentric mapping $f_1 : F \to K$ such that $f_1 \simeq f$. By the inductive assumption we have $\mathcal{L}\text{-Ind}\,P \leq n-1$. On the other hand, by Lemmas 1.28 and 1.24, $P \in \text{Part}(f_1, L(K)) = \text{Part}(f, L(K))$. Thus $\mathcal{L}\text{-Ind}\,X \leq n$. ∎

2.5. Proposition. *If Y is closed in X, then* $\mathcal{L}\text{-Ind}\,Y \leq \mathcal{L}\text{-Ind}\,X$.

Proof. Induction on $\mathcal{L}\text{-Ind}\,X$. ∎

Applying induction and Proposition 2.5 we get

2.6. Proposition. *Let X be the discrete union of subspaces X_α, $\alpha \in A$. Then $\mathcal{L}\text{-Ind}\,X \leq n$ if and only if $\mathcal{L}\text{-Ind}\,X_\alpha \leq n$ for every $\alpha \in A$.* ∎

2.7. Proposition. *Let X be a hereditarily normal space and let Y be a subspace of X such that $\mathcal{L}\text{-Ind}\,Y \leq n \geq 0$. Then for every $f \in PC(X,L)$, $L \in \mathcal{L}$, there exists an L-partition P of f such that $\mathcal{L}\text{-Ind}(P \cap Y) \leq n-1$.*

Proof. Let $\operatorname{dom} f = F$. Since $\mathcal{L}\text{-}\operatorname{Ind} Y \le n$, there exist an open subset V of Y and a mapping $f_1 : V \cup F \to L$ such that $f_1|_F = f$ and $\mathcal{L}\text{-}\operatorname{Ind} Q \le n-1$, where $Q = Y \setminus V$. By Proposition 1.40 there exist an open subset U of X and a mapping $f_2 : U \to L$ such that $V \cup F \subset U$ and $f_1 \simeq f_2|_{V\cup F}$. Put $P = X \setminus U$. Then $P \in \operatorname{Part}(f_2|_F, L) = \operatorname{Part}(f, L)$ by Lemma 1.24. On the other hand, $P \cap Y \subset Q$. Hence, by Proposition 2.5, $\mathcal{L}\text{-}\operatorname{Ind}(P \cap Y) \le \mathcal{L}\text{-}\operatorname{Ind} Q \le n-1$. ■

2.8. Theorem. *If a hereditarily normal space X is represented as the union of two subspaces X_1 and X_2, then*

$$\mathcal{L}\text{-}\operatorname{Ind} X \le \mathcal{L}\text{-}\operatorname{Ind} X_1 + \mathcal{L}\text{-}\operatorname{Ind} X_2 + 1.$$

Proof. The assertion is obvious if one of the subspaces is empty. So we assume that $X_1 \neq \emptyset \neq X_2$ and apply induction on $p = m + n \ge 0$, where $\operatorname{Ind} X_1 = m$ and $\operatorname{Ind} X_2 = n$. We consider only the inductive step $p - 1 \to p$, since the case $p = 0$ is considered by the same argument. Let $f \in PC(X, L), L \in \mathcal{L}$. By Proposition 2.7 there exists an L-partition P of f such that $\mathcal{L}\text{-}\operatorname{Ind}(P \cap X_1) \le m-1$. The set $P \cap X_2$ is closed in X_2. Applying Proposition 2.5 we get $\mathcal{L}\text{-}\operatorname{Ind}(P \cap X_2) \le \mathcal{L}\text{-}\operatorname{Ind} X_2 = n$. Hence

$$\mathcal{L}\text{-}\operatorname{Ind}(P \cap X_1) + \mathcal{L}\text{-}\operatorname{Ind}(P \cap X_2) \le m - 1 + n = p - 1.$$

By the inductive assumption, $\mathcal{L}\text{-}\operatorname{Ind} P \le m+n$. Thus $\mathcal{L}\text{-}\operatorname{Ind} X \le m+n+1$. ■

2.9. Corollary. *If a hereditarily normal space X can be represented as the union of $n+1$ subspaces $X_1, \dots, X_{n+1}$ such that $\mathcal{L}\text{-}\operatorname{Ind} X_i \le 0$, $i = 1, \dots, n+1$, then $\mathcal{L}\text{-}\operatorname{Ind} X \le n$.* ■

Applying a standard argument (see, for example, [3, proof of Theorem 2.2.10]) one can prove the following statements.

2.10. Theorem. *For every space X we have $\mathcal{K}\text{-}\operatorname{Ind} \beta X = \mathcal{K}\text{-}\operatorname{Ind} X$.*

2.11. Theorem. *For every space X we have $\mathcal{L}\text{-}\operatorname{Ind} \beta X = \mathcal{L}\text{-}\operatorname{Ind} X$.*

To prove these theorems we use Lemma 1.7 and Theorem 1.6 respectively.

3. Comparison of dimensions. Since Lemma 1.30 holds for every normal space X, an analysis of the proof of Theorem 2.4 shows that

$$\mathcal{L}_\tau\text{-}\operatorname{Ind} X \le \mathcal{L}\text{-}\operatorname{Ind} X \tag{3.1}$$

for every (normal) space X and every class $\mathcal{L}$ of polyhedra.

3.1. Question. Does the equality

$$\mathcal{L}_\tau\text{-}\operatorname{Ind} X = \mathcal{L}\text{-}\operatorname{Ind} X \tag{3.2}$$

hold for an arbitrary space X?

A partial answer to Question 3.1 is given by

3.2. Proposition. *If $\mathcal{L}_\tau\text{-}\operatorname{Ind} X = 0$, then $\mathcal{L}\text{-}\operatorname{Ind} X = 0$.*

To prove Proposition 3.2 we use the argument of the second part of the proof of Theorem 2.4. We have a partition P there of dimension $\le n-1 = -1$. Hence P is empty and u is a cover of X. Consequently, we can construct a $(u|F)$-barycentric mapping f_1 for a normal space X.

3.3. Proposition. *If* $\mathcal{K}_1 \subset \mathcal{K}_2$, *then* $\mathcal{K}_1\text{-Ind}\,X \le \mathcal{K}_2\text{-Ind}\,X$. ∎

3.4. Proposition. *If* $\mathcal{L}_1 \subset \mathcal{L}_2$, *then* $\mathcal{L}_1\text{-Ind}\,X \le \mathcal{L}_2\text{-Ind}\,X$. ∎

Propositions 3.3 and 3.4 yield

$$\sup\{K\text{-Ind}\,X : K \in \mathcal{K}\} \le \mathcal{K}\text{-Ind}\,X, \tag{3.3}$$

$$\sup\{L\text{-Ind}\,X : L \in \mathcal{L}\} \le \mathcal{L}\text{-Ind}\,X. \tag{3.4}$$

3.5. Question. Is it true that

$$\mathcal{K}\text{-Ind}\,X = \sup\{K\text{-Ind}\,X : K \in \mathcal{K}\}, \qquad \mathcal{L}\text{-Ind}\,X = \sup\{L\text{-Ind}\,X : L \in \mathcal{L}\}?$$

3.6. Proposition. *If* $\mathcal{L}_1 \le_h \mathcal{L}_2$, *then*

$$\mathcal{L}_1\text{-Ind}\,X \le \mathcal{L}_2\text{-Ind}\,X \tag{3.5}$$

for every hereditarily normal space X.

