COLLOQUIUM MATHEMATICUM

VOL. 120 2010 NO. 2

FINITE-DIMENSIONAL TWISTED GROUP ALGEBRAS OF SEMI-WILD REPRESENTATION TYPE

BY

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Abstract. Let G be a finite group, K a field of characteristic $p > 0$, and $K^\lambda G$ the twisted group algebra of G over K with a 2-cocycle $\lambda \in Z^2(G, K^*)$. We give necessary and sufficient conditions for $K^\lambda G$ to be of semi-wild representation type in the sense of Drozd. We also introduce the concept of projective K-representation type for a finite group (tame, semi-wild, purely semi-wild) and we exhibit finite groups of each type.

Introduction. Let K be a field, $\widehat{K}$ the separable closure of K, and A, B algebras over K. Denote by $\mathrm{Bim}(A, B)$ the set of all A-B-bimodules M such that M is a free right module of finite rank over B. Let $F = \widehat{K}\langle t_1, t_2\rangle$ be the free associative $\widehat{K}$-algebra of polynomials in two non-commutting indeterminates t_1, t_2 with coefficients in $\widehat{K}$. We recall from [9] that a finite-dimensional K-algebra A is said to be of *semi-wild representation type* (briefly, A is semi-wild) if there is $M \in \mathrm{Bim}(A, F)$ such that for any $N \in \mathrm{Bim}(F, \widehat{K})$ there exist only a finite number of non-isomorphic $N_i \in \mathrm{Bim}(F, \widehat{K})$ with $M \otimes_F N \cong M \otimes_F N_i$. If, moreover, $M \otimes_F N \cong M \otimes_F N'$ implies $N \cong N'$, we say that the algebra A is of *wild representation type* (briefly, A is wild).

We recall that the paper by Simson [22] gives various notions of wildness of an algebra A and discusses relations between them. A detailed description of basic concepts of the theory of tame and wild representation types of algebras over an algebraically closed field can be found in monographs by Simson [21] and Simson and Skowroński [23].

Let K be a field of characteristic $p > 0$, G a finite group and $p \mid |G|$. Higman [14] proved that the group algebra KG is of finite representation type if and only if Sylow p-subgroups of G are cyclic. Bashev [5] and Heller and Reiner [13] have determined indecomposable representations of KG in the case when $p = 2$ and G is the group of type $(2, 2)$. Kruglyak showed in [17] that if $p > 2$ and G is a non-cyclic p-group, then KG is wild. Brenner [8] has

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2010 *Mathematics Subject Classification*: Primary 16G60; Secondary 20C20, 20C25.
Key words and phrases: modular projective representation, modular representation, projective representation, semi-wild twisted group algebra, twisted group algebra, wild twisted group algebra.

DOI: 10.4064/cm120-2-8

shown that KG is wild if $p = 2$, G is a non-cyclic 2-group and $|G : G'| \neq 4$. Bondarenko [6] and Ringel [20] have independently shown that KG is not wild if $p = 2$ and G is a dihedral 2-group. Finally, Bondarenko and Drozd [7] have established that KG is not wild either if $p = 2$ and G is a semidihedral 2-group or a quaternion 2-group.

Let K be an algebraically closed field, A a finite-dimensional K-algebra, G a finite group and AG the group algebra of G over A. The representation type of AG is determined by Meltzer and Skowroński [18, 19] and by Skowroński [24, 25].

Let G be a finite group, G_p a Sylow p-subgroup of G, K a field of characteristic p and $\lambda \in Z^2(G, K^*)$. We recall from [4] that the twisted group algebra $K^\lambda G$ is of finite representation type if and only if the algebra $K^\lambda G_p$ is uniserial.

In this paper we determine the algebras $K^\lambda G$ of semi-wild representation type. We also introduce the concept of projective K-representation type for G and exhibit finite groups of each type.

Let us briefly present the main results of the paper. Let G be a finite group, G_p a Sylow p-subgroup of G, G'_p the commutant of G_p, s the number of invariants of the abelian group G_p/G'_p, and C_p a Sylow p-subgroup of the commutant G' of G. We assume that $C_p \subset G_p$. Denote by D the subgroup of G_p such that $G'_p \subset D$ and $D/G'_p = \operatorname{soc}(G_p/G'_p)$. Let K be a field of characteristic p and let

$$i(K) = \begin{cases} n & \text{if } [K : K^p] = p^n, \\ \infty & \text{if } [K : K^p] = \infty. \end{cases}$$

In Section 1, we describe twisted group algebras of semi-wild representation type. Let $\lambda \in Z^2(G, K^*)$ and $d = \dim_K(K^\lambda G_p / \operatorname{rad} K^\lambda G_p)$. Suppose that $p \neq 2$ and if $|G'_p| = p$, $pd = |G_p : G'_p|$ then D is abelian. We prove that the algebra $K^\lambda G$ is semi-wild if and only if the subalgebra $K^\lambda G_p$ is not uniserial (Theorem 1.16). If $p = 2$ and one of the following conditions holds:

(i) $4d < |G_2 : G'_2|$;
(ii) $4d = |G_2 : G'_2|$, $|G'_2| \geq 4$ and $K^\lambda G_2 / K^\lambda G_2 \cdot \operatorname{rad} K^\lambda G'_2$ is a uniserial algebra;
(iii) $d = |G_2 : G'_2|$ and $|G'_2 : G''_2| \neq 4$,

then the algebra $K^\lambda G$ is semi-wild if and only if the subalgebra $K^\lambda G_2$ is not uniserial (Theorem 1.17).

We say that a finite group G is of *purely semi-wild* [*wild*] *projective K-representation type* if $K^\lambda G$ is of semi-wild [wild] representation type for any $\lambda \in Z^2(G, K^*)$.

In Section 2, we exhibit finite groups of purely semi-wild projective K-representation type at characteristic $p \neq 2$. Assume that if $|C_p| = p$, $s =$

$i(K)+1$ and D is non-abelian, then $\exp D = p^2$. We prove (Theorem 2.7) that a group G is of purely semi-wild projective K-representation type if and only if one of the following conditions is satisfied:

(i) C_p is a non-cyclic group;
(ii) $s \geq i(K)+2$;
(iii) $s = i(K)+1$, $C_p = G'_p = \langle c \rangle$, $|c| \geq p^2$ and $g^p \in \langle c^p \rangle$ for every $g \in D$;
(iv) $s = i(K)+1$, $C_p = G'_p$, $|G'_p| = p$ and D is an elementary abelian p-group.

As a consequence we obtain the following two corollaries:

1. Let $p \neq 2$ and $[K : K^p] = \infty$. A group G is of purely semi-wild projective K-representation type if and only if C_p is non-cyclic.
2. Let $p \neq 2$ and G be a finite group such that G_p is abelian. Then G is of purely semi-wild projective K-representation type if and only if C_p is non-cyclic or $s \geq i(K)+2$.

In Section 3, we characterize finite groups of purely semi-wild projective K-representation type at characteristic $p = 2$. Let G be a finite group such that G_2 is abelian and C_2 is cyclic. We show in Theorem 3.2 that G is of purely semi-wild projective K-representation type if and only if one of the following conditions is satisfied:

(i) $s \geq i(K)+3$;
(ii) $s = i(K)+2$ and $|C_2| \geq 4$;
(iii) $s = i(K)+2$, $|C_2| \leq 2$ and G_2 has at most one invariant equal to 2.

We also prove that if $|C_2 : C'_2| \neq 4$ and $[K : K^2] = \infty$, then the group G is of purely semi-wild projective K-representation type if and only if C_2 is not cyclic (Corollary 3.4).

Preliminaries. Throughout this paper, we use the following notations: K is a field of characteristic $p > 0$; K^* is the multiplicative group of K; $K^p = \{\alpha^p : \alpha \in K\}$; G is a finite group, G' is the commutant of G and G'' is the commutant of G'; G_p is a Sylow p-subgroup of G, C_p a Sylow p-subgroup of G', G'_p the commutant of G_p, s the number of invariants of G_p/G'_p. Moreover, we assume that $C_p \subset G_p$, hence $G'_p \subset C_p$. Let $Z(G)$ be the center of G, e the identity element of G, $|g|$ the order of $g \in G$ and $\operatorname{soc} B$ the socle of an abelian p-group B. We denote by $Z^2(G, K^*)$ the group of all K^*-valued normalized 2-cocycles of the group G, where we assume that G acts trivially on K^* (see [15, Chapter 2] and [16, Chapter 1]).

Given a cocycle $\lambda : G \times G \to K^*$ in $Z^2(G, K^*)$, we denote by $K^\lambda G$ the twisted group algebra of G over K with the 2-cocycle λ. A K-basis $\{u_g : g \in G\}$ of $K^\lambda G$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$ is called *natural* (corresponding to λ). If H is a subgroup of G, we often use the same symbol for an element $\lambda : G \times G \to K^*$ of $Z^2(G, K^*)$ and its restriction to $H \times H$. In this case, $K^\lambda H$ is a subalgebra of $K^\lambda G$.

Let G be a finite p-group. Denote by $\operatorname{rad} K^\lambda G$ the radical of $K^\lambda G$. We set $\overline{K^\lambda G} = K^\lambda G / \operatorname{rad} K^\lambda G$. We recall that in this case $\overline{K^\lambda G}$ is a finite purely inseparable field extension of K [16, p. 74]. Given $\lambda \in Z^2(G, K^*)$, the kernel $\operatorname{Ker}(\lambda)$ of λ is the union of all cyclic subgroups $\langle g \rangle$ of G such that the restriction of λ to $\langle g \rangle \times \langle g \rangle$ is a coboundary. We recall from [3, p. 268] that $G' \subset \operatorname{Ker}(\lambda)$, $\operatorname{Ker}(\lambda)$ is a normal subgroup of G and the restriction of λ to $\operatorname{Ker}(\lambda) \times \operatorname{Ker}(\lambda)$ is a coboundary (see also [2, p. 197] for a simple proof). Up to cohomology in $Z^2(G, K^*)$, we have $\lambda_{g,a} = \lambda_{a,g} = 1$ for all $g \in G$ and $a \in \operatorname{Ker}(\lambda)$. In what follows, we assume that every cocycle $\lambda \in Z^2(G, K^*)$ under consideration satisfies this condition.

