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DYNAMICALLY DEFINED CANTOR SETS UNDER THE CONDITIONS OF MCDUFF'S CONJECTURE

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Abstract. We prove that if the Cantor set K, dynamically defined by a function $S \in C^{1+\alpha}$, satisfies the conditions of McDuff's conjecture then it cannot be C^1-minimal.

1. Introduction and main result. If $f : S^1 \to S^1$ is a C^1-diffeomorphism without periodic points, then there exists a unique set $\Omega(f) \subset S^1$ minimal for f (we say that $\Omega(f)$ is C^1-*minimal* for f). In this case $\Omega(f)$ is either a Cantor set or S^1. Examples of C^1-minimal Cantor sets are Denjoy's examples and their conjugates ([1]). In [2] McDuff proves that the usual middle thirds Cantor set is not C^1-minimal, and in [3] A. Norton also proves that affine Cantor sets are not C^1-minimal.

Let K be a Cantor subset of the circle and let $K^c = \bigcup I_j$, where I_j is a connected component of K^c . We define the *spectrum* of K, denoted by E_K, as the ordered set $\{\lambda_i\}$ ($\lambda_{i+1} < \lambda_i$), where λ_i is the length of I_j for some j. In [2] McDuff conjectures that if $\lambda_n/\lambda_{n+1} \nrightarrow 1$ then the Cantor set K is not C^1-minimal.

Let $I_1, \ldots, I_k$, $k \geq 2$, be pairwise disjoint compact intervals in $\mathbb{R}$, and let L be a compact interval containing their union $I \equiv I_1 \cup \cdots \cup I_k$. Define $\mathcal{S}^r(I_1, \ldots, I_k, L)$, $r \geq 0$, to be the set of C^r functions $S : I \to L$ such that for $j = 1, \ldots, k$, $S(I_j) = L$. For $S \in \mathcal{S}^r(I_1, \ldots, I_k, L)$ define

$$C_S = \{x \in I : S^k(x) \in I \text{ for all } k \in \mathbb{Z}^+\}.$$

Note that not every C_S is a Cantor set (see Figure 1) and $C_{S^2} = C_S$. Also note that every Cantor set K is a C_S Cantor set for some function $S \in C^0$. We say that a Cantor set K is *dynamically defined* by S if $K = C_S$ for some $S \in \mathcal{S}^r(I_1, \ldots, I_k, L)$. If $S \in \mathcal{S}^r(I_1, \ldots, I_k, L)$ and $|S'(x)| > 1$ for all $x \in I$, then C_S is a Cantor set, and these sets are called *hyperbolic*. If S' is locally constant, C_S is called *affine*, and if S' is globally constant, C_S is called *linear*. In [3] Norton proves that affine Cantor sets are not C^1-minimal. In this work we will consider $S \in \mathcal{S}^{1+\alpha}(I_1, \ldots, I_k, L)$ (here S is not necessarily

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monotone) such that C_S is a Cantor set with $\lambda_n/\lambda_{n+1} \nrightarrow 1$. We will prove that these Cantor sets are not C^1-minimal. Our long-term goal is to prove McDuff's conjecture, that is why we require $\lambda_n/\lambda_{n+1} \nrightarrow 1$. For this work we identified the end points of L, so we can suppose that $K \subset S^1$.

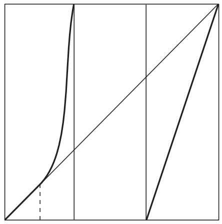

Fig. 1. Note that $K_1 \subset C_S$

We prove the following result:

THEOREM A. *Let K be a Cantor set dynamically defined by $S \in \mathcal{S}^0(I_1, \ldots, I_k, L)$ such that $\lambda_n/\lambda_{n+1} \nrightarrow 1$. If there exist a fixed point x_0 of S and an open interval J containing x_0 such that $S|_{J\cap I} \in C^{1+\alpha}$ for some $\alpha > 0$, then K is not C^1-minimal.*