Proof. We apply induction on $\mathcal{L}_2\text{-Ind}\,X = n \ge -1$. For $n = -1$ the assertion is obvious. Let $\mathcal{L}_2\text{-Ind}\,X = n \ge 0$ and let $f \in PC(X, L_1)$ for some $L_1 \in \mathcal{L}_1$. We have to find a partition $P \in \text{Part}(f, L_1)$ with $\mathcal{L}_1\text{-Ind}\,P \le n-1$.

Since $\mathcal{L}_1 \le_h \mathcal{L}_2$ there exists $L_2 \in \mathcal{L}_2$ such that $L_1 \le_h L_2$, i.e. there exist mappings $\alpha : L_1 \to L_2$ and $\beta : L_2 \to L_1$ with $\beta \circ \alpha \simeq \text{id}_{L_1}$. Let

$$g = \alpha \circ f : \text{dom}\, f \to L_2.$$

Then $g \in PC(X, L_2)$. Since $\mathcal{L}_2\text{-Ind}\,X = n$, there exists a partition $P \in \text{Part}(g, L_2)$ with $\mathcal{L}_2\text{-Ind}\,P \le n-1$. Then $P \in \text{Part}(\beta \circ g, L_1)$. But $\beta \circ g = (\beta \circ \alpha) \circ f \simeq f$, because $\beta \circ \alpha \simeq \text{id}_{L_1}$. Consequently, $P \in \text{Part}(f, L_1)$ in view of Lemma 1.24. On the other hand, by the inductive assumption we have $\mathcal{L}_1\text{-Ind}\,P \le \mathcal{L}_2\text{-Ind}\,P \le n-1$. ∎

3.7. Corollary. *If* $\mathcal{L}_1 \simeq \mathcal{L}_2$, *then*

$$\mathcal{L}_1\text{-Ind}\,X = \mathcal{L}_2\text{-Ind}\,X \tag{3.6}$$

for every hereditarily normal space X. ∎

3.8. Question. Does equality (3.6) hold for an arbitrary space whenever $\mathcal{L}_1 \simeq \mathcal{L}_2$?

Theorem 1.4 and Corollary 3.7 yield

3.9. Proposition. *For every non-empty class* $\mathcal{R}$ *of* ANR*-compacta there exists a class* $\mathcal{L} = \mathcal{L}(\mathcal{R})$ *of compact polyhedra such that* $\mathcal{R}\text{-Ind}\,X = \mathcal{L}\text{-Ind}\,X$ *for every hereditarily normal space* X. ∎

So, when we investigate the $\mathcal{L}$-Ind dimension of hereditarily normal spaces, we can consider only classes $\mathcal{L}$ consisting of compact polyhedra.

3.10. Lemma. *Let $\Phi=(F_1,\dots,F_m)\in\operatorname{Exp}_K(X)$ and let $u=(U_1,\dots,U_m)$ be a K-neighbourhood of Φ. Then every partition P in X between $F=\bigcup\Phi$ and $X\setminus\bigcup u$ is a K-partition of Φ.*

Proof. There exist open sets U and V such that

$$U\sqcup P\sqcup V=X \tag{3.7}$$

and

$$F\subset U\subset U\cup P\subset\bigcup u.$$

We define a new K-neighbourhood $u_1=(U_1^1,\dots,U_m^1)$ of Φ as follows:

$$U_1^1=(U_1\cap U)\cup V,\qquad U_j^1=U_j\cap U,\qquad j=2,\dots,m.$$

Then $P=X\setminus\bigcup u_1$. ∎

3.11. Lemma. *Let $f\in PC(X,L)$ and let W be a neighbourhood of $F=$ $\operatorname{dom}f$ such that $X\subset W\in\operatorname{Part}(f,L)$. Then every partition P in X between F and $X\setminus W$ is an L-partition of f.*

Proof. There exist open sets U and V satisfying (3.7) and $F\subset U\subset U\cup P\subset W$. Since $X\setminus W\in\operatorname{Part}(f,L)$, there exists a mapping $f_1:W\to L$ such that $f_1|_F=f$. We define an extension f_2 of f putting $f_2|_U=f_1$ and $f_2(W)=\mathrm{pt}\in L$. Then $\operatorname{dom}f_2=X\setminus P$, so $P\in\operatorname{Part}(f,L)$. ∎

3.12. Theorem. *For every $\mathcal K$, $\mathcal L$, and X we have*

$$\mathcal K\text{-}\operatorname{Ind}X\le\operatorname{Ind}X, \tag{3.8}$$

$$\mathcal L\text{-}\operatorname{Ind}X\le\operatorname{Ind}X. \tag{3.9}$$

Proof. We prove (3.8) by induction on $n=\operatorname{Ind}X$. For $n=-1$ the assertion is obvious. Let $X=n\ge 0$ and let $\Phi\in\operatorname{Exp}_K X$, $K\in\mathcal K$. By Lemma 3.10 there exists a K-partition P of Φ with $\operatorname{Ind}P\le n-1$. By the inductive assumption we have $\mathcal K$-$\operatorname{Ind}P\le\operatorname{Ind}P$. Consequently, $\mathcal K$-$\operatorname{Ind}X\le n$. To prove (3.9) we apply Lemma 3.11 instead of Lemma 3.10. ∎

In connection with Theorem 3.12 two problems arise.

Problem 1. For what classes $\mathcal K$ of complexes,

$$\mathcal K\text{-}\operatorname{Ind}X=\operatorname{Ind}X\qquad\text{for every } X?$$

Problem 2. For what classes $\mathcal L$ of ANR-compacta,

$$\mathcal L\text{-}\operatorname{Ind}X=\operatorname{Ind}X\qquad\text{for every } X?$$

To solve Problem 1 we need the following statement.

3.13. Lemma. *If $\mathcal L$ consists of connected compacta, then* $\mathcal L$-$\operatorname{Ind}I=0$.

Proof. If L is a connected ANR-compactum, then it is path-connected, and consequently $L\in AE(I)$. Hence $\mathcal L$-$\operatorname{Ind}I=0$. ∎

The next theorem solves Problem 1.

3.14. Theorem. *The equality $\mathcal{K}$-$\operatorname{Ind} X = \operatorname{Ind} X$ holds for every space X if and only if $\mathcal{K}$ contains a disconnected complex.*

Proof. Necessity is a consequence of Lemma 3.13 and Theorem 2.4.