Assume now that G is a finite group, $\lambda \in Z^2(G, K^*)$, μ is the restriction of λ to $G_p \times G_p$ and $H_p = \operatorname{Ker}(\mu)$. Then $C_p \subset H_p$ (see [16, p. 48]).

Let G be an abelian p-group, $\overline{G} = \operatorname{soc} G$, $\lambda \in Z^2(G, K^*)$, and $S = \operatorname{Ker}(\lambda)$. If $K^\lambda G$ is not a field and $\overline{G} \not\subset S$, then there exists a direct product decomposition $G = H \times \langle c_1 \rangle \times \cdots \times \langle c_m \rangle$ such that $H \neq \{e\}$, $K^\lambda H$ is a field and $K^\lambda D_j$ is not a field for every $j \in \{1, \dots, m\}$, where $D_j = H \times \langle c_j \rangle$. The algebra $K^\lambda G$ is not uniserial if and only if $m \geq 2$ (see [4, p. 176]). If $\overline{G} \subset S$ then $K^\lambda \overline{G}$ is the group algebra of $\overline{G}$ over K. In this case $K^\lambda G$ is not a uniserial algebra if and only if G is a non-cyclic group. Assume that $G = B \times C$, where C is a cyclic group and $K^\lambda C$ is not a field. The algebra $K^\lambda G$ is not uniserial if and only if $K^\lambda B$ is not a field (see [4, pp. 175–176]).

Let G be a p-group, H a normal subgroup of G, $\lambda \in Z^2(G, K^*)$, $H \subset \operatorname{Ker}(\lambda)$ and $T = G/H$. We put $\mu_{xH,yH} = \lambda_{x,y}$ for all $x, y \in G$. Then $\mu \in Z^2(T, K^*)$. Assume that $\{u_g : g \in G\}$ is a natural K-basis of $K^\lambda G$ corresponding to λ, and $\{v_{gH} : g \in G\}$ is a natural K-basis of $K^\mu T$ corresponding to μ. The formula

$$f\Big(\sum_{g \in G} \alpha_g u_g\Big) = \sum_{g \in G} \alpha_g v_{gH}$$

defines a K-algebra epimorphism $f : K^\lambda G \to K^\mu T$ with the kernel $U = K^\lambda G \cdot \operatorname{rad} K^\lambda H$ (see [15, p. 14] or [16, p. 88]). Hence, $K^\lambda G / U \cong K^\mu T$. We recall that

$$\operatorname{rad} K^\lambda H = \bigoplus_{h \in H \setminus \{e\}} K(u_h - u_e)$$

is called the *augmentation ideal* of the group algebra $K^\lambda H$. If $G' \subset H$ then $K^\mu T$ is a commutative algebra. Let $d = \dim_K \overline{K^\lambda G}$. Then d divides $|G : G'|$. Moreover, if G is an abelian p-group and $K^\lambda G$ is not a uniserial algebra, then $p^2 d$ is a divisor of $|G|$.

If V is a finite-dimensional vector space over K and $\Gamma : G \to \mathrm{GL}(V)$ a projective representation of G with a 2-cocycle $\lambda \in Z^2(G, K^*)$, we refer to Γ as a λ-*representation* of G over K. Let $\mathrm{PGL}(V) = \mathrm{GL}(V)/K^* \cdot 1_V$ and $\pi : \mathrm{GL}(V) \to \mathrm{PGL}(V)$ be the canonical group homomorphism. The kernel of the composite homomorphism $\pi \circ \Gamma : G \to \mathrm{PGL}(V)$ is called the *kernel* of Γ and is denoted by $\mathrm{Ker}(\Gamma)$. Let G be a p-group and K be a field of characteristic p. If Γ is an irreducible λ-representation of G over K then $\mathrm{Ker}(\Gamma) = \mathrm{Ker}(\lambda)$ (see [2, p. 198]).

We recall from [26, p. 129] that a set of elements $\alpha_1, \ldots, \alpha_m$ of a field K is called *p-independent* if the p^m monomials $\alpha_1^{i_1} \ldots \alpha_m^{i_m}$ with $0 \le i_r < p$ for every $r \in \{1, \ldots, m\}$ are linearly independent over the subfield K^p of K. A subset $\{\beta_1, \ldots, \beta_n\}$ of K is said to be a *p-basis* if it is a p-independent set and $K = K^p(\beta_1, \ldots, \beta_n)$. In this case $[K : K^p] = p^n$.

It is not difficult to verify that a subset $\{\alpha_1, \ldots, \alpha_m\}$ of the field K is p-independent if and only if the K-algebra

$$K[t]/(t^p - \alpha_1) \otimes_K \cdots \otimes_K K[t]/(t^p - \alpha_m) \tag{1}$$

is a field. It follows that $i(K)$ (definition on p. 278) is the supremum of the set that consists of 0 and all positive integers m such that a K-algebra of the form (1) is a field for some $\alpha_1, \ldots, \alpha_m \in K$.

The reader is referred to [15] and [16] for basic facts and notation from the theory of projective representations of finite groups and to [1] and [11] for terminology, notation and introduction to the representation theory of finite-dimensional algebras over a field.

1. Twisted group algebras of semi-wild representation type. Let G be a finite group, K a field of characteristic p, $\widehat{K}$ the separable closure of K, $\lambda \in Z^2(G, K^*)$ and $A = K^\lambda G$. Following Drozd [9, p. 109], we say that A is of *tame representation type* (briefly, A is tame) if there exists a family $\mathcal{S} = \{M_i \in \mathrm{Bim}(A, \widehat{K}[t]) : i \in I\}$ of A-$\widehat{K}[t]$-bimodules M_i with the following properties:

- for every positive integer d the set $\mathcal{S}$ has a finite number of M_i of $\widehat{K}[t]$-rank d;
- for every indecomposable module V over the algebra $\widehat{A} = A \otimes_K \widehat{K}$ there exist $M_i \in \mathcal{S}$, $\alpha \in \widehat{K}$ and positive integer n such that

$$V \cong M_i \otimes_{\widehat{K}[t]} \widehat{K}[t]/(t - \alpha)^n.$$

Here $\mathrm{Bim}(A, \widehat{K}[t])$ is the set of all A-$\widehat{K}[t]$-bimodules M such that M is a free right module of finite rank over the algebra $\widehat{K}[t]$. By [9, p. 110], no algebra $K^\lambda G$ is both semi-wild and tame.

Now we collect basic facts we use throughout the paper.

LEMMA 1.1. *Let G be a finite group and $\lambda \in Z^2(G, K^*)$.*

(i) *If the algebra $K^\lambda G$ is tame then the subalgebra $K^\lambda H$ is also tame for any subgroup H of G. If the subalgebra $K^\lambda H$ of $K^\lambda G$ is semi-wild then the algebra $K^\lambda G$ is semi-wild.*

(ii) *Let G_p be a Sylow p-subgroup of G. If the subalgebra $K^\lambda G_p$ of $K^\lambda G$ is tame then the algebra $K^\lambda G$ is also tame. If the algebra $K^\lambda G$ is semi-wild then the subalgebra $K^\lambda G_p$ is semi-wild as well.*

Proof. Apply [7, Proposition 2]. ∎

LEMMA 1.2 ([7, p. 24]). *Let G be a p-group. The group algebra KG is wild if and only if G is non-cyclic and $|G : G'| \neq 4$. Otherwise KG is tame.*

LEMMA 1.3 ([4, p. 175]). *Let G be a finite group and $\lambda \in Z^2(G, K^*)$. The algebra $K^\lambda G$ is of finite representation type if and only if $K^\lambda G_p$ is uniserial.*

LEMMA 1.4 ([12, p. 119]). *Let G be an abelian p-group and T a subgroup of $\operatorname{soc} G$. Then there exists a direct decomposition $G = A \times B$ such that $\operatorname{soc} B = T$.*

LEMMA 1.5. *Let G be a non-abelian p-group with $G' = \langle c\rangle$, $H = \langle c^p\rangle$, and $\lambda \in Z^2(G, K^*)$.*

(i) *$V := K^\lambda G \cdot \operatorname{rad} KH$ is an ideal of $K^\lambda G$ and there is an algebra isomorphism $K^\lambda G/V \cong K^\mu T$, where $T = G/H$, $T' = \langle cH\rangle$ and $\mu_{xH,yH} = \lambda_{x,y}$ for all $x, y \in G$.*

(ii) *The algebra $K^\lambda G$ is uniserial if and only if $K^\lambda G/V$ is uniserial.*

Proof. We have $G' \subset \operatorname{Ker}(\lambda)$, $H \lhd G$ and $(G/H)' = G'/H = \langle cH\rangle$. The set V is an ideal of $K^\lambda G$ and $K^\lambda G/V \cong K^\mu T$. We put $\widehat{K^\lambda G} = K^\lambda G/V$ and $\widehat{w} = w + V$ for every $w \in K^\lambda G$. Since V is a nilpotent ideal, $\operatorname{rad} \widehat{K^\lambda G} = (\operatorname{rad} K^\lambda G)/V$. If $K^\lambda G$ is uniserial then $\operatorname{rad} K^\lambda G = K^\lambda G \cdot \theta$ for some $\theta \in \operatorname{rad} K^\lambda G$ (see [11, p. 170]). It follows that $\operatorname{rad} \widehat{K^\lambda G} = \widehat{K^\lambda G} \cdot \widehat{\theta}$. Hence $\widehat{K^\lambda G}$ is a uniserial algebra.

