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AUTOMORPHISMS OF THE ALGEBRA OF OPERATORS IN $\mathbb{L}^p$ PRESERVING CONDITIONING

BY

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Abstract. Let α be an isometric automorphism of the algebra $\mathbb{B}_p$ of bounded linear operators in $\mathbb{L}^p[0,1]$ ($p \geq 1$). Then α transforms conditional expectations into conditional expectations if and only if α is induced by a measure preserving isomorphism of $[0,1]$.

For $\mathbb{L}^p[0,1]$ (real or complex), let $\mathbb{B}_p$ be the algebra of bounded linear operators in $\mathbb{L}^p$ ($p \geq 1$). Let $\mathcal{E}$ denote the set of conditional expectations acting in $\mathbb{L}^p$.

The theory of automorphisms of $\mathbb{B}_2$ is well developed in a general set-up of C^*-algebras. There is no such theory for $\mathbb{B}_p$, $p \neq 2$. The von Neumann algebra structure of $\mathbb{B}_2$ is entirely different in comparison with that of $\mathbb{B}_p$, $p \neq 2$. That is why it is of interest to distinguish some classes of automorphisms in $\mathbb{B}_p$ enjoying the properties common for all $p \geq 1$. The following result is of this type.

THEOREM. *If α is an isometric automorphism of $\mathbb{B}_p$ ($p \geq 1$) then the following conditions are equivalent:*

$$\alpha(\mathcal{E}) \subset \mathcal{E}, \tag{1}$$

$$\alpha(x) = UxU^{-1}, \quad x \in \mathbb{B}_p, \quad \textit{with} \quad Uf = f \circ \beta, \tag{2}$$

where β is a measure preserving isomorphism of the unit Lebesgue interval.

Proof. Let α be of the form (2). For a fixed σ-field G of subsets of $[0,1]$, let us put

$$\alpha(\mathbb{E}^G)f = g, \quad f \in \mathbb{L}^p,$$

i.e. $[\mathbb{E}^G(f \circ \beta^{-1})] \circ \beta = g$.

Thus $\mathbb{E}^G(f \circ \beta^{-1}) = g \circ \beta^{-1}$, which means that $g \circ \beta^{-1}$ is G-measurable and $\int_A f \circ \beta^{-1} = \int_A g \circ \beta^{-1}$, $A \in G$. Consequently, g is $\beta^{-1}G$-measurable and $\int_{\beta^{-1}A} f = \int_{\beta^{-1}A} g$, $A \in G$, which means that $g = \mathbb{E}^{\beta^{-1}G} f$ so $\alpha(\mathbb{E}^G) = \mathbb{E}^{\beta^{-1}G}$.

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Now, let us assume (1) and take a partition of $[0,1]$, say $(A_1,\ldots,A_N)$. All subsets of $[0,1]$ are supposed to be measurable. We shall use the following notation. For $A\subset[0,1]$, we set

$$\widehat{A}=(\cdot\widehat{\chi}_A)\widehat{\chi}_A,\qquad\text{where } \widehat{\chi}_A=|A|^{-1/2}\chi_A.$$

Then $\sum_{k=1}^N \widehat{A}_k$ is a conditional expectation operator which is transformed by α to be a conditional expectation

$$\sum_{k=1}^N \widehat{B}_k,\qquad\text{where } (B_1,\ldots,B_N) \text{ is a partition of } [0,1],\ \widehat{B}_k=\alpha(\widehat{A}_k).$$

Indeed, it is enough to show that, for $A\subset[0,1]$,

$$(*)\qquad \alpha(\widehat{A})=\widehat{B}\qquad\text{for some } B\subset[0,1].$$

To prove $(*)$, take $A\subset[0,1]$. Then $\Delta=\widehat{A}+\widehat{A^c}$ is a two-dimensional conditional expectation which is transformed by α to $\alpha(\Delta)=\alpha(\widehat{A})+\alpha(\widehat{A^c})$; the latter is also a two-dimensional conditional expectation (obviously, $\dim\alpha(\Delta)\ge 2$, and $\dim\alpha(\Delta)\ge 3$ would imply $\dim\Delta=\dim\alpha^{-1}\alpha(\Delta)\ge 3$). Thus $\alpha(\Delta)=P_1+P_2=\widehat{B}+\widehat{B^c}$, where $P_1=\alpha(\widehat{A})$, $P_2=\alpha(\widehat{A^c})$ are mutually orthogonal one-dimensional projections and $B\subset[0,1]$. Let us write $P_1f=a_1(f)f_1$, $P_2f=a_2(f)f_2$, where a_j are linear functionals, $f_j\in\mathbb{L}^p$. The equality $\mathcal{E}_{B,B^c}\alpha(\Delta)=\alpha(\Delta)$ implies that f_1 and f_2 are linear combinations of χ_B and χ_{B^c}. By the orthogonality of P_1 and P_2 it follows that the disjoint supports of f_1 and f_2 coincide with B or B^c, and consequently $P_1=\widehat{B}$, $P_2=\widehat{B^c}$ (or the other way round).

For $A\subset[0,1]$, we denote by UA the set such that

$$\alpha(\widehat{A})=\widehat{UA}.$$

In particular, since $\alpha(\mathbb{E})=\mathbb{E}$, $U[0,1]=[0,1]$ a.e. We shall show that $|UA|=|A|$ for $A\subset[0,1]$.