Sufficiency. In view of Theorem 3.12 it suffices to show that

$$\operatorname{Ind} X \le \mathcal{K}\text{-}\operatorname{Ind} X. \tag{3.10}$$

We shall prove (3.10) by induction on $n = \mathcal{K}$-$\operatorname{Ind} X$. The assertion is obvious for $n = -1$. Assume that $\mathcal{K}$-$\operatorname{Ind} X = n \ge 0$. Let F_1 and F_2 be disjoint closed subsets of X. We have to find a partition P between F_1 and F_2 with $\operatorname{Ind} P \le n-1$.

Take a disconnected complex $K = K_1 \sqcup K_2 \in \mathcal{K}$. We can enumerate its vertices as $v(K) = (a_1, \ldots, a_m)$ so that $a_1 \in K_1$ and $a_2 \in K_2$. Let $\Phi = (F_1, F_2, F_3, \ldots, F_m)$, where $F_3 = \cdots = F_m = \emptyset$. Then $\Phi \in \operatorname{Exp}_K(X)$. Since $\mathcal{K}$-$\operatorname{Ind} X = n$, there exists a K-neighbourhood $u = (U_1, \ldots, U_m)$ of Φ such that $\mathcal{K}$-$\operatorname{Ind} P \le n-1$, where $P = X \setminus (U_1 \cup \cdots \cup U_m)$. Let

$$A_i = \{j \in \{1, \ldots, m\} : a_j \in K_i\}, \qquad V_i = \bigcup \{U_j : j \in A_i\}, \qquad i = 1, 2.$$

Since the embedding $N(u) \to K$ is generated by the correspondence $U_j \mapsto a_j$, we have

$$V_1 \cap V_2 = \emptyset, \qquad F_1 \subset V_1, \qquad F_2 \subset V_2.$$

Hence $P = X \setminus (V_1 \cup V_2)$ is a partition between F_1 and F_2. By the inductive assumption we have $\operatorname{Ind} P \le \mathcal{K}$-$\operatorname{Ind} P \le n - 1$. ∎

The next theorem gives a partial solution of Problem 2. It is a corollary of Theorems 2.4 and 3.14.

3.15. Theorem. *The equality $\mathcal{L}$-$\operatorname{Ind} X = \operatorname{Ind} X$ holds for every hereditarily normal space X if and only if $\mathcal{L}$ contains a disconnected compactum.* ∎

3.16. Question. Is it true that $\mathcal{L}$-$\operatorname{Ind} X = \operatorname{Ind} X$ for every space X whenever $\mathcal{L}$ contains a disconnected compactum?

Question 3.16 has a positive answer if the next question has a positive answer.

3.17. Question. Is it true that $\mathcal{L}_1$-$\operatorname{Ind} X \le \mathcal{L}_2$-$\operatorname{Ind} X$ for every space X whenever $\mathcal{L}_1 \le_h \mathcal{L}_2$?

In connection with Theorem 3.12 another two problems arise.

Problem 3. For what classes $\mathcal{K}$ of complexes, $\mathcal{K}$-$\operatorname{Ind} X < \infty \Rightarrow \operatorname{Ind} X < \infty$?

Problem 4. For what classes $\mathcal{L}$ of ANR-compacta, $\mathcal{L}$-$\operatorname{Ind} X < \infty \Rightarrow \operatorname{Ind} X < \infty$?

3.18. Theorem. *The inequality $\mathcal{K}$-$\dim X \le \mathcal{K}$-$\operatorname{Ind} X$ holds for every space X and every class $\mathcal{K}$.*

To prove Theorem 3.18 we need some additional information.

3.19. LEMMA. *Let $X = Y \sqcup Z$, $\alpha = (A_1, \dots, A_m)$ be a sequence of subsets of Y, and $\beta = (B_1, \dots, B_m)$ be a sequence of subsets of Z such that $N(\alpha), N(\beta) \subset K$. Let $\gamma = (C_1, \dots, C_m)$, where $C_j = A_j \cup B_j$. Then $N(\gamma) \subset K$.*

Proof. For $a_{j_1}, \dots, a_{j_r} \in v(K)$ we denote by $K(a_{j_1}, \dots, a_{j_r}) \equiv K_1$ the biggest subcomplex of K with $v(K_1) = (a_{j_1}, \dots, a_{j_r})$. We have to prove that

$$C_{j_1} \cap \dots \cap C_{j_r} \neq \emptyset \ \Rightarrow \ K(a_{j_1}, \dots, a_{j_r}) \text{ is a simplex.}$$

Let $x \in C_{j_1} \cap \dots \cap C_{j_r}$. If $x \in Y$, then $x \in A_{j_1} \cap \dots \cap A_{j_r}$, and consequently $K(a_{j_1}, \dots, a_{j_r})$ is a simplex, because $N(\alpha) \subset K$. If $x \in Z$, then $x \in B_{j_1} \cap \dots \cap B_{j_r}$, and so $K(a_{j_1}, \dots, a_{j_r})$ is a simplex, since $N(\beta) \subset K$. ■

Lemma 3.19 yields

3.20. LEMMA. *Let Y be a subspace of a space X, $\alpha = (A_1, \dots, A_m)$ be a sequence of subsets of X, and $\beta = (B_1, \dots, B_m)$ be a sequence of subsets of Y such that $N(\alpha), N(\beta) \subset K$ and $A_j \cap Y \subset B_j$, $j = 1, \dots, m$. Let $C_j = A_j \cup B_j$ and $\gamma = (C_1, \dots, C_m)$. Then $N(\gamma) \subset K$.* ■

Proof of Theorem 3.18. We apply induction on $\mathcal{K}\text{-Ind}\, X = n \geq -1$. If $n = -1$ the assertion is obvious. Assume that we have proved it for all X with $\mathcal{K}\text{-Ind}\, X = k \leq n - 1 \geq -1$ and let $\mathcal{K}\text{-Ind}\, X = n \geq 0$.

We have to prove that every sequence $(K_1, \dots, K_{n+1})$, $K_i \in \mathcal{K}$, is inessential in X. Take an arbitrary sequence $(\Phi_1, \dots, \Phi_{n+1})$, $\Phi_i \in \mathrm{Exp}_{K_i}(X)$. We are looking for K_i-partitions P_i of Φ_i such that $P_1 \cap \dots \cap P_{n+1} = \emptyset$. Since $\mathcal{K}\text{-Ind}\, X = n$, there exists a K_{n+1}-partition P_{n+1} of Φ_{n+1} such that $\mathcal{K}\text{-Ind}\, P_{n+1} \leq n - 1$. Let $\Phi_i = (F_1^i, \dots, F_{m_i}^i)$ and $F_i = F_1^i \cup \dots \cup F_{m_i}^i$. Since $\mathcal{K}\text{-Ind}\, P_{n+1} \leq n - 1$, by the inductive assumption we have $\mathcal{K}\text{-dim} P_{n+1} \leq n - 1$. Hence the sequence $(\Phi_1 | P_{n+1}, \dots, \Phi_n | P_{n+1})$ is inessential in P_{n+1}, and consequently there exist partitions $Q_i \in \mathrm{Part}(\Phi_i | P_{n+1}, K_i)$ with $Q_1 \cap \dots \cap Q_n = \emptyset$. By Lemma 1.7 there exist sets V_i open in P_{n+1} such that