2. Previous results. The following proposition contains facts that can be easily verified.

REMARK 1. If K is a Cantor set with $K = C_S$ for $S \in \mathcal{S}^1(I_1, \ldots, I_k, L)$, then:

(1) Each I_i contains at least one fixed point of S.
(2) If x_0 is a fixed point of S then $|S'(x_0)| \geq 1$ and therefore $(S^2)'(x_0) \geq 1$.
(3) The set of fixed points of S is finite.
(4) For each fixed point x_0 of S, there exists δ_0 such that:
 (i) If $x \in (x_0, x_0 + \delta_0] \cap I$ then $S^2(x) > x$.
 (ii) If $x \in [x_0 - \delta_0, x_0) \cap I$ then $S^2(x) < x$.

LEMMA 1. *Let K be a Cantor set with $K = C_S$ for $S \in \mathcal{S}^1(I_1, \ldots, I_k, L)$ and $\lambda_n/\lambda_{n+1} \nrightarrow 1$. If x_0 is a fixed point of S then $(S^2)'(x_0) > 1$.*

Proof. Suppose that there exists a fixed point x_0 such that $(S^2)'(x_0) \leq 1$. From Remark 1(2) we have $(S^2)'(x_0) = 1$. As $\lambda_n/\lambda_{n+1} \nrightarrow 1$ there exist $\{n_j\}$ and $\varepsilon > 0$ such that

$$\frac{\lambda_{n_j}}{\lambda_{n_j+1}} > 1 + \varepsilon. \tag{1}$$

Let $\delta > 0$ be such that:

- if $x \in (x_0 - \delta, x_0 + \delta) \cap I$ then $1 - \varepsilon/2 \le (S^2)'(x) \le 1 + \varepsilon/2$;
- the function $F = S^2|_{[x_0-\delta,x_0+\delta]\cap I}$ is increasing;
- $\delta < \delta_0$ where δ_0 is from Remark 1(4).

Let T be a connected component of K^c such that $T \subset [x_0 - \delta, x_0 + \delta]$, and let $T_k = F^{-k}(T)$, $k \ge 0$, and $\lambda_{m_k} = |T_k|$. Since $\delta < \delta_0$ if $i \ne j$ we see that $T_i \ne T_j$, therefore $\lambda_{m_k} \to 0$. By the definition of T_k we have $\lambda_{m_k} = \lambda_{m_{k+1}} F'(\theta_{k+1})$ with $\theta_{k+1} \in T_{k+1} \subset [x_0 - \delta, x_0 + \delta]$. Then

$$F'(\theta_{k+1}) = \frac{\lambda_{m_k}}{\lambda_{m_{k+1}}} \le 1 + \frac{\varepsilon}{2}. \tag{2}$$

Consider λ_{m_1} and let $\lambda_{n_{j_0}}$ be such that $\lambda_{n_{j_0}} < \lambda_{m_1}$. As $\lambda_{m_k} \to 0$, there exists

$$k_0 = \max\{k \in \mathbb{N} : \lambda_{m_k} \ge \lambda_{n_{j_0}}\}.$$

The definition of k_0 gives

$$\lambda_{m_{k_0}} \ge \lambda_{n_{j_0}} > \lambda_{n_{j_0}+1} \ge \lambda_{m_{k_0+1}}. \tag{3}$$

Therefore

$$\frac{\lambda_{m_{k_0}}}{\lambda_{m_{k_0+1}}} \overset{(3)}{\ge} \frac{\lambda_{n_{j_0}}}{\lambda_{n_{j_0}+1}} \overset{(1)}{>} 1 + \varepsilon,$$

and this contradicts (2). So $(S^2)'(x_0) > 1$. ∎

We say that a covering $\{\mathcal{J}_i\}$ ($\mathcal{J}_i = [\alpha_i, \beta_i]$, $\beta_{i+1} < \alpha_i \le \beta_i$) of E_K is an *ε-covering* (with $\varepsilon > 0$) if $\alpha_i/\beta_{i+1} = 1 + \varepsilon$.