Indeed, let $\|\cdot\|_\infty$ denote the norm in $\mathbb{B}_p$. Let us take two projections $\widehat{A}$ and $\mathbb{E}=\widehat{[0,1]}$. Then $(\mathbb{E}\widehat{A})f=(f,\widehat{\chi}_A)(\widehat{\chi}_A,1)1$, so

$$\|\mathbb{E}\widehat{A}\|_\infty=|A|^{1/2}\sup_{\|f\|_p\le 1}|(f,\widehat{\chi}_A)|=|A|^{1/q},\qquad \frac{1}{p}+\frac{1}{q}=1.$$

Similarly,

$$\|\alpha(\mathbb{E}\widehat{A})\|=\|\mathbb{E}\alpha(\widehat{A})\|_\infty=|UA|^{1/q}.$$

Thus $|A|=|UA|$ for $A\subset[0,1]$.

Let $(A_1,\ldots,A_N)$ be a partition of $[0,1]$. Then $\alpha(\sum_{k=1}^N\widehat{A}_k)=\sum_{k=1}^N\widehat{UA_k}$, and $(UA_1,\ldots,UA_N)$ is a partition of $[0,1]$ with $|A_j|=|UA_j|$. It turns out that U : Borel $[0,1]\to$ Borel $[0,1]$ is a measure preserving regular set-isomorphism. This means that U enjoys the properties

1° $U(A^{c}) = (UA)^{c}$,
2° $U(\bigcup_{k=1}^{\infty} A_k) = \bigcup_{k=1}^{\infty} UA_k$ for disjoint A_k,
3° $|UA| = |A|$, $A \subset [0,1]$.

The set transformation U induces a unique linear operator, also denoted by U, on the space of measurable functions such that

$$U\chi_A = \chi_{UA}, \qquad A \subset [0,1]$$

(cf. [1], [2]).

In our case (of the measure space being the unit Lebesgue interval) U is induced by a measure preserving isomorphism β of $[0,1]$, i.e. $Uf = f \circ \beta$ ([3, Chapter 15]). Consequently,

$$\begin{aligned}\alpha(\widehat{A})f &= (f, \widehat{\chi}_A \circ \beta)\widehat{\chi}_A \circ \beta = (f \circ \beta^{-1}, \widehat{\chi}_A)\widehat{\chi}_A \circ \beta \\ &= [\widehat{A}(f \circ \beta^{-1})] \circ \beta = U\widehat{A}U^{-1}f.\end{aligned}$$

Thus (2) holds for the operators $\widehat{A}$, $A \in \mathrm{Borel}\,[0,1]$. A fairly standard approximation shows that α is of the form (2) for compact operators in $\mathbb{L}^2[0,1]$. To conclude the proof it is enough to show that if α and $\widetilde{\alpha}$ are two automorphisms of $\mathbb{B}_p$ which coincide on the subalgebra of compact operators then they coincide on the whole algebra $\mathbb{B}_p$.

Let us assume the contrary, i.e. that $\alpha \neq \widetilde{\alpha}$. Then there would exist an operator $y \in \mathbb{B}_p$ and a vector $f \in \mathbb{L}^p$ such that $\alpha(y)f \neq \widetilde{\alpha}(y)f$. Let (h_n) be the Haar system on $[0,1]$, and let $S_n f = \sum_{k=0}^{n} (f, h_k)h_k$. The operators $T_n = \alpha^{-1}(S_n)$ are compact, so $\alpha(yT_n) = \widetilde{\alpha}(yT_n)$, $n = 1, 2, \ldots$. Thus $(\alpha(y) - \widetilde{\alpha}(y))S_n f = 0$ for $n = 1, 2, \ldots$. Letting $n \to \infty$ we get $(\alpha(y) - \widetilde{\alpha}(y))f = 0$, which contradicts the assumption. Consequently, α of the form (2) is the unique automorphism of $\mathbb{B}_p$ satisfying (1), which ends the proof. ∎

Let us remark that the Lebesgue unit interval can be replaced by a more general measure space (X, μ). It is enough to assume that it enjoys the property

(∗) any regular set-isomorphism in (X, μ) is induced by a point transformation.

For details and examples we refer to [3, Chapter 15].

In the case of a general probability space we have to confine ourselves to a set-isomorphism instead of a measure preserving transformation. This means that, keeping the notation of our theorem, we can say that the isomorphism $\alpha : \mathbb{B}_p \to \mathbb{B}_p$ preserves (globally) the conditional expectations if and only if $\alpha(\cdot) = U \cdot U^{-1}$, where U is the unique linear extension of a regular set-isomorphism.

Concluding this note it is worth noticing that our theorem gives, in a sense, an algebraic characterization of the measure preserving transformations. This suggests the investigations in operator algebras. For example, taking a von Neumann algebra $\mathcal{M}$ with a finite trace, say, one can consider the inner automorphisms $x \mapsto v \times v^{-1}$ of $\mathcal{M}$ globally preserving the conditional expectations of $\mathcal{M}$. In this context, the unitary operator v is a *non-commutative counterpart* of the operator $u_\alpha : f \mapsto f \circ \alpha$ which is the automorphism of the *commutative* algebra $\mathbb{L}^\infty$.

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