$$Q_i \subset V_i \subset P_{n+1} \setminus F_i, \quad i = 1, \dots, n, \tag{3.11}$$

$$V_1 \cap \dots \cap V_n = \emptyset. \tag{3.12}$$

In view of the definition of the K_i-partitions Q_i there exist sequences $u_i = (U_1^1, \dots, U_{m_i}^i)$ of open subsets of P_{n+1} such that

$$F_j^i \cap P_{n+1} \subset U_j^i, \quad j = 1, \dots, m_i, \tag{3.13}$$

$$U_1^i \cup \dots \cup U_{m_i}^i = P_{n+1} \setminus Q_i, \tag{3.14}$$

$$N(u_i) \subset K_i, \quad i = 1, \dots, n. \tag{3.15}$$

Put $H_i = P_{n+1} \setminus V_i$. The sequences $u_i | H_i$ are open coverings of H_i in view of (3.11) and (3.14). Shrinking them to closed coverings we get sequences

$\Phi_i^0 = ({}^0F_1^i, \ldots, {}^0F_{m_i}^i)$ of closed sets such that

$$F_j^i \cap P_{n+1} \subset {}^0F_j^i \subset U_j^i, \quad j = 1, \ldots, m_i, \tag{3.16}$$

$${}^0F_1^i \cup \cdots \cup {}^0F_{m_i}^i = H_i, \quad i = 1, \ldots, n. \tag{3.17}$$

From (3.15) and (3.16) it follows that

$$N(\Phi_i^0) \subset K_i, \quad i = 1, \ldots, n. \tag{3.18}$$

Put $\Phi_i^1 = ({}^0F_1^i \cup F_1^i, \ldots, {}^0F_{m_i}^i \cup F_{m_i}^i)$. According to (3.18) and Lemma 3.20 we have $N(\Phi_i^1) \subset K_i$, $i = 1, \ldots, n$. Take arbitrary K_i-neighbourhoods $w_i = (W_1^i, \ldots, W_{m_i}^i)$ of Φ_i^1 in X and put $P_i = X \setminus \bigcup w_i$. Then $P_1 \cap \cdots \cap P_n \subset X \setminus P_{n+1}$ because of (3.12) and (3.17). ∎

From the definition we get

3.21. Proposition. $\mathcal{K}$-Ind $X = 0 \Leftrightarrow \mathcal{K}$-dim $X = 0$. ∎

Corollary 2.9 and Proposition 3.21 imply

3.22. Proposition. *If a hereditarily normal space X can be represented as the union of $n+1$ subspaces $X_1, \ldots, X_{n+1}$ such that $\mathcal{K}$-dim $X_i \le 0$, $i = 1, \ldots, n+1$, then $\mathcal{K}$-Ind $X \le n$.* ∎

Theorems 1.17, 1.37, 3.18, and Proposition 3.22 yield

3.23. Theorem. *If X is metrizable space, then $\mathcal{K}$-Ind $X = \mathcal{K}$-dim X.* ∎

Theorem 3.23 is a generalization of a theorem by M. Katětov [10] and K. Morita [12] for the classical dimensions dim and Ind.

We conclude this section with another application of Lemmas 3.19 and 3.20, which we will need in Section 5.

3.24. Theorem. *Let $f : X \to Y$ be a mapping of a compact Hausdorff space X onto a space Y with* dim$Y = 0$. *Then*

$$\mathcal{K}\text{-dim}\, X \le \sup\{\mathcal{K}\text{-dim}\, f^{-1}(y) : y \in Y\}.$$

Proof. It suffices to consider the case

$$\sup\{\mathcal{K}\text{-dim}\, f^{-1}(y) : y \in Y\} = n < \infty. \tag{3.19}$$

Let $\Phi_i = (F_1^i, \ldots, F_{m_i}^i) \in \mathrm{Exp}_{K_i}(X)$, $K_i \in \mathcal{K}$, $i = 1, \ldots, n+1$. For $y \in Y$, put

$$\Phi_i^y = (F_1^i \cap f^{-1}(y), \ldots, F_{m_i}^i \cap f^{-1}(y)). \tag{3.20}$$

Since $\mathcal{K}$-dim $f^{-1}(y) \le n$, there exist partitions $P_i^y \in \mathrm{Part}(\Phi_i^y, K_i)$ such that

$$P_1^y \cap \cdots \cap P_{n+1}^y = \emptyset, \quad y \in Y. \tag{3.21}$$

This means that there exist families $v_i^y = (V_{i,1}^y, \dots, V_{i,m_i}^y)$, $i = 1, \dots, n+1$, of open subsets of $f^{-1}(y)$ such that

$$F_j^i \cap f^{-1}(y) \subset V_{i,j}^y, \quad j = 1, \dots, m_i, \tag{3.22}$$

$$N(v_i^y) \subset K_i, \quad y \in Y, \tag{3.23}$$

$$v^y = v_1^y \cup \dots \cup v_{n+1}^y \in \operatorname{cov}(f^{-1}(y)). \tag{3.24}$$

We can shrink the covering v^y to a closed covering

$$\Phi^y = \{F_{i,j}^y : i = 1, \dots, n+1;\ j = 1, \dots, m_i\}$$

so that

$$F_j^i \cap f^{-1}(y) \subset F_{i,j}^y \subset V_{i,j}^y. \tag{3.25}$$

Put ${}^i\Phi^y = (F_{i,1}^y, \dots, F_{i,m_i}^y)$. From (3.23) and (3.25) it follows that

$$N({}^i\Phi^y) \subset K_i, \quad i = 1, \dots, n+1. \tag{3.26}$$

Put ${}^1F_{i,j}^y = F_{i,j}^y \cup F_j^i$ and ${}^i\Phi_1^y = ({}^1F_{i,1}^y, \dots, {}^1F_{i,m_i}^y)$. From (3.25), (3.26), and Lemms 3.20 it follows that

$$N({}^i\Phi_1^y) \subset K, \quad i = 1, \dots, n+1. \tag{3.27}$$

By Lemma 1.7 and (3.27) there exist families $w_i^y = (W_{i,1}^y, \dots, W_{i,m_i}^y)$ of open subsets of X such that

$${}^1F_{i,j}^y \subset W_{i,j}^y, \quad j = 1, \dots, m_i, \tag{3.28}$$

$$N(w_i^y) \subset K_i, \quad i = 1 \dots, n+1. \tag{3.29}$$

Put $W_y = \bigcup\{W_{i,j}^y : i = 1, \dots, n+1;\ j = 1, \dots, m_i\}$. Since $\bigcup \Phi^y = f^{-1}(y)$, from (3.28) we get $f^{-1}(y) \subset W_y$. Hence there exists a neighbourhood Oy of y such that