Note that there exists an ε-covering if and only if $\lambda_n/\lambda_{n+1} \nrightarrow 1$.

For the proof of the following theorem see [4, Theorem 1.4].

THEOREM 1. *If $\{\mathcal{J}_i\}$ is an ε-covering of the spectrum of a Cantor set K and β_i/α_i is constant, then the Cantor set K is not C^1-minimal.*

For the proof of the following proposition see [2, Corollary 3.2].

PROPOSITION 1. *If K is C^1-minimal, then any $x \in K$ is contained in an arbitrarily small open arc A such that $A \cap K$ is also C^1-minimal.*

3. Proof of Theorem A. Let $F = S^2$ (recall that $K = C_F = C_S$). By hypothesis there exists a fixed point x_0 for F and an interval J such that $F|_{J\cap I} \in C^{1+\alpha}$, with $x_0 \in J$. So there exists $k > 0$ such that

$$|F'(t_1) - F'(t_2)| \le k|t_1 - t_2|^\alpha, \quad \forall t_1, t_2 \in J. \tag{4}$$

As $\lambda_n/\lambda_{n+1} \nrightarrow 1$ there exist $\{n_j\}$ and $\varepsilon > 0$ such that

$$\frac{\lambda_{n_j}}{\lambda_{n_j+1}} > 1 + \varepsilon.$$

By Lemma 1 we have $F'(x_0) = \eta > 1$. Without loss of generality we may assume that $1 + \varepsilon < \eta$. Let $\varepsilon_0 > 0$ be such that $\eta - \varepsilon_0 > 1$.

Remark 2. We take δ such that:

(1) $[x_0-\delta, x_0+\delta] \subset J$.
(2) $F'|_{[x_0-\delta,x_0+\delta]\cap I} > \eta - \varepsilon_0$.
(3) $\displaystyle\prod_{r=1}^{+\infty}\left[1-\frac{k}{\eta}\left[\frac{2\delta}{(\eta-\varepsilon_0)^r}\right]^\alpha\right] > \frac{1}{\sqrt[3]{1+\varepsilon}}$.
(4) $\displaystyle\prod_{r=1}^{+\infty}\left[1+\frac{k}{\eta}\left[\frac{2\delta}{(\eta-\varepsilon_0)^r}\right]^\alpha\right] < \sqrt[3]{1+\varepsilon}$.

For simplicity we will denote $F|_{[x_0-\delta,x_0+\delta]\cap I}$ by F.

Let U be any open interval such that $x_0 \in U \subset [x_0-\delta, x_0+\delta]$ and let $K_0 = K \cap U$. We will prove that K_0 is not C^1-minimal. Then by Proposition 1 it will follow that K is not C^1-minimal.

Let $A_0 = [x_0-\delta, x_0+\delta]$ and define $A_n = F^{-n}(A_0)$ and $K_n = F^{-n}(K_0)$. From Remark 2(2) we have

$$|A_n| \le \frac{2\delta}{(\eta-\varepsilon_0)^n}, \qquad \forall n \in \mathbb{N}. \tag{5}$$

Lemma 2. *For each $\lambda > 0$ there exist a, b with $0 < a < b < \lambda$, $b/a = \sqrt[3]{1+\varepsilon}$ such that*

$$\left[\bigcup_{i=0}^{+\infty}(a\eta^i, b\eta^i)\right] \cap E_{K_0} = \emptyset.$$

Proof. Let n_{j_0} be such that $\lambda_{n_{j_0}} < \lambda$ and let a and b be such that $\lambda_{n_{j_0+1}} < a < b < \lambda_{n_{j_0}}$ with

$$\frac{\lambda_{n_{j_0}}}{b} = \frac{b}{a} = \sqrt[3]{1+\varepsilon} \tag{6}$$

(note that $a/\lambda_{n_{j_0+1}} \ge \sqrt[3]{1+\varepsilon}$).