Conversely, assume that $\widehat{K^\lambda G}$ is uniserial. Then $\operatorname{rad} \widehat{K^\lambda G} = \widehat{K^\lambda G} \cdot \widehat{\theta}$ for some $\theta \in \operatorname{rad} K^\lambda G$. Consequently, for any $w \in \operatorname{rad} K^\lambda G$, there is $z \in K^\lambda G$ such that $w + V = (z + V)(\theta + V)$. It follows that $w - z\theta \in (\operatorname{rad} K^\lambda G)^2$, since $u_c^p - u_e = (u_c - u_e)^p \in (\operatorname{rad} K^\lambda G)^2$. This implies

$$w + (\operatorname{rad} K^\lambda G)^2 = [z + (\operatorname{rad} K^\lambda G)^2] \cdot [\theta + (\operatorname{rad} K^\lambda G)^2].$$

Therefore the radical of $K^\lambda G/(\operatorname{rad} K^\lambda G)^2$ is a principal left ideal. Hence $K^\lambda G/(\operatorname{rad} K^\lambda G)^2$ is a uniserial algebra. But $K^\lambda G/(\operatorname{rad} K^\lambda G)^2$ is uniserial if and only if $K^\lambda G$ is uniserial (see [11, p. 172]). Consequently, $K^\lambda G$ is a uniserial algebra. ∎

LEMMA 1.6. *Let G be a p-group, $|G'| = p$ and $\lambda \in Z^2(G, K^*)$. Suppose that $K^\lambda G$ is not a uniserial algebra and $K^\lambda G/K^\lambda G \cdot \operatorname{rad} KG'$ is a uniserial algebra. If H is an abelian subgroup of G, $G' \subset H$ and $K^\lambda H/K^\lambda H \cdot \operatorname{rad} KG'$ is not a field, then $K^\lambda H$ is not a uniserial algebra either.*

Proof. Let $G' = \langle c \rangle$. If $H = A \times \langle c \rangle$ then $K^\lambda H \cong K^\lambda A \otimes_K KG'$ and $K^\lambda A$ is not a field, since $K^\lambda A \cong K^\lambda H/K^\lambda H \cdot \operatorname{rad} KG'$. Therefore $K^\lambda H$ is not a uniserial algebra.

Now we assume that $\langle c \rangle$ is not a direct factor of H. By Lemma 1.4, there exists a direct product decomposition $H = \langle h_1 \rangle \times \cdots \times \langle h_n \rangle$, where $c \in \langle h_n \rangle$ and $|c| < |h_n|$. Assume $K^\lambda H$ is uniserial algebra, $|h_n| = p^{t+1}$ and $c = h_n^{p^t}$. Then

$$K^\lambda H = \bigoplus_{i_1=0}^{|h_1|-1} \cdots \bigoplus_{i_n=0}^{|h_n|-1} K u_{h_1}^{i_1} \ldots u_{h_n}^{i_n},$$

where

$$u_{h_j}^{|h_j|} = \delta_j u_e, \qquad \delta_j \in K^* \quad \text{for } j = 1, \ldots, n-1,$$
$$u_{h_n}^{p^t} = \alpha u_c, \qquad \alpha \in K^*,$$

and

$$F := \bigoplus_{i_1=0}^{|h_1|-1} \cdots \bigoplus_{i_{n-1}=0}^{|h_{n-1}|-1} K u_{h_1}^{i_1} \ldots u_{h_{n-1}}^{i_{n-1}}$$

is a field. Since $K^\lambda H/K^\lambda H(u_c - u_e)$ is not a field, we have $\alpha u_e = \theta^p$ for some $\theta \in F$. From

$$(\theta^{-1} u_{h_n}^{p^{t-1}} - u_e)^p = u_c - u_e$$

it follows that $u_c - u_e \in (\operatorname{rad} K^\lambda H)^2$. Because $K^\lambda G$ is a local algebra, we have $\operatorname{rad} K^\lambda H \subset \operatorname{rad} K^\lambda G$. Hence $K^\lambda G(u_c - u_e) \subset (\operatorname{rad} K^\lambda G)^2$. By hypothesis, the algebra $K^\lambda G/K^\lambda G(u_c - u_e)$ is uniserial. Arguing as in the proof of Lemma 1.5, we show that $K^\lambda G/(\operatorname{rad} K^\lambda G)^2$ is a uniserial algebra. Then $K^\lambda G$ is also uniserial, a contradiction. ∎

LEMMA 1.7. *Let G be a non-abelian p-group with non-cyclic commutant. Then G contains a normal subgroup H such that $H \subset G'$ and $(G/H)' = G'/H$ is an elementary abelian group of type (p, p).*

Proof. By our assumption, G'/G'' is a non-cyclic abelian p-group. Denote by S the subgroup of G' generated by G'' and all elements x^p, where $x \in G'$. Clearly, S is a normal subgroup of G and G'/S is a non-cyclic elementary abelian p-group. Since $(G/S)' = G'/S$, in what follows we assume that G' is a non-cyclic elementary abelian p-group.

Let $|G'| = p^n$ and $n > 2$. Choose $a \neq e$ in G' such that $a \in Z(G)$. Then the commutant of $G/\langle a \rangle$ is a non-cyclic elementary abelian group of order p^{n-1}. If $n - 1 > 2$, we inductively continue the above construction. ∎

Let H be a subgroup of a group G. If M is a $K^\lambda G$-module, we denote by M_H the module M viewed as a $K^\lambda H$-module. If N is a $K^\lambda H$-module, then $N^G = K^\lambda G \otimes_{K^\lambda H} N$ is called the induced $K^\lambda G$-module.

LEMMA 1.8. *Let G be a finite group, H a subgroup of G and $\lambda \in Z^2(G, K^*)$. If $H \subset Z(G)$ and $K^\lambda H$ is a wild algebra, then $K^\lambda G$ is also wild.*

Proof. Let V_1 and V_2 be $K^\lambda H$-modules. As $(V_i^G)_H \cong V_i \oplus \cdots \oplus V_i$ ($|G:H|$ summands) for $i = 1, 2$, by the Krull–Schmidt Theorem the modules V_1^G and V_2^G are isomorphic if and only if the modules V_1 and V_2 are isomorphic.

Let $\widehat{K}$ be the separable closure of K and $F = \widehat{K}\langle t_1, t_2\rangle$. By definition of a wild algebra, there exists a bimodule $M \in \operatorname{Bim}(K^\lambda H, F)$ such that, for any $N, N' \in \operatorname{Bim}(F, \widehat{K})$, $M \otimes_F N \cong M \otimes_F N'$ implies $N \cong N'$. We put $W = K^\lambda G \otimes_{K^\lambda H} M$. If $W \otimes_F N \cong W \otimes_F N'$ for some $N, N' \in \operatorname{Bim}(F, \widehat{K})$, then

$$(M \otimes_F N)^G \cong (M \otimes_F N')^G.$$

This implies $M \otimes_F N \cong M \otimes_F N'$, hence $N \cong N'$. Thus $K^\lambda G$ is wild. ■

LEMMA 1.9. *Let G be an abelian p-group, $\lambda \in Z^2(G, K^*)$, $d = \dim_K \overline{K^\lambda G}$ and $4d \neq |G|$. If $K^\lambda G$ is not a uniserial algebra then it is wild.*

Proof. Let $G = H \times \langle b_1\rangle \times \cdots \times \langle b_m\rangle$, $D_j = H \times \langle b_j\rangle$ for $j = 1, \dots, m$, where H may be the trivial group. Assume that $K^\lambda H$ is a field and $K^\lambda D_j$ is not a field for every $j \in \{1, \dots, m\}$. Then $m \geq 2$, since $K^\lambda G$ is not uniserial. First, we assume that $m \geq 3$ in the case $p = 2$. Let $B = \langle b_1\rangle \times \cdots \times \langle b_m\rangle$, $\overline{G} = \operatorname{soc} G$, $\overline{H} = \operatorname{soc} H$ and $\overline{B} = \operatorname{soc} B$. Then $K^\lambda \overline{G} \cong K^\lambda \overline{H} \otimes K\overline{B}$. Since $|\overline{B}| > 4$, by Lemmas 1.2 and 1.8, the algebra $K^\lambda \overline{G}$ is wild. It follows, by Lemma 1.8, that so is $K^\lambda G$.

Next we suppose that $p = 2$ and $G = H \times \langle b_1\rangle \times \langle b_2\rangle$. Denote by F the field $K^\lambda H$. We have $K^\lambda G = F^\lambda B$, where $B = \langle b_1\rangle \times \langle b_2\rangle$. Let $\{u_g : g \in G\}$ be a natural K-basis of $K^\lambda G$. Then

$$K^\lambda G = \bigoplus_{i_1=0}^{2^{n_1}-1} \bigoplus_{i_2=0}^{2^{n_2}-1} F u_{b_1}^{i_1} u_{b_2}^{i_2}, \qquad u_{b_j}^{2^{n_j}} = \gamma_j u_e \quad \text{for } j \in \{1, 2\},$$

where $2^{n_j} = |b_j|$, $\gamma_j \in K^*$. Since

$$\dim_F(F^\lambda B/\operatorname{rad} F^\lambda B) = \frac{d}{[F:K]}, \qquad |G| = [F:K] \cdot |B| \quad \text{and} \quad 4d \neq |G|,$$

we have $4 \cdot \dim_F(F^\lambda B/\operatorname{rad} F^\lambda B) \neq |B|$. The algebra $F^\lambda B$ is not uniserial, hence $\gamma_j \in F^2$ for every j and $4 \cdot \dim_K \overline{F^\lambda B} < |B|$.

Assume that $n_1 \geq 2$, $n_2 \geq 2$, $\gamma_1 = \delta_1^2$, $\gamma_2 = \delta_2^2$, where $\delta_1, \delta_2 \in F^*$ and $\delta_1 \notin F^2$. We set

$$c_j = b_j^{2^{n_j - 2}} \qquad \text{for } j = 1, 2.$$

Then $C = \langle c_1\rangle \times \langle c_2\rangle$ is of type $(4,4)$. Denote by θ_j a root of the polynomial $X^2 - \delta_j$ in the algebraic closure of the field F. Then $[F(\theta_1,\theta_2) : F] < 4$, hence $\delta_2 \in F(\theta_1)^2$, that is, $\delta_2 = (\rho_1+\rho_2\theta_1)^2 = \rho_1^2+\rho_2^2\delta_1$ for some $\rho_1,\rho_2 \in F$. Put

$$v_{c_1} = u_{c_1}, \quad v_{c_2} = (\rho_1+\rho_2 u_{c_1})^{-1}u_{c_2}.$$

We have $v_{c_2}^4 = u_e$. Hence, if $D = H\times C$ then

$$K^\lambda D = F^\mu C = \bigoplus_{i_1=0}^{3}\bigoplus_{i_2=0}^{3} F v_{c_1}^{i_1}v_{c_2}^{i_2}, \qquad v_{c_1}^4 = \gamma_1 u_e, \qquad v_{c_2}^4 = u_e.$$

We set $T = H\times\langle c_1^2\rangle\times\langle c_2\rangle$ and $N = \langle c_1^2\rangle\times\langle c_2\rangle$. Then for the algebra

$$K^\mu T = \bigoplus_{i_1=0}^{1}\bigoplus_{i_2=0}^{3} K^\lambda H\cdot v_{c_1^2}^{i_1}v_{c_2}^{i_2}, \qquad v_{c_1^2}^2 = \gamma_1 u_e, \qquad v_{c_2}^4 = u_e,$$

we have $K^\mu T \cong K^\lambda H \otimes_K KN$. Since $|N| = 8$, by Lemmas 1.2 and 1.8, the algebra $K^\mu T$ is wild. By Lemma 1.8, so is $K^\lambda D$. Applying again Lemma 1.8, we conclude that $K^\lambda G$ is wild.