$$f^{-1}(y) \subset f^{-1}Oy \subset W_y. \tag{3.30}$$

The covering $\{Oy : y \in Y\}$ of Y admits a refinement $\gamma = \{G_1, \dots, G_r\}$ consisting of pairwise disjoint clopen sets. For every $s = 1, \dots, r$ fix a point $y(s)$ so that $G_s \subset Oy(s)$. Put

$$U_{i,j}^s = W_{i.j}^{y(s)} \cap f^{-1}G_s, \quad s = 1, \dots, r, \tag{3.31}$$

$$u_i^s = (U_{i,1}^s, \dots, U_{i,m_i}^s), \quad i = 1, \dots, n+1. \tag{3.32}$$

From (3.29) it follows that

$$N(u_i^s) \subset K_i. \tag{3.33}$$

Let $U_{i,j} = U_{i,j}^1 \cup \dots \cup U_{i,j}^r$ and $u_i = (U_{i,1}, \dots, U_{i,m_i})$. From Lemma 3.19 and (3.33) we get

$$N(u_i) \subset K_i. \tag{3.34}$$

From (3.28), (3.30), and (3.31) it follows that

(3.35) $$F_j^i \subset U_{i,j},$$

(3.36) $$u_1 \cup \cdots \cup u_{n+1} \in \mathrm{cov}(X).$$

Put $P_i = X \setminus \bigcup u_i$. Then conditions (3.34)–(3.36) imply that $P_i \in \mathrm{Part}(\Phi_i, K_i)$ and $P_1 \cap \cdots \cap P_{n+1} = \emptyset$. ∎

4. Fully closed mappings. Let $f : X \to Y$ be a mapping and $A \subset X$. Recall that the set

$$f^{\#}A = \{y \in Y : f^{-1}(y) \subset A\} = Y \setminus f(X \setminus A)$$

is said to be the *small image* of A. If α is a family of subsets of X then we put $f^{\#}\alpha = \{f^{\#}A : A \in \alpha\}$.

4.1. Definition ([4]). A continuous surjective mapping $f : X \to Y$ is called *fully closed* if for every point $y \in Y$ and for every finite family u of open sets in X with $f^{-1}(y) \subset \bigcup u$, the set $\{y\} \cup \bigcup f^{\#}u$ is a neighbourhood of y.

Obviously, every fully closed mapping is closed.

4.2. Proposition. *If $f : X \to Y$ is a fully closed mapping and u is a finite open cover of X, then the set $Y \setminus \bigcup f^{\#}u$ is discrete.* ∎

4.3. Proposition. *If $f : X \to Y$ is a fully closed mapping and $Z \subset Y$, then the mapping $f|_{f^{-1}(Z)} : f^{-1}(Z) \to Z$ is fully closed.* ∎

4.4. Proposition. *If $f : X \to Y$ and $g : Y \to Z$ are mappings whose composition $g \circ f$ is fully closed, then g is also fully closed.* ∎

4.5. For a mapping $f : X \to Y$ and an arbitrary set $M \subset Y$, we put

$$M^f = \{f^{-1}y : y \in Y \setminus M\} \cup \{\{x\} : x \in f^{-1}M\}.$$

The family M^f is an upper semicontinuous decomposition of the space X. We denote the quotient space with respect to this decomposition by Y_f^M and the corresponding quotient mapping $X \to Y_f^M$ by f_M. Since the decomposition M^f refines the decomposition corresponding to the mapping f, there exists a unique mapping $\pi_f^M : Y_f^M \to Y$ such that $f = \pi_f^M \circ f_M$. The mapping π_f^M is continuous, because f is continuous and f^M is quotient. If $M = \emptyset$, then $Y_f^\emptyset = Y$, $f_\emptyset = f$, $\pi_f^\emptyset = \mathrm{id}_Y$.

4.6. Proposition ([7]). *For a closed surjective mapping $f : X \to Y$ of a regular space X to a regular space Y, the following conditions are equivalent:*

(1) *f is fully closed;*

(2) *for any set $M \subset Y$, the space Y_f^M is regular.* ∎

4.7. Proposition ([7]). *If $f : X \to Y$ is a fully closed mapping and $M \subset Y$, then both mappings f_M and π_f^M are fully closed.* ■

4.8. Proposition. *If $f : X \to Y$ is a closed surjective mapping of a normal space X onto a T_1-space Y, then Y is a normal space.* ■

Propositions 4.6–4.8 yield

4.9. Proposition. *If $f : X \to Y$ is a fully closed mapping between normal spaces, then Y_f^M is a normal space for any $M \subset Y$.* ■

4.10. Definition. A family $\mathcal{M}$ of subsets of Y is said to be a *direction* in Y if it satisfies the following conditions:

0) $\emptyset \in \mathcal{M}$;

1) $\mathcal{M}$ is a covering of Y;

2) if $M_1, M_2 \in \mathcal{M}$, then there exists $M \in \mathcal{M}$ such that $M_1 \cup M_2 \subset M$.

4.11. *The inverse system $S_{\mathcal{M}}^f$.* Let $f : X \to Y$ be a fully closed mapping and let $\mathcal{M}$ be a direction in Y. If $M_1, M_2 \in \mathcal{M}$ and $M_1 \subset M_2$, then the decomposition M_2^f refines the decomposition M_1^f. Hence there exists a unique mapping $\pi_{M_1}^{M_2} : Y_f^{M_2} \to Y_f^{M_1}$ such that $\pi_f^{M_2} = \pi_f^{M_1} \circ \pi_{M_1}^{M_2}$. It is easy to check that if $M_1 \subset M_2 \subset M_3$, $M_i \in \mathcal{M}$, then

$$\pi_{M_1}^{M_3} = \pi_{M_1}^{M_2} \circ \pi_{M_2}^{M_3}.$$

So the family $S_{\mathcal{M}}^f = \{Y_f^M, \pi_{M'}^M, \mathcal{M}\}$ is an inverse system. We denote by π_M the limit projection $\lim S_{\mathcal{M}}^f \to Y_f^M$.

4.12. Theorem. *Let $f : Y \to Y$ be a fully closed mapping between compact Hausdorff spaces and let $\mathcal{M}$ be a direction in Y. Then f_M is homeomorphic to the limit projection π_M of the inverse system $S_{\mathcal{M}}^f$, $M \in \mathcal{M}$.*

The proof is a routine.

For a mapping $f : X \to Y$ the number $\mathcal{L}\text{-}\dim f$ is defined as follows:

$$\mathcal{L}\text{-}\dim f = \sup\{\mathcal{L}\text{-}\dim f^{-1}(y) : y \in Y\}.$$

4.13. Theorem ([9]). *If $f : X \to Y$ is a fully closed mapping between compact spaces, then*

$$\mathcal{L}\text{-}\dim X \leq \max\{\mathcal{L}\text{-}\dim Y, \mathcal{L}\text{-}\dim f\}. \ ■$$

In applications, fully closed mappings appear as resolutions.

4.14. Definition ([7]). Given a space X, spaces Y_x, and continuons mappings $h_x : X \setminus \{x\} \to Y_x$ for each point $x \in X$, a *resolution* of (the set) X (at each point x to the space Y_x by means of the mappings h_x) is the set

$$R(X) \equiv R(X, Y_x, h_x) = \bigcup\{\{x\} \times Y_x : x \in X\}.$$

The mapping $\pi = \pi_X : R(X) \to X$ taking (x, y) to x is called the *resolution mapping* or simply the *resolution*.

We define a topology on $R(X)$. Given a triple (U, x, V), where U is an open subset of X, $x \in U$, and V is an open subset of Y_x, put

$$U \otimes_x V = \{x\} \times V \cup \pi^{-1}(U \cap h_x^{-1}(V)).$$

The family of sets of the form $U \otimes_x V$ is the base for a topology on $R(X)$ called the *resolution topology*.

4.15. THEOREM ([5]). *If X and all Y_x are compact Hausdorff spaces, then $R(X)$ is also a compact Hausdorff space, π is fully closed, and each fibre $\pi^{-1}(x)$ is homeomorphic to Y_x. Moreover, $R(X)$ is first countable if and only if X and all Y_x are first countable.* ∎

4.16. DEFINITION. A closed mapping $f : X \to Y$ is called *atomic* if $F = f^{-1}f(F)$ for every closed $F \subset X$ such that $f(F)$ is a continuum (connected closed non-singleton).

4.17. DEFINITION. A closed mapping $f : X \to Y$ is said to be *ring-like* if, for any point $x \in X$ and any neighbourhoods Ox and $Of(x)$, the set $Of(x) \cap f^{\#}Ox$ contains a partition between $f(x)$ and $Y \setminus Of(x)$.

4.18. PROPOSITION. *Every ring-like mapping is atomic.* ∎

A number of applications of resolutions are based on the following statement.

4.19. LEMMA ([6]). *Let X be a first countable connected compact Hausdorff space and let Y_x, $x \in X$, be AR-compacta. Then we can choose mappings $h_x : X \setminus \{x\} \to Y_x$ so that*

(i) *the resolution $\pi_X : R(X) \to X$ is a ring-like mapping.*

If X is perfectly normal and hereditarily separable then, under the continuum hypothesis, the mappings h_x can be chosen so that, in addition to (i)*,*

(ii) *the space $R(X)$ is perfectly normal and hereditarily separable.* ∎

4.20. *Reduced resolution.* Applying the construction from 4.5 to the mapping $\pi : R(X) \to X$ and a set $M \subset X$ we get a space $R^M(X)$ and mappings $\pi_M : R(X) \to R^M(X)$ and $\pi^M : R^M(X) \to X$ such that $\pi = \pi^M \circ \pi_M$ and

$$(\pi^M)^{-1}(x) = \pi^{-1}(x) \quad \text{for } x \in M, \tag{4.1}$$

$$|(\pi^M)^{-1}(x)| = 1 \quad \text{for } x \in X \setminus M. \tag{4.2}$$

The space $R^M(X)$ is called a *reduced resolution* of the resolution $R(X)$ with respect to M.

4.21. *The inverse system $S^{\pi}_{\mathcal{M}}$.* If $M_1 \subset M_2 \subset X$, then there exists a unique mapping $\pi^{M_2}_{M_1} : R^{M_2}(X) \to R^{M_1}(X)$ such that $\pi^{M_2} = \pi^{M_1} \circ \pi^{M_2}_{M_1}$. If $\mathcal{M}$

is a direction in X, then according to 4.11 the family $S^{\pi}_{\mathcal{M}} = \{R^M(X), \pi^M_{M'}, \mathcal{M}\}$ is an inverse system.

Theorems 4.12 and 4.15 yield

4.22. Theorem. *Let $\pi : R(X) \to R$ be a resolution of a Hausdorff compact space X and let $\mathcal{M}$ be a direction in X. Then π_M is homeomorphic to the limit projection $\lim S^{\pi}_{\mathcal{M}} \to R^M(X)$ of the inverse system $S^{\pi}_{\mathcal{M}}$, $M \in \mathcal{M}$.* ∎

5. Compact spaces with non-coinciding dimensions. The main result of this section is

5.1. Theorem.

(i) *For an arbitrary complex K with $K * K$ non-contractible and any $n \geq 2$ there exists a separable first countable compact Hausdorff space X_n such that*

$$K\text{-}\dim X_n = n < 2n-1 \leq K\text{-}\mathrm{Ind}\, X_n \leq 2n. \tag{5.1}$$

(ii) *Under the continuum hypothesis there exists a perfectly normal space X^0_n with properties from* (i).

To prove Theorem 5.1 we need some auxiliary information.

Just from the definition we get

5.2. Proposition. *Let $f : X \to Y$ be a ring-like mapping and let $U \subset X$ be an open subset. Then $\mathrm{ind}_y(Y \setminus f^{\#}U) \leq 0$ for every $y \in f(U) \setminus f^{\#}U$.* ∎

The next statement is an immediate consequence of Proposition 5.2.

5.3. Proposition. *Let $f : X \to Y$ be a ring-like mapping and let $U_1, \ldots, U_m$ be open subsets of X. Then*

$$\mathrm{ind}(f(U_1) \cup \cdots \cup f(U_m) \setminus (f^{\#}U_1 \cup \cdots \cup f^{\#}U_m)) \leq 0. \ \blacksquare$$

5.4. Proposition. *Let X be a compactum with K-$\dim X = k \geq 1$ and let $R(X)$ be the resolution from Lemma* 4.19 *with $Y_x = I^m$, $x \in X$, and*

$$m \geq n = K\text{-}\dim I^m \geq k. \tag{5.2}$$

Then K-$\mathrm{Ind}\, R(X) \geq k + n - 1$.