Now assume that $n_1 \geq 2$, $n_2 \geq 1$, and $\gamma_1 = \delta_1^4$, $\gamma_2 = \delta_2^2$ for some $\delta_1,\delta_2 \in F^*$. Let

$$c_1 = b_1^{2^{n_1-2}}, \quad c_2 = b_2^{2^{n_2-1}}.$$

Then $C = \langle c_1\rangle\times\langle c_2\rangle$ is of type $(4,2)$ and $F^\lambda C$ is the group algebra of the group C over the field F. We put $D = H\times C$. We have $K^\lambda D = F^\lambda C \cong K^\lambda H\otimes_K KC$. By Lemmas 1.2 and 1.8, $K^\lambda D$ is wild. Hence, in view of Lemma 1.8, $K^\lambda G$ is also wild. ∎

LEMMA 1.10. *Let G be a non-abelian p-group, $\lambda \in Z^2(G,K^*)$, and $d = \dim_K \overline{K^\lambda G}$. Assume that $pd < |G:G'|$ if $p \neq 2$, and $4d < |G:G'|$ if $p = 2$. If $K^\lambda G$ is not a uniserial algebra then it is semi-wild.*

Proof. Let $\{u_g : g\in G\}$ be a natural K-basis of the algebra $K^\lambda G$, $U = K^\lambda G\cdot \operatorname{rad} KG'$, $\widetilde{K^\lambda G} = K^\lambda G/U$ and $\tilde w = w+U$ for every $w\in K^\lambda G$. Suppose that $G/G' = \langle a_1G'\rangle\times\cdots\times\langle a_mG'\rangle$, where $|a_jG'| = p^{s_j}$ for $j = 1,\dots,m$. Then

$$\widetilde{K^\lambda G} = \bigoplus_{i_1=0}^{l_1}\dots\bigoplus_{i_m=0}^{l_m} K\tilde u_{a_1}^{i_1}\cdots\tilde u_{a_m}^{i_m},$$

where $l_j = p^{s_j}-1$,

$$\tilde u_{a_j}^{p^{s_j}} = \gamma_j\tilde u_e, \quad \gamma_j\in K^* \quad \text{for } j = 1,\dots,m.$$

The algebra $\widetilde{K^\lambda G}$ is a twisted group algebra of the non-cyclic abelian p-group G/G' over the field K.

If $\widetilde{K^\lambda G}$ is not a uniserial algebra then, by Lemma 1.9, $\widetilde{K^\lambda G}$ is wild. Then so is $K^\lambda G$. Assume now that $\widetilde{K^\lambda G}$ is a uniserial algebra and the K-subalgebra

$$F = \bigoplus_{i_1=0}^{l_1} \cdots \bigoplus_{i_{m-1}=0}^{l_{m-1}} K\tilde{u}_{a_1}^{i_1} \cdots \tilde{u}_{a_{m-1}}^{i_{m-1}}$$

is a field. We have $F = (K^\lambda D + U)/U$, where D is the subgroup of G generated by G' and the elements $a_1, \dots, a_{m-1}$. Evidently,

$$\widetilde{K^\lambda G} = \bigoplus_{i_m=0}^{p^{s_m}-1} F\tilde{u}_{a_m}^{i_m}.$$

Since $\dim_K \widetilde{K^\lambda G} = |G : G'|$, $\dim_K(\widetilde{K^\lambda G}/\operatorname{rad}\widetilde{K^\lambda G}) = d$ and $d < |G : G'|$, the algebra $\widetilde{K^\lambda G}$ is not a field. There exists an element

$$\rho = \sum_{i_1=0}^{l_1} \cdots \sum_{i_{m-1}=0}^{l_{m-1}} \alpha_{i_1,\dots,i_{m-1}} u_{a_1}^{i_1} \dots u_{a_{m-1}}^{i_{m-1}}, \tag{2}$$

where $\alpha_{i_1,\dots,i_{m-1}} \in K$, $l_j = p^{s_j} - 1$ for every $j \in \{1, \dots, m-1\}$, such that $\tilde{\rho}^{p^r} = \gamma_m^{-1}\tilde{u}_e$ with r satisfying one of the following two conditions:

(i) if $p \neq 2$ then $2 \leq r \leq s_m$, and $\tilde{\rho} \notin F^p$ in the case $r < s_m$;
(ii) if $p = 2$ then $3 \leq r \leq s_m$, and $\tilde{\rho} \notin F^2$ in the case $r < s_m$.

We have $d = |D : G'| \cdot p^{s_m - r}$, hence $dp^r = |G : G'|$.

In view of Lemmas 1.1, 1.2, 1.5 and 1.7, we can assume that $|G'| = p$ for $p \neq 2$, while G' is the elementary abelian group of type $(2, 2)$ or the group of order 2 for $p = 2$. Denote by H the subgroup of G generated by G' and the elements

$$a_1^p, \dots, a_{m-1}^p, a_m^{p^{s_m - r + 1}}.$$

We show that H is abelian. Assume that $p = 2$ and $G' = \langle c_1 \rangle \times \langle c_2 \rangle$, where $|c_1| = 2$, $|c_2| = 2$ and $c_1 \in Z(G)$. Since the center of $G/\langle c_1 \rangle$ contains $c_2\langle c_1 \rangle$, we have $g^{-1}c_2 g = c_2 c_1^i$ for any $g \in G$. This implies $c_2 g^2 = g^2 c_2$ for every $g \in G$. If $h \in G$ then $g^{-1}hg = hc_1^r c_2^s$ for some $r, s \in \{0, 1\}$. It follows that $g^{-2}hg^2 = hc_1^{is}$ and $g^{-2}h^2g^2 = h^2$. In the case $G' = \langle c_1 \rangle$, $|c_1| = p$ we obtain $g^{-1}c_1 g = c_1$, $g^{-1}hg = hc_1^r$, $g^{-p}hg^p = h$ for all $g, h \in G$.

Let S be the subgroup of H generated by G' and the elements $a_1^p, \dots, a_{m-1}^p$. Let $T = S/G'$ and

$$w = \sum_{i_1=0}^{l_1} \cdots \sum_{i_{m-1}=0}^{l_{m-1}} \alpha_{i_1,\dots,i_{m-1}}^p u_{a_1}^{pi_1} \dots u_{a_{m-1}}^{pi_{m-1}} \quad \text{(see (2))}.$$

Then $w \in K^\lambda S$ and

$$(wu_{a_m}^{p^{s_m-r+1}})^{p^{r-1}} \equiv u_e \pmod{K^\lambda H \cdot \mathrm{rad}\, KG'}.$$

It follows that $K^\lambda H/K^\lambda H \cdot \mathrm{rad}\, KG'$ is the group algebra of the cyclic group of order p^{r-1} over the field $L = (K^\lambda S + K^\lambda H \cdot \mathrm{rad}\, KG')/K^\lambda H \cdot \mathrm{rad}\, KG'$. Clearly, $L \cong K^\lambda S/K^\lambda S \cdot \mathrm{rad}\, KG' \cong K^\mu T$, where $\mu_{xG',yG'} = \lambda_{x,y}$ for all $x, y \in S$. Thus, $\dim_K \overline{K^\lambda H} = |T|$.

By Lemma 1.6, the algebra $K^\lambda H$ is not uniserial. Since $|H| = |T| \cdot |G'| \cdot p^{r-1}$ and $|G'| \cdot p^{r-1} \neq 4$, Lemma 1.9 shows that $K^\lambda H$ is wild. By Lemma 1.1, $K^\lambda G$ is semi-wild. ∎

LEMMA 1.11. *Let G be a non-abelian 2-group, $|G'| \geq 4$, K a field of characteristic 2, $\lambda \in Z^2(G, K^*)$, $d = \dim_K \overline{K^\lambda G}$ and $4d = |G : G'|$. If $K^\lambda G$ is not a uniserial algebra and $K^\lambda G/K^\lambda G\,\mathrm{rad}\, KG'$ is uniserial, then $K^\lambda G$ is of semi-wild representation type.*

Proof. Here we follow the proof of Lemma 1.10, and we keep the same notations with p and 2 interchanged. There exists an element ρ of the form (2) such that $\tilde{\rho}^4 = \gamma_m^{-1}\tilde{u}_e$, where $\tilde{\rho} \notin F^2$ if $4 < s_m$.

If G' is a non-cyclic group, we shall assume, by Lemma 1.7, that G' is the elementary abelian group of type $(2, 2)$. Let $G' = \langle c \rangle$, $B = \langle c^4 \rangle$, $N = \langle c^2 \rangle$, $V = K^\lambda G(u_c^2 - u_e)$ and $W = K^\lambda G(u_c^4 - u_e)$. By Lemma 1.5, $K^\lambda G/V$ is not a uniserial algebra. Since

$$(K^\lambda G/W)/(V/W) \cong K^\lambda G/V,$$

the algebra $K^\lambda G/W$ is not uniserial either. Moreover, $K^\lambda G/W \cong K^\nu \widehat{G}$, where $\widehat{G} = G/B$ and $\nu_{xB,yB} = \lambda_{x,y}$ for all $x, y \in G$. We also have

$$\widehat{G}' = G'/B = \langle cB \rangle, \quad |\widehat{G} : \widehat{G}'| = |G : G'| \quad \text{and} \quad \dim_K \overline{K^\nu \widehat{G}} = d.$$

This implies that $|G'| = 4$.