Proof. We apply induction on k. Let $k = 1$. Take an arbitrary point $x \in X$. Then

$$K\text{-}\mathrm{Ind}\, R(X) \overset{2.5}{\geq} K\text{-}\mathrm{Ind}(\pi^{-1}(x)) = K\text{-}\mathrm{Ind}\, I^m \overset{3.23}{=} K\text{-}\dim I^m \overset{(5.2)}{=} n = k+n-1.$$

Assume that the assertion holds for dimensions K-$\dim X$ less than $k \geq 2$ and consider a space X with K-$\dim X = k$. There exists $\Phi = (F_1, \ldots, F_m) \in \mathrm{Exp}_K(X)$ such that

$$K\text{-}\mathrm{Ind}\, P \geq k-1 \quad \text{for an arbitrary } K\text{-partition } P \text{ of } \Phi. \tag{5.3}$$

Put $\Psi = (\pi^{-1}F_1, \dots, \pi^{-1}F_m)$. Then $\Psi \in \mathrm{Exp}_K(R(X))$. Let $O\Psi = (U_1, \dots, U_m)$, be an arbitrary K-neighbourhood of Ψ existing by Lemma 1.7. The sequence $O\Phi = (\pi^{\#}U_1, \dots, \pi^{\#}U_m)$ is a K-neighbourhood of Φ. Then

$$P = X \setminus (\pi^{\#}U_1 \cup \dots \cup \pi^{\#}U_m) \tag{5.4}$$

is a K-partition of Φ. In view of (5.3) we have

$$K\text{-Ind}\, P \geq k - 1 \geq 1. \tag{5.5}$$

Put $U = U_1 \cup \dots \cup U_m$ and $Q = R(X) \setminus U$. Then Q is a K-partition of Ψ. Let

$$G = \pi^{\#}U \setminus (\pi^{\#}U_1 \cup \dots \cup \pi^{\#}U_m). \tag{5.6}$$

By (5.4) we have

$$P = G \sqcup f(Q). \tag{5.7}$$

Since X is a compactum, from Theorem 3.23 and (5.5) it follows that

$$K\text{-dim}\, P \geq k - 1 \geq 1. \tag{5.8}$$

On the other hand,

$$K\text{-dim}\, G \leq \dim G \leq 0 \tag{5.9}$$

by Theorems 1.17, 1.35, and Proposition 5.3. Consequently, from (5.7)–(5.9) and Proposition 1.33 it follows that $K\text{-dim}\, f(Q) \geq k-1$. Hence by Theorem 3.24 there exists a continuum $C \subset \pi(Q)$ such that $K\text{-dim}\, C \geq k - 1$. Then

$$K\text{-Ind}\, \pi^{-1}(C) \geq n + k - 2 \tag{5.10}$$

by the inductive assumption. Since π is ring-like mapping, we have $\pi^{-1}(C) \subset Q$ by Proposition 4.18. Thus from (5.10) it follows that $K\text{-Ind}\, Q \geq n+k-2$. But Q is an arbitrary K-partition of Ψ. Consequently, $K\text{-Ind}\, R(X) \geq n + k - 1$. ∎

5.5. Lemma. *Let X be a hereditarily normal space and let Y be a closed subspace such that $\mathcal{K}\text{-Ind}(X \setminus Y) \leq n \geq 0$. Then for every $\Phi \in \mathrm{Exp}_K(X)$, $K \in \mathcal{K}$, and every $Q \in \mathrm{Part}(\Phi|Y, K)$ there exists a K-partition P of Φ such that*

$$P \cap Y = Q, \tag{5.11}$$

$$\mathcal{K}\text{-Ind}(P \setminus Y) \leq n - 1. \tag{5.12}$$

Proof. Let $\Phi = (F_1, \dots, F_m)$ and $F = F_1 \cup \dots \cup F_m$. There exists a family $v = (V_1, \dots, V_m)$ of open subsets of Y such that

$$F_j \cap Y \subset V_j, \quad j = 1, \dots, m, \tag{5.13}$$

$$V_1 \cup \dots \cup V_m = Y \setminus Q, \tag{5.14}$$

$$N(v) \subset K. \tag{5.15}$$

The family v is an open covering of a normal space $Y_0 = Y \setminus Q$. Hence there exists a family $h = (H_1, \dots, H_m)$ of closed subsets of Y_0 such that

$$F_j \cap Y \subset H_j \subset V_j, \quad j = 1, \dots, m, \tag{5.16}$$

$$H_1 \cup \dots \cup H_m = Y \setminus Q, \tag{5.17}$$

$$N(h) \subset K. \tag{5.18}$$

Since Y_0 is a closed subset of the space $X_0 = X \setminus Q$, the sets $F_j^1 = F_j \cup H_j$ are closed in X_0. Put $\Phi_1 = (F_1^1, \dots, F_m^1)$. Since $\Phi \in \mathrm{Exp}_K(X)$, conditions (5.16), (5.18), and Lemma 3.20 imply that

$$N(\Phi_1) \subset K. \tag{5.19}$$

By (5.19) and Lemma 1.7 there exists a family $u = (U_1, \dots, U_m)$ of open subsets of X_0 such that

$$F_j^1 \subset U_j, \quad j = 1, \dots, m, \tag{5.20}$$

$$N(u) = N(\Phi_1) \subset K. \tag{5.21}$$

Since X_0 is normal, there exists a family $u_1 = (U_1^1, \dots, U_m^1)$ of open subsets of X_0 such that

$$F_j^1 \subset U_j^1 \subset \overline{U_j^1}^{X_0} \subset U_j, \quad j = 1, \dots, m. \tag{5.22}$$

Put $E_j = \overline{U_j^1}^{X_0} \setminus Y$ and $e = (E_1, \dots, E_m)$. From (5.21) it follows that

$$N(e) \subset K. \tag{5.23}$$

Since $\mathcal{K}\text{-}\mathrm{Ind}(X \setminus Y) \le n$, condition (5.23) implies the existence of a family $w = (W_1, \dots, W_m)$ of open subsets of $X \setminus Y$ such that

$$E_j \subset W_j, \quad j = 1, \dots, m, \tag{5.24}$$

$$N(w) \subset K, \tag{5.25}$$

$$\mathcal{K}\text{-}\mathrm{Ind}(X \setminus (Y \cup W_1 \cup \dots \cup W_m)) \le n - 1. \tag{5.26}$$

Put $U_j^2 = U_j^1 \cup W_j$ and $u_2 = (U_1^2, \dots, U_m^2)$. As unions of open sets, U_j^2 are open subsets of X_0, and hence of X. Conditions (5.21), (5.25), and Lemma 3.20 imply that $N(u_2) \subset K$. Moreover, from (5.22) and (5.24) it follows that

$$F_j \subset U_j^2, \quad j = 1, \dots, m.$$

Hence u_2 is a K-neighbourhood of Φ. Put $U_j^3 = U_j^2 \setminus Q$ and $u_3 = (U_1^3, \dots, U_m^3)$. Since $Q \cap F = \emptyset$, u_3 is a K-neighbourhood of Φ. We claim that

$$P = X \setminus (U_1^3 \cup \dots \cup U_m^3) \tag{5.27}$$

is the required partition. To check (5.11) it suffices to show that

$$Y \setminus (U_1^2 \cup \dots \cup U_m^2) \subset Q.$$

But this follows from (5.17) and (5.22). As for (5.12), it will be a consequence of (5.27), as soon as we prove that

$$P \setminus Y = X \setminus (Y \cup W_1 \cup \cdots \cup W_m). \tag{5.28}$$

By (5.27) we have $P \setminus Y = X \setminus (Y \cup U_1^3 \cup \cdots \cup U_m^3)$. But since $Q \subset Y$, we have $Y \cup U_1^3 \cup \cdots \cup U_m^3 = Y \cup U_1^2 \cup \cdots \cup U_m^2 = Y \cup W_1 \cup \cdots \cup W_m$ in view of (5.22) and (5.24). Thus equality (5.28) is proved. ∎

5.6. Proposition. *Let X be a compactum with K-$\dim X = k \geq 0$ and let $R(X)$ be the resolution from Lemma* 4.19, $Y_x = I^m$, $x \in X$, *and*

$$m \geq n = K\text{-}\dim I^m \geq k. \tag{5.29}$$

Then K-$\operatorname{Ind} R(X) \leq k + n$.