Denote by H the subgroup of G generated by G' and the elements

$$a_1^2, \ldots, a_{m-1}^2, a_m^{2^{s_m-1}}.$$

We show that H is abelian. In case G' is of type $(2, 2)$, this was established in the proof of Lemma 1.10. Let $G' = \langle c \rangle$ and $|c| = 4$. Then $c^2 \in Z(G)$. If $a \in G$ then $a^{-1}ca = c^i$, where $i \equiv 1 \pmod 2$. Let $b \in G$ and $b^{-1}ab = ac^r$. Then $b^{-1}a^2b = a^2c^{r(1+i)}$, and therefore $b^{-2}a^2b^2 = a^2$. We also have $a^{-2}ca^2 = c$.

Let S be the subgroup of H generated by G' and the elements $a_1^2, \ldots, a_{m-1}^2$. Let $T = S/G'$. The quotient algebra $K^\lambda H/K^\lambda H \cdot \mathrm{rad}\, KG'$ is the group algebra of the cyclic group of order 2 over the field $L \cong K^\mu T$. By Lemma 1.6, the algebra $K^\lambda H$ is not uniserial. Since $|H| = |T| \cdot 2|G'|$ and $|G'| = 4$, Lemma 1.9 shows that $K^\lambda H$ is wild. It follows, by Lemma 1.1, that $K^\lambda G$ is semi-wild. ∎

LEMMA 1.12. *Let $p \neq 2$, G be a p-group with cyclic commutant and D the subgroup of G such that $G' \subset D$ and $D/G' = \mathrm{soc}(G/G')$. Then $|D'| \leq p$.*

Proof. Let $G' = \langle c \rangle$, $|c| = p^m$ and $m \geq 2$. If $g \in D$ then $g^{-1}cg = c^r$, where $r \equiv 1 \pmod{p^{m-1}}$. It follows that $g^{-1}c^p g = c^p$. Let $a, b \in D$, $a^{-1}ca = c^r$ and $b^{-1}ab = ac^i$. Then $b^{-1}a^p b = a^p c^{it}$, where $t = 1 + r + \cdots + r^{p-1}$. It is easy to see that $t \equiv p \pmod{p^m}$. Hence $b^{-1}a^p b = a^p c^{ip}$.

Let $H = \langle c^p \rangle$. If $a^p \in H$ then $b^{-1}a^p b = a^p$, and we conclude that $ip \equiv 0 \pmod{p^m}$. If $a^p \notin H$ then we may assume that $a^p = c$. This implies that $b^{-1}ab = a^{1+ip}$. We have $b^p = a^{pj}$ for some j, since $b^p \in G'$. Therefore $b^{-p}ab^p = a$ and

$$b^{-p}ab^p = a^{(1+pi)^p}.$$

Hence, $(1 + pi)^p \equiv 1 \pmod{p^{m+1}}$. Thus $pi \equiv 0 \pmod{p^m}$ and $[a, b]^p = e$. ∎

LEMMA 1.13. *Let $p \neq 2$, G be a non-abelian p-group, $\overline{K}$ a non-perfect field of characteristic p, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G}$. Moreover, assume that $K^\lambda G$ is not a uniserial algebra, $pd = |G : G'|$ and $|G'| > p$. Then $K^\lambda G$ is a semi-wild algebra.*

Proof. If G' is non-cyclic then, by Lemmas 1.1 and 1.2, the algebra $K^\lambda G$ is semi-wild. Let $G' = \langle c \rangle$ and $T = \langle c^p \rangle$. Denote by D the subgroup of G such that $G' \subset D$ and $D/G' = \mathrm{soc}(G/G')$. By Lemma 1.12, D/T is abelian. In view of Lemma 1.5, we can assume that $|G'| = p$ and D is an abelian group. By Lemma 1.6, $K^\lambda D$ is not uniserial, since $K^\lambda G/K^\lambda G \cdot \mathrm{rad}\, KG'$ is uniserial and $K^\lambda D/K^\lambda D \cdot \mathrm{rad}\, KG'$ is not a field. According to Lemma 1.9, $K^\lambda D$ is wild. By Lemma 1.1, $K^\lambda G$ is semi-wild. ∎

PROPOSITION 1.14. *Let G be a finite group, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G_p}$. Assume that G_p is abelian and $|G_p| \neq 4d$. The algebra $K^\lambda G$ is semi-wild if and only if $K^\lambda G_p$ is not uniserial.*

Proof. If $K^\lambda G_p$ is not uniserial then, by Lemma 1.9, $K^\lambda G_p$ is wild. Hence, by Lemma 1.1, $K^\lambda G$ is semi-wild. If $K^\lambda G_p$ is uniserial then, by Lemma 1.3, $K^\lambda G$ is of finite representation type. ∎

PROPOSITION 1.15. *Let G be a finite group, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G_p}$. If $d = |G_p : G'_p|$ and $|G'_p : G''_p| \neq 4$, then the algebra $K^\lambda G$ is semi-wild if and only if G'_p is a non-cyclic group.*

Proof. Since $d = |G_p : G'_p|$, the algebra $K^\lambda G_p/K^\lambda G_p \cdot \mathrm{rad}\, KG'_p$ is a field and $K^\lambda G_p \cdot \mathrm{rad}\, KG'_p$ is the radical of $K^\lambda G_p$. If G'_p is cyclic then $K^\lambda G_p$ is uniserial, thus, by Lemma 1.3, $K^\lambda G$ is of finite representation type. Hence, $K^\lambda G$ is not semi-wild. If G'_p is a non-cyclic group then, by Lemma 1.2, $K^\lambda G'_p = KG'_p$ is a wild algebra. In view of Lemma 1.1, it follows that $K^\lambda G$ is semi-wild. ∎

We are now able to prove one of the main results of this paper.

Theorem 1.16. *Let $p \neq 2$, G be a finite group, $\lambda \in Z^2(G, K^*)$, μ the restriction of λ to $G_p \times G_p$ and $d = \dim_K \overline{K^\lambda G_p}$. Denote by D the subgroup of G_p such that $G'_p \subset D$ and $D/G'_p = \mathrm{soc}(G_p/G'_p)$. Assume that if $|G'_p| = p$ and $pd = |G_p : G'_p|$, then $\mathrm{Ker}(\mu) \neq G'_p$ or D is abelian. The algebra $K^\lambda G$ is of semi-wild representation type if and only if the subalgebra $K^\lambda G_p$ is not uniserial.*

Proof. If $K^\lambda G_p$ is uniserial then, by Lemma 1.3, $K^\lambda G$ is of finite representation type. Suppose that $K^\lambda G_p$ is not uniserial. If $pd < |G_p : G'_p|$ then, by Lemma 1.10, $K^\lambda G_p$ is semi-wild. If $pd = |G_p : G'_p|$ and $|G'_p| > p$ then, by Lemma 1.13, $K^\lambda G_p$ is also semi-wild. Let $d = |G_p : G'_p|$. Then $K^\lambda G_p / K^\lambda G_p \cdot \mathrm{rad}\, KG'_p = \overline{K^\lambda G_p}$, therefore G'_p is non-cyclic. According to Lemma 1.2, KG'_p is wild. By Lemma 1.1, in all cases $K^\lambda G$ is of semi-wild representation type.

Now assume that $pd = |G_p : G'_p|$ and $|G'_p| = p$. Then $\mathrm{Ker}(\mu) \neq G'_p$ or D is abelian. Let $H_p = \mathrm{Ker}(\mu)$ and $H_p \neq G'_p$. Denote by U the ideal $K^\lambda G_p \cdot \mathrm{rad}\, KH_p$ of the algebra $K^\lambda G_p$ and by $\widetilde{K^\lambda G_p}$ the quotient algebra $K^\lambda G_p/U$. Since $G'_p \subset H_p$, $G'_p \neq H_p$, $\dim_K \widetilde{K^\lambda G_p} = |G_p : H_p|$ and $\widetilde{K^\lambda G_p}/\mathrm{rad}\, \widetilde{K^\lambda G_p} \cong \overline{K^\lambda G_p}$, we have $|G_p : H_p| = d$ and $U = \mathrm{rad}\, K^\lambda G_p$. The algebra $K^\lambda G_p$ is not uniserial, hence U is not a principal left ideal. This implies that H_p is non-cyclic. By Lemma 1.2, KH_p is wild. According to Lemma 1.1, $K^\lambda G$ is semi-wild.

Finally, we examine the case when D is abelian. Since $K^\lambda G_p / K^\lambda G_p \cdot \mathrm{rad}\, KG'_p$ is a uniserial algebra and $K^\lambda D / K^\lambda D \cdot \mathrm{rad}\, KG'_p$ is not a field, Lemma 1.6 shows that $K^\lambda D$ is a non-uniserial algebra. By Lemma 1.9, $K^\lambda D$ is wild and, by Lemma 1.1, $K^\lambda G$ is semi-wild. ∎

Theorem 1.17. *Let G be a finite group, K a field of characteristic 2, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G_2}$. Assume that one of the following three conditions holds:*

(i) *$4d < |G_2 : G'_2|$;*
(ii) *$4d = |G_2 : G'_2|$, $|G'_2| \geq 4$ and $K^\lambda G_2 / K^\lambda G_2 \cdot \mathrm{rad}\, KG'_2$ is a uniserial algebra;*
(iii) *$d = |G_2 : G'_2|$ and $|G'_2 : G''_2| \neq 4$.*

The algebra $K^\lambda G$ is semi-wild if and only if $K^\lambda G_2$ is not uniserial.

Proof. Apply Lemmas 1.1, 1.3, 1.9 – 1.11 and Proposition 1.15. ∎

Proposition 1.18. *Let G be a finite group, K a field of characteristic 2, $\lambda \in Z^2(G, K^*)$, μ the restriction of λ to $G_2 \times G_2$ and $H_2 = \mathrm{Ker}(\mu)$. Assume that H_2 is non-cyclic and $|H_2 : H'_2| \neq 4$. Then $K^\lambda G$ is semi-wild.*

Proof. Apply Lemmas 1.1 and 1.2. ∎

Proposition 1.19. *Assume that $p \neq 2$ and keep the notation of Theorem 1.16. Assume also that if $|C_p| = |G'_p| = p$ and $s \leq i(K) + 1$, then D is abelian. Then, for every $\lambda \in Z^2(G, K^*)$, $K^\lambda G$ is of finite or semi-wild representation type.*

Proof. Let $\lambda \in Z^2(G, K^*)$, $d = \dim_K \overline{K^\lambda G_p}$, $pd = |G_p : G'_p|$ and $U = K^\lambda G_p \cdot \operatorname{rad} KG'_p$. We have $K^\lambda G_p/U \cong K^\nu H_p$, where $H_p = G_p/G'_p$ and $\nu_{xG'_p, yG'_p} = \lambda_{x,y}$ for all $x, y \in G_p$. Since $\dim_K K^\nu H_p = |H_p| = pd$ and $d = \dim_K \overline{K^\nu H_p}$, $K^\nu H_p$ is uniserial. Hence $s \leq i(K) + 1$. Denote by μ the restriction of λ to $G_p \times G_p$. We have $C_p \subset \operatorname{Ker}(\mu)$. If $|G'_p| = p$ and $C_p \neq G'_p$, then $\operatorname{Ker}(\mu) \neq G'_p$. Now apply Lemma 1.3 and Theorem 1.16. ∎

2. Groups of purely semi-wild projective representation type at characteristic $p \neq 2$. We say that a finite group G is of *tame projective K-representation type* if $K^\lambda G$ is of tame representation type for every $\lambda \in Z^2(G, K^*)$. A group G is said to be of *semi-wild projective K-representation type* if $K^\lambda G$ is of semi-wild representation type for some $\lambda \in Z^2(G, K^*)$. A group G is defined to be of *purely semi-wild projective K-representation type* if $K^\lambda G$ is of semi-wild representation type for any $\lambda \in Z^2(G, K^*)$.

Proposition 2.1. *Let G be a finite group and K a field of characteristic $p \geq 2$. The group G is of semi-wild projective K-representation type if and only if G_p is non-cyclic and $|G_p : G'_p| \neq 4$. Otherwise G is of tame projective K-representation type.*

Proof. If G_p is non-cyclic and $|G_p : G'_p| \neq 4$, then, by Lemmas 1.1 and 1.2, KG is semi-wild. Hence G is of semi-wild projective K-representation type. Assume that G_p is cyclic. For every $\lambda \in Z^2(G, K^*)$, the subalgebra $K^\lambda G_p$ of $K^\lambda G$ is uniserial. It now follows from Lemma 1.3 that $K^\lambda G$ is of finite representation type. Now let $p = 2$, G_2 be non-cyclic and $|G_2 : G'_2| = 4$. For every $\lambda \in Z^2(G, K^*)$, there exists a finite purely inseparable field extension F of K such that $F \otimes_K K^\lambda G_2 \cong FG_2$. By Lemma 1.2, FG_2 is a tame algebra. This implies that $K^\lambda G_2$ is tame (see [10, p. 247]). Applying Lemma 1.1, we conclude that $K^\lambda G$ is tame for any $\lambda \in Z^2(G, K^*)$. Hence G is of tame projective K-representation type. ∎

Proposition 2.2. *Let G be a finite group and K a perfect field of characteristic $p \geq 2$. The group G is of purely semi-wild projective K-representation type if and only if G_p is non-cyclic and $|G_p : G'_p| \neq 4$. Otherwise G is of tame projective K-representation type.*

Proof. Since K is a perfect field, $K^\lambda G_p$ is the group algebra of G_p over K for every $\lambda \in Z^2(G, K^*)$ (see [15, p. 90] or [16, p. 43]). If G_p is non-cyclic and

$|G_p : G'_p| \neq 4$, then, by Lemma 1.2, KG_p is wild. It follows, by Lemma 1.1, that $K^\lambda G$ is semi-wild for any λ. Hence G is of purely semi-wild projective K-representation type. If G_p is cyclic or $p = 2$ and $|G_2 : G'_2| = 4$, then, in view of Proposition 2.1, G is of tame projective K-representation type. ∎

Let G be a finite group, G' the commutant of G, G_p a Sylow p-subgroup of G and C_p a Sylow p-subgroup of G'. We assume that $C_p \subset G_p$. Then $G'_p \subset C_p$, and hence $C_p \triangleleft G_p$. We have $G_pG'/G' \cong G_p/C_p$, since $G_p \cap G' = C_p$. The group G_pG'/G' is the Sylow p-subgroup of the abelian group G/G'. Denote by A a normal subgroup of G_p such that $C_p \subset A$. Let $\psi : G \to G/G'$ be the canonical homomorphism, $\chi : G/G' \to G_pG'/G'$ a projector, and $\phi : G_pG'/G' \to G_p/A$ the epimorphism defined by $\phi(xG') = xA$ for any $x \in G_p$. Then

$$f := \phi\chi\psi : G \to G_p/A \tag{3}$$

is a surjective group homomorphism. Moreover, the restriction of f to G_p is the canonical homomorphism $\pi : G_p \to G_p/A$.

Lemma 2.3. *Assume that G is a finite group, $H = G_p/A$, $f : G \to H$ is the epimorphism* (3)*, $\nu \in Z^2(H, K^*)$ and $\lambda_{a,b} = \nu_{f(a),f(b)}$ for any $a, b \in G$.*

(i) *$\lambda \in Z^2(G, K^*)$ and $\lambda_{x,y} = \lambda_{y,x} = 1$ for all $x \in G_p$, $y \in A$.*
(ii) *If μ is the restriction of λ to $G_p \times G_p$, then $\mu_{a,b} = \nu_{\pi(a),\pi(b)}$ for all $a, b \in G_p$ and $\mathrm{Ker}(\mu) = \pi^{-1}(\mathrm{Ker}(\nu))$.*
(iii) *If $V = K^\lambda G_p \cdot \mathrm{rad}\, KA$ then V is an ideal of $K^\lambda G_p$ and $K^\lambda G_p/V \cong K^\nu H$.*

Proof. Statements (i) and (iii) are obvious.

(ii) By Proposition 2.1 of [2], $\mathrm{Ker}(\nu) = \mathrm{Ker}(\Gamma)$, where Γ is an irreducible ν-representation of the group H over the field K. Since $\Gamma \circ \pi$ is an irreducible μ-representation of the group G_p over K, we get $\mathrm{Ker}(\mu) = \pi^{-1}\big(\mathrm{Ker}(\nu)\big)$. ∎

Proposition 2.4. *Let G be a finite group such that C_p is cyclic, A a cyclic subgroup of G_p, $C_p \subset A$ and r the number of invariants of G_p/A. If $r \leq i(K)$ then there exists a cocycle $\lambda \in Z^2(G, K^*)$ such that $K^\lambda G_p$ is a uniserial algebra.*

Proof. Let $H = G_p/A$. Since $r \leq i(K)$, there exists a cocycle $\nu \in Z^2(H, K^*)$ such that $K^\nu H$ is a field. In view of Lemma 2.3, there exists a cocycle $\lambda \in Z^2(G, K^*)$ satisfying the following conditions:

- if μ is the restriction of λ to $G_p \times G_p$ then $\mathrm{Ker}(\mu) = A$;
- if $V = K^\lambda G_p \cdot \mathrm{rad}\, KA$ then V is the radical of $K^\lambda G_p$ and $K^\lambda G_p/V \cong K^\nu H$.

Since A is cyclic, V is a principal left ideal and therefore $K^\lambda G_p$ is uniserial. ∎

COROLLARY 2.5. *Let G be a finite group with C_p cyclic. If G is of purely semi-wild projective K-representation type, then $G_p/C_p = \langle a_1C_p\rangle \times \cdots \times \langle a_rC_p\rangle$, where $r \geq i(K)+1$ and if $r = i(K)+1$ then $C_p \not\subset \langle a_j\rangle$ for every $j \in \{1,\ldots,r\}$.*

Proof. Assume that $r = i(K)+1$ and $C_p \subset \langle a_{j_0}\rangle$ for some $j_0 \in \{1,\ldots,r\}$. Let $A = \langle a_{j_0}\rangle$. Since the number of invariants of G_p/A is at most $i(K)$, there exists, by Proposition 2.4, a cocycle $\lambda \in Z^2(G,K^*)$ such that $K^\lambda G_p$ is a uniserial algebra. Hence G is not of purely semi-wild projective K-representation type. ∎

LEMMA 2.6. *Let G be an elementary abelian p-group, s the number of invariants of G, K a field of characteristic p and $\lambda \in Z^2(G,K^*)$. If $s = i(K)+r$ then $K^\lambda G \cong K^\lambda D \otimes_K KT$, where $G = D\times T$ and $|T| \geq p^r$.*

Proof. Since $s > i(K)$, $K^\lambda G$ is not a field. Assume that $K^\lambda G$ is not the group algebra of G over K. There exists a direct product decomposition $G = D\times\langle c_1\rangle\times\cdots\times\langle c_m\rangle$ such that $K^\lambda D$ is a field and $K^\lambda T_j$ is not a field for every $j \in \{1,\ldots,m\}$, where $T_j = D\times\langle c_j\rangle$. It follows that $K^\lambda G \cong K^\lambda D \otimes_K KT$, where $T = \langle c_1\rangle\times\cdots\times\langle c_m\rangle$. The number of invariants of the group D is at most $i(K)$, hence $m \geq r$. ∎

Now we are able to prove the main result of this section.