Proof. We apply induction on k. Let $k = 0$ and $\Phi = (F_1, \ldots, F_m) \in \operatorname{Exp}_K(R(X))$. Let $\mathcal{M}$ be the family of all finite subsets of X, i.e. $\mathcal{M} = \operatorname{Fin}(X) \cup \{\emptyset\}$. By Theorem 4.22 there exists a finite set $M = \{x_1, \ldots, x_l\} \subset X$ such that

$$N(\pi_M(\Phi)) = N(\Phi). \tag{5.30}$$

Put $Z = (\pi^M)^{-1} M$ and $Y = R^M(X) \setminus Z$. The set $Z = (\pi^M)^{-1}\{x_1, \ldots, x_l\}$ is homeomorphic to the disjoint union of l copies of I^m according to (4.1). Hence

$$n \overset{(5.29)}{=} K\text{-}\dim Z \overset{3.23}{=} K\text{-}\operatorname{Ind} Z. \tag{5.31}$$

On the other hand, Y is homeomorphic to $X \setminus M$ by (4.2). Thus

$$K\text{-}\operatorname{Ind} Y = K\text{-}\operatorname{Ind}(X \setminus M) \overset{3.23}{=} K\text{-}\dim(X \setminus M) \leq K\text{-}\dim X = 0. \tag{5.32}$$

From (5.31) it follows that there exists a partition $Q \in \operatorname{Part}(\pi_M(\Phi)|Z, K)$ with K-$\operatorname{Ind} Q \leq n - 1$. According to (5.32) and Lemma 5.5 there exists a K-partition P of $\pi_M(\Phi)$ such that

$$P \cap Z = Q, \qquad K\text{-}\operatorname{Ind}(P \setminus Z) \leq -1.$$

Consequently, $P \subset Z$ and $P = Q$.

But if $P \in \operatorname{Part}(\pi_M(\Phi), K)$, then $P_1 = \pi_M^{-1}(P) \in \operatorname{Part}(\Phi, K)$. From (4.1) it follows that

$$\pi_M|_{\pi^{-1}(M)} : \pi^{-1}(M) \to (\pi^M)^{-1}(M)$$

is a homeomorphism. So K-$\operatorname{Ind} P_1 = K$-$\operatorname{Ind} P = K$-$\operatorname{Ind} Q \leq n - 1$. Thus K-$\operatorname{Ind} R(X) \leq k + n$ for $k = 0$.

Assume that our assertion holds for all compacta X with K-$\dim X \leq k - 1 \geq 0$ and consider a compactum X with K-$\dim X = k$. Let $\Phi \in \operatorname{Exp}_K(R(X))$. Repeating the previous proof we find a finite set $M \subset X$ with $N(\pi_M(\Phi)) = N(\Phi)$ and a K-partition P of $\pi_M(\Phi)$ such that

$$K\text{-}\operatorname{Ind}(P \setminus Z) \leq k - 1.$$

As $\pi^M|_{P\setminus Z}$ is a homeomorphism, $K\text{-}\dim \pi^M(P \setminus Z) = K\text{-}\mathrm{Ind}\, \pi^M(P \setminus Z) \le k-1$. Consequently, $K\text{-}\dim \pi^M(P) \le K\text{-}\dim(M \cup \pi^M(P \setminus Z)) \le k-1$, because M is finite. By the inductive assumption ($X = \pi^M(P)$) we have

$$\dim \pi^{-1}(\pi^M(P)) \le n + k - 1.$$

But $\pi_M^{-1}(P) \subset \pi^{-1}(\pi^M(P))$. Thus $P_1 \equiv \pi_M^{-1}(P)$ is a K-partition of Φ with $K\text{-}\dim P_1 \le n + k - 1$. Hence $K\text{-}\dim X \le n + k$. ∎

Proof of Theorem 5.1. By Theorem 1.39 there is m such that $K\text{-}\dim I^m = n$. Put $X_n = R(X)$, where $R(X)$ is a resolution from Lemma 4.19(i) with $Y_x = I^m$, $x \in X$. Then the required properties of X_n are consequences of Theorems 4.13, 4.15, Proposition 4.8, Lemma 4.19, and Propositions 5.4 and 5.6 with $k = n$.

For X_n^0 we apply Lemma 4.19(ii) instead of Lemma 4.19(i). ∎

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REFERENCES

[1] R. Cauty, *Sur le prolongement des fonctions continues à valeurs dans les CW-complexes*, C. R. Acad. Sci. Paris 274 (1972), A35–A37.

[2] A. N. Dranishnikov, *Extension of mappings into CW-complexes*, Mat. Sb. 182 (1991), 1300–1310 (in Russian).

[3] R. Engelking, *Theory of Dimensions. Finite and Infinite*, Sigma Ser. Pure Math. 10, Heldermann, Lemgo, 1995.

[4] V. V. Fedorchuk, *On mappings not reducing dimension*, Dokl. Akad. Nauk SSSR 185 (1969), 54–57 (in Russian).

[5] —, *A bicompactum all of whose infinite closed subsets are n-dimensional*, Math. USSR-Sb. 25 (1976), 37–57.

[6] —, *On the dimension of hereditarily normal spaces*, Proc. London Math. Soc. 34 (1978), 163–175.

[7] —, *Fully closed maps and their applications*, J. Math. Sci. 136 (2006), 4201–4292.

[8] —, *Finite dimensions modulo simplicial complexes and ANR-compacta*, Mat. Vesnik 61 (2009), 25–52.

[9] —, *Several remarks on dimensions modulo ANR-compacta*, Topology Appl. 157 (2010), 716–723.

[10] M. Katětov, *On the dimension of non-separable spaces I*, Czechoslovak Math. J. 2 (1952), 333–368.

[11] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982.

[12] K. Morita, *Normal families and dimension theory for metric spaces*, Math. Ann. 128 (1954), 350–362.

[13] —, *On generalizations of Borsuk's homotopy extension theorem*, Fund. Math. 88 (1975), 1–6.

[14] M. Starbird, *The Borsuk homotopy extension without binormality condition*, ibid. 87 (1975), 207–211.

[15] J. E. West, *Mapping Hilbert cube manifolds to ANRs. A solution of a conjecture of Borsuk*, Ann. of Math. 106 (1977), 1–18.

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