THEOREM 2.7. *Let $p \neq 2$, G be a finite group and D the subgroup of G_p such that $G'_p \subset D$ and $D/G'_p = \operatorname{soc}(G_p/G'_p)$. Assume that if $|C_p| = p$, $s = i(K)+1$ and D is non-abelian then $\exp D = p^2$. Then G is of purely semi-wild projective K-representation type if and only if one of the following four conditions is satisfied:*

- (i) *C_p is a non-cyclic group;*
- (ii) *$s \geq i(K)+2$;*
- (iii) *$s = i(K)+1$, $C_p = G'_p = \langle c\rangle$, $|c| \geq p^2$ and $g^p \in \langle c^p\rangle$ for every $g \in D$;*
- (iv) *$s = i(K)+1$, $C_p = G'_p$, $|G'_p| = p$ and D is an elementary abelian p-group.*

Proof. First we prove that if G satisfies one of conditions (i)–(iv), then it is of purely semi-wild projective K-representation type. Let $\lambda \in Z^2(G,K^*)$. The algebra $K^\lambda G_p$ contains the group algebra $KC_p = K^\lambda C_p$. If C_p is non-cyclic then, by Lemma 1.2, KC_p is wild. In view of Lemma 1.1, $K^\lambda G$ is semi-wild.

Assume that $s \geq i(K)+2$. Since s is also the number of invariants of the group $\bar{D} = D/G'_p$, by Lemma 2.6, we have $K^\mu\bar{D} \cong K^\mu B_1 \otimes_K KB_2$, where $\bar{D} = B_1\times B_2$ and $|B_2| \geq p^2$; moreover, $\mu_{xG'_p,yG'_p} = \lambda_{x,y}$ for any $x,y \in D$.

According to Lemmas 1.2 and 1.8, the algebra $K^\mu \overline{D}$ is wild, hence $K^\lambda D$ is also wild, and it follows from Lemma 1.1 that $K^\lambda G$ is semi-wild.

Now we suppose that (iii) holds. Let $T = \langle c^p \rangle$. By Lemma 1.12, $H = D/T$ is an abelian group. Let $D/G'_p = \langle b_1 G'_p \rangle \times \cdots \times \langle b_s G'_p \rangle$, $s = i(K) + 1$. Then $H = \langle cT \rangle \times \langle b_1 T \rangle \times \cdots \times \langle b_s T \rangle$. If $V = K^\lambda D \cdot \operatorname{rad} KT$ then $K^\lambda D/V \cong K^\mu H$, where $\mu_{xT,yT} = \lambda_{x,y}$ for all $x, y \in D$. Denote by $\overline{H}$ the socle of H. By Lemma 2.6, $K^\mu \overline{H} \cong K^\mu N_1 \otimes_K KN_2$, where $\overline{H} = N_1 \times N_2$ and $|N_2| \geq p^2$. In view of Lemmas 1.2 and 1.8, $K^\mu \overline{H}$ is wild. Applying again Lemma 1.8 we deduce that $K^\mu H$ is wild, hence so is $K^\lambda D$. By Lemma 1.1, $K^\lambda G$ is semi-wild.

If G satisfies (iv) then D is a direct product of $s + 1 = i(K) + 2$ cyclic groups of order p. According to Lemmas 2.6, 1.2 and 1.8, $K^\lambda D$ is wild. Hence, by Lemma 1.1, $K^\lambda G$ is semi-wild.

Conversely, let G be of purely semi-wild projective K-representation type. If $C_p = \langle c \rangle$ then, by Corollary 2.5, $G_p/C_p = \langle a_1 C_p \rangle \times \cdots \times \langle a_r C_p \rangle$, where $r \geq i(K) + 1$. We also have $s \geq r$. Let $r = i(K) + 1$. By Corollary 2.5, $C_p \neq \{e\}$ and

$$a_j^{|a_j C_p|} \in \langle c^p \rangle$$

for any $j \in \{1, \ldots, r\}$. Let $C_p \neq G'_p$ and $T = \langle c^p \rangle$. Then $G'_p \subset T$ and

$$G_p/T = \langle cT \rangle \times \langle a_1 T \rangle \times \cdots \times \langle a_r T \rangle.$$

Since $G_p/T \cong (G_p/G'_p)/(T/G'_p)$, the number of invariants of the group G_p/T is at most s. This implies $r + 1 \leq s$. Hence $s \geq i(K) + 2$. This means that (ii) holds. Assume now that $C_p = G'_p$. If $|c| \geq p^2$, we have (iii). If $|c| = p$ then $g^p = e$ for every $g \in D$. By hypothesis, D is abelian and (iv) follows. ■

Corollary 2.8. *Let $p \neq 2$, G be a finite group and $[K : K^p] = \infty$. The group G is of purely semi-wild projective K-representation type if and only if C_p is non-cyclic.*

Corollary 2.9. *Let $p \neq 2$, G be a finite group such that G_p is abelian. The group G is of purely semi-wild projective K-representation type if and only if C_p is non-cyclic or $s \geq i(K) + 2$.*

3. Groups of purely semi-wild projective representation type at characteristic 2. In this section we assume that K is a field of characteristic 2, G a finite group, $2 \mid |G|$, G_2 a Sylow 2-subgroup of G, C_2 a Sylow 2-subgroup of the commutant G' of G and $C_2 \subset G_2$. Denote by s the number of invariants of the abelian group G_2/G'_2.

Lemma 3.1. *Let G be an abelian 2-group of exponent 4, $\overline{G} = \operatorname{soc} G$, $s = i(K) + 2$ and $\lambda \in Z^2(G, K^*)$. If G has at most one invariant equal to 2* *then one of the following conditions holds:*

(i) $K^\lambda \overline{G} \cong K^\lambda B \otimes_K KC$, *where* $\overline{G} = B \times C$ *and* $|C| \geq 8$*;*
(ii) $K^\lambda G \cong K^\lambda D \otimes_K KT$, *where* $G = D \times T$ *and* $|T| \geq 8$.

Proof. If K is a perfect field then $K^\lambda G = KG$ (see [15, p. 90] or [16, p. 43]). In this case $i(K) = 0$, $G = T$ and (ii) holds. Let K be a non-perfect field, $m = i(K)$, $G = \langle a_1 \rangle \times \cdots \times \langle a_s \rangle$ and

$$K^\lambda G = \bigoplus_{i_1=0}^{|a_1|-1} \cdots \bigoplus_{i_s=0}^{|a_s|-1} K u_{a_1}^{i_1} \ldots u_{a_s}^{i_s}, \quad u_{a_j}^{|a_j|} = \gamma_j u_e,$$

where $\gamma_j \in K^*$ for $j = 1, \ldots, s$. If $K^\lambda \overline{G}$ is not the group algebra of $\overline{G}$ over K, then $K^\lambda \overline{G} \cong K^\lambda B \otimes_K KC$, where $K^\lambda B$ is a field and $|B| \leq 2^m$. Since $s = m + 2$, we get $|C| \geq 4$. If $|C| > 4$, then (i) holds. Assume that $|C| = 4$. Renumbering $a_1, \ldots, a_s$ if needed, we may suppose that $\gamma_1, \ldots, \gamma_m$ are 2-independent. Let $D = \langle a_1 \rangle \times \cdots \times \langle a_m \rangle$.

First we consider the case when $|a_j| = 4$ for all $j \in \{1, \ldots, m\}$. Denote by θ_j a root of the polynomial $X^4 - \gamma_j$ for $j = 1, \ldots, m$. For any $\delta \in K^*$, the set $\{\gamma_1, \ldots, \gamma_m, \delta\}$ is not 2-independent. Therefore $\delta = \theta^2$ for some $\theta \in K(\theta_1^2, \ldots, \theta_m^2)$. We have

$$\theta = \sum_{i_1=0}^{1} \cdots \sum_{i_m=0}^{1} \beta_{i_1,\ldots,i_m} \theta_1^{2i_1} \ldots \theta_m^{2i_m},$$

where $\beta_{i_1,\ldots,i_m} \in K$. Since $\beta_{i_1,\ldots,i_m} = \rho_{i_1,\ldots,i_m}^2$ for some $\rho_{i_1,\ldots,i_m} \in K(\theta_1^2, \ldots, \theta_m^2)$, we obtain

$$\theta = \Big(\sum_{i_1=0}^{1} \cdots \sum_{i_m=0}^{1} \rho_{i_1,\ldots,i_m} \theta_1^{i_1} \ldots \theta_m^{i_m} \Big)^2.$$

Hence $\delta = z^4$ for some $z \in K(\theta_1, \ldots, \theta_m)$. The element z is of the form

$$z = \sum_{j_1=0}^{3} \cdots \sum_{j_m=0}^{3} \alpha_{j_1,\ldots,j_m} \theta_1^{j_1} \ldots \theta_m^{j_m}, \quad \alpha_{j_1,\ldots,j_m} \in K.$$

We put

$$w = \sum_{j_1=0}^{3} \cdots \sum_{j_m=0}^{3} \alpha_{j_1,\ldots,j_m} u_{a_1}^{j_1} \ldots u_{a_m}^{j_m}.$$

Then $w \in K^\lambda D$ and $w^4 = \delta u_e$. It follows that $K^\lambda G \cong K^\lambda D \otimes_K KT$, where $T = \langle a_{m+1} \rangle \times \langle a_{m+2} \rangle$ and $|T| \geq 8$.

Now we examine the case when $|a_m| = 2$. By previous arguments, we may assume that $\gamma_1, \ldots, \gamma_{m-1}, \gamma_i$ are 2-dependent for every $i \in \{m+1, m+2\}$. Denote by θ_j a root of the polynomial $X^4 - \gamma_j$ for every $j \in \{1, \ldots, m-1\}$

and by θ_m a root of the polynomial $X^2 - \gamma_m$. Then $\gamma_i = \rho_i^2$ for some

$$\rho_i = \sum_{i_1=0}^{1} \cdots \sum_{i_m=0}^{1} \alpha_{i,i_1,\dots,i_{m-1}} \theta_1^{2i_1} \dots \theta_{m-1}^{2i_{m-1}},$$

where $\alpha_{i,i_1,\dots,i_{m-1}} \in K$. But $\alpha_{i,i_1,\dots,i_{m-1}} = w_{i,i_1,\dots,i_{m-1}}^2$, where $w_{i,i_1,\dots,i_{m-1}}$ is an element of the field $K(\theta_1^2, \dots, \theta_{m-1}^2, \theta_m)$. This implies $\rho_i = \delta_i^2$ for some $\delta_i \in K(\theta_1, \dots, \theta_{m-1}, \theta_m)$, hence $\gamma_i = \delta_i^4$. Consequently, $\gamma_i u_e = v_i^4$ for some $v_i \in K^\lambda D$. Therefore $K^\lambda G \cong K^\lambda D \otimes_K KT$, where $T = \langle a_{m+1}\rangle \times \langle a_{m+2}\rangle$ and $|T| = 16$. ∎

Our final main result of this paper is the following theorem.

THEOREM 3.2. *Let K be a field of characteristic 2 and G a finite group such that G_2 is abelian and C_2 is cyclic. Then G is of purely semi-wild projective K-representation type if and only if one of the following conditions is satisfied:*

(i) $s \geq i(K) + 3$;
(ii) $s = i(K) + 2$ *and* $|C_2| \geq 4$;
(iii) $s = i(K) + 2$, $|C_2| \leq 2$ *and* G_2 *has at most one invariant equal to* 2.

Proof. By our assumption, we have $C_2 = \langle c\rangle$. To prove the necessity, we assume that G is of purely semi-wild projective K-representation type. By Corollary 2.5, $G_2/C_2 = \langle a_1C_2\rangle \times \dots \times \langle a_rC_2\rangle$, where $r \geq i(K) + 1$. We also have $s \geq r$. Let $r = i(K) + 1$. By Corollary 2.5, $C_2 \neq \{e\}$ and

$$a_j^{|a_jC_2|} \in \langle c^2\rangle$$

for every $j \in \{1, \dots, r\}$. Let $T = \langle c^2\rangle$. Then $G_2/T = \langle cT\rangle \times \langle a_1T\rangle \times \dots \times \langle a_rT\rangle$. It follows that $r + 1 \leq s$. Consequently, $s \geq i(K) + 2$. Assume that $s = i(K) + 2$ and $G_2 = A \times B$, where A is the group of type $(2, 2)$ and $C_2 \subset A$. There exists $\nu \in Z^2(B, K^*)$ such that $K^\nu B$ is a field. In view of Lemma 2.3, there exists $\lambda \in Z^2(G, K^*)$ satisfying the following condition: if μ is the restriction of λ to $G_2 \times G_2$ then $\mathrm{Ker}(\mu) = A$. The algebra $K^\lambda G_2$ is the group algebra of A over the field $K^\nu B$. By Lemma 1.2, $K^\lambda G_2$ is tame. Hence, by Lemma 1.1, $K^\lambda G$ is also tame.

To prove the sufficiency, we assume that $s \geq i(K) + 3$. Denote by $\overline{G}_2$ the socle of the group G_2. By Lemma 2.6, $K^\lambda \overline{G}_2 \cong K^\lambda D \otimes_K KT$, where $\overline{G}_2 = D \times T$ and $|T| \geq 8$. In view of Lemmas 1.2 and 1.8, the algebra $K^\lambda \overline{G}_2$ is wild, and it follows from Lemma 1.8 that so is $K^\lambda G_2$ for all $\lambda \in Z^2(G, K^*)$.

Now we assume that $s = i(K) + 2$. Let $H = \{g \in G_2 : g^4 = e\}$. If G_2 has at most one invariant equal to 2 then, by Lemmas 3.1, 1.2 and 1.8, $K^\lambda H$ is wild. Applying again Lemma 1.8, we deduce that $K^\lambda G_2$ is wild for any $\lambda \in Z^2(G, K^*)$. Suppose that $|C_2| \geq 4$. There exists a direct decomposition

$H = A \times B$ such that $B \subset C_2$ and $|B| = 4$. Let $N = \operatorname{soc} A \times B$. By Lemma 2.6, $K^\lambda N \cong K^\lambda D \otimes_K KT$, where $B \subset T$ and $|T| = 8$. According to Lemmas 1.2 and 1.8, $K^\lambda G_2$ is wild for any $\lambda \in Z^2(G, K^*)$. ∎

PROPOSITION 3.3. *Let G be a finite group such that one of the following conditions is satisfied:*

(i) *C_2 is non-cyclic and $|C_2 : C_2'| \neq 4$;*
(ii) *$s \geq i(K) + 3$;*
(iii) *$s = i(K) + 2$ and $|C_2 : G_2'| \geq 4$;*
(iv) *$s = i(K) + 2$ and G_2/G_2' has at most one invariant equal to 2.*

Then G is of purely semi-wild projective K-representation type.

Proof. For any $\lambda \in Z^2(G, K^*)$, $K^\lambda G$ contains the group algebra $KC_2 = K^\lambda C_2$ and $K^\lambda G_2/K^\lambda G_2 \cdot \operatorname{rad} KG_2' \cong K^\mu H$, where $H = G_2/G_2'$. It remains to apply Lemmas 1.1, 1.2 and Theorem 3.2. ∎

COROLLARY 3.4. *Let G be a finite group and K a field of characteristic 2. Assume that $|C_2 : C_2'| \neq 4$ and $[K : K^2] = \infty$. The group G is of purely semi-wild projective K-representation type if and only if C_2 is not cyclic.*

Proof. Apply Corollary 2.5 and Proposition 3.3. ∎

LEMMA 3.5. *Let G be a non-abelian 2-group with cyclic commutant G' and D the subgroup of G such that $G' \subset D$ and $D/G' = \operatorname{soc}(G/G')$.*

(i) *If $|G' \cap Z(D)| \geq 4$ then $|D'| \leq 4$.*
(ii) *If $G' \subset Z(D)$ then $|D'| \leq 2$.*

Proof. (i) Let $G' = \langle c \rangle$ and $|c| = 2^m$. If $m = 2$ then $G' \subset Z(D)$. First we examine the case when $m > 2$. If $g \in D$ then $g^{-1}cg = c^r$, where $r \equiv 1 \pmod{2^{m-1}}$. It follows that $g^{-1}c^2g = c^2$. Suppose that $a, b \in D$, $a^{-1}ca = c^r$ and $b^{-1}ab = ac^i$. Then $b^{-1}a^2b = a^2c^{i(1+r)}$. Let $H = \langle c^2 \rangle$. If $a^2 \in H$ then $b^{-1}a^2b = a^2$, hence $i(1 + r) \equiv 0 \pmod{2^m}$. This implies $2i \equiv 0 \pmod{2^m}$. If $a^2 \notin H$, we may assume that $a^2 = c$ and $r = 1$. Then $b^{-1}c^2b = c^2$ and $b^{-1}c^2b = c^2c^{4i}$, which yields $c^{4i} = e$.

(ii) Now let $G' \subset Z(D)$. Then $b^{-1}a^2b = a^2c^{2i}$ and $b^{-1}a^2b = a^2$, hence $c^{2i} = e$. ∎

PROPOSITION 3.6. *Let G be a non-abelian 2-group with cyclic commutant $G' = \langle c \rangle$ and D the subgroup of G such that $G' \subset D$ and $D/G' = \operatorname{soc}(G/G')$. Assume that $s = i(K) + 1$ and $|G' : D'| \geq 4$. The group G is of purely semi-wild projective K-representation type if and only if $g^2 \in \langle c^2 \rangle$ for every $g \in D$.*

Proof. Let $N = \langle c^2 \rangle$. Since $|G' : N| = 2$ and $|G' : D'| \geq 4$, we have $D' \subset N$. Suppose that G is of purely semi-wild projective K-representation type. By Corollary 2.5, $D/G' = \langle d_1G' \rangle \times \cdots \times \langle d_sG' \rangle$, where $d_j^2 \in N$ for every $j \in \{1, \dots, s\}$. Since $D/N = \langle cN \rangle \times \langle d_1 N \rangle \times \cdots \times \langle d_s N \rangle$, $g^2 \in N$ for every $g \in D$.

Conversely, assume that $h^2 \in N$ for every $h \in D$. If $g \in D$ then $g^{-1}cg = c^r$, where $c^{r-1} \in D'$. Let $g^2 = c^{2t}$, $t \in \mathbb{Z}$. We get

$$(gc^{-t})^2 = g^2 c^{-rt-t} = c^{t(1-r)}.$$

Therefore $(gc^{-t})^2 \in D'$. As a consequence,

$$D/G' = \langle x_1 G' \rangle \times \cdots \times \langle x_s G' \rangle,$$

where $x_j^2 \in D'$ for every $j \in \{1, \dots, s\}$. Let $T = \langle c^4 \rangle$. Then $D' \subset T$ and

$$D/T = \langle cT \rangle \times \langle x_1 T \rangle \times \cdots \times \langle x_s T \rangle.$$

Let $H = D/T$, $\lambda \in Z^2(G, K^*)$ and $\mu_{xT,yT} = \lambda_{x,y}$ for all $x, y \in D$. Then $\mu \in Z^2(H, K^*)$ and $\langle cT \rangle \subset \operatorname{Ker}(\mu)$. Let $Q = \langle x_1 T \rangle \times \cdots \times \langle x_s T \rangle$. According to Lemma 2.6, $K^\mu Q \cong K^\mu Q_1 \otimes_K KQ_2$, where $|Q_2| \geq 2$. It follows that $K^\mu H \cong K^\mu H_1 \otimes_K KH_2$, where $H_1 = Q_1$ and $H_2 = \langle cT \rangle \times Q_2$. Since $|H_2| \geq 8$, we deduce from Lemmas 1.2 and 1.8 that $K^\mu H$ is wild. Therefore, $K^\lambda D$ is wild for any $\lambda \in Z^2(G, K^*)$. Applying Lemma 1.1, we conclude that G is of purely semi-wild projective K-representation type. ■

Corollary 3.7. *Let G be a non-abelian 2-group with cyclic commutant $G' = \langle c \rangle$ of order 2^m, D the subgroup of G such that $G' \subset D$ and $D/G' = \operatorname{soc}(G/G')$. Assume that $s = i(K) + 1$ and one of the following conditions holds:*

(i) *$m \geq 4$ and $|G' \cap Z(D)| \geq 4$;*
(ii) *$m = 3$ and $G' \subset Z(D)$.*

The group G is of purely semi-wild projective K-representation type if and only if $g^2 \in \langle c^2 \rangle$ for every $g \in D$.

Proof. Apply Lemma 3.5 and Proposition 3.6. ■

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Received 27 October 2009;
revised 21 December 2009 (5305)