Suppose that there exist $l \in E_{K_0}$ and $n_0 \in \mathbb{N}$ such that $l \in (a\eta^{n_0}, b\eta^{n_0})$. Let T be a connected component of $A_0 \setminus K_0$ with $|T| = l$. We will show that the existence of T implies the existence of a connected component T_{n_0} of $A_0 \setminus K_0$ with $|T_{n_0}| \in (\lambda_{n_{j_0+1}}, \lambda_{n_{j_0}})$, and this is a contradiction. By the definition of A_n and K_n there exists a connected component T_{n_0} of $A_{n_0} \setminus K_{n_0} \subset A_0 \setminus K_0$ such that $F^{n_0}(T_{n_0}) = T$. We will estimate $|T_{n_0}|$ in terms of $|T|$.

There exists $T_1 \subset A_1$ such that $F(T_1) = T$. So

$$|F(T_1)| = |F'(x_1)|\,|T_1| \quad \text{with } x_1 \in T_1. \tag{7}$$

On the other hand, as $F \in C^{1+\alpha}$, replacing t_1 by x_1 and t_2 by x_0 in (4) we have

$$\left|\frac{F'(x_1)}{\eta}-1\right| \le \frac{k}{\eta}|x_1-x_0|^\alpha \le \frac{k}{\eta}|A_1|^\alpha \overset{(5)}{\le} \frac{k}{\eta}\left(\frac{2\delta}{\eta-\varepsilon_0}\right)^\alpha.$$

Then

$$1-\frac{k}{\eta}\left(\frac{2\delta}{\eta-\varepsilon_0}\right)^\alpha \le \frac{F'(x_1)}{\eta} \le 1+\frac{k}{\eta}\left(\frac{2\delta}{\eta-\varepsilon_0}\right)^\alpha \tag{8}$$

so

$$|T_1| \overset{(7)}{=} \frac{|F(T_1)|}{F'(x_1)} \overset{(8)}{\le} \frac{|T|}{\eta\big[1-\frac{k}{\eta}\big(\frac{2\delta}{\eta-\varepsilon_0}\big)^\alpha\big]}$$

and

$$|T_1| \overset{(7)}{=} \frac{|F(T_1)|}{F'(x_1)} \overset{(8)}{\ge} \frac{|T|}{\eta\big[1+\frac{k}{\eta}\big(\frac{2\delta}{\eta-\varepsilon_0}\big)^\alpha\big]};$$

therefore

$$\frac{|T|}{\eta\big[1+\frac{k}{\eta}\big(\frac{2\delta}{\eta-\varepsilon_0}\big)^\alpha\big]} \le |T_1| \le \frac{|T|}{\eta\big[1-\frac{k}{\eta}\big(\frac{2\delta}{\eta-\varepsilon_0}\big)^\alpha\big]}. \tag{9}$$

Proceeding inductively shows that

$$\frac{|T|}{\eta^{n_0}\prod_{r=1}^{n_0}\big[1+\frac{k}{\eta}\big[\frac{2\delta}{(\eta-\varepsilon_0)^r}\big]^\alpha\big]} \le |T_{n_0}| \le \frac{|T|}{\eta^{n_0}\prod_{r=1}^{n_0}\big[1-\frac{k}{\eta}\big[\frac{2\delta}{(\eta-\varepsilon_0)^r}\big]^\alpha\big]}. \tag{10}$$

As $a\eta^{n_0} \le |T| \le b\eta^{n_0}$, we see that

$$\frac{a}{\prod_{r=1}^{n_0}\big[1+\frac{k}{\eta}\big[\frac{2\delta}{(\eta-\varepsilon_0)^r}\big]^\alpha\big]} \le |T_{n_0}| \le \frac{b}{\prod_{r=1}^{n_0}\big[1-\frac{k}{\eta}\big[\frac{2\delta}{(\eta-\varepsilon_0)^r}\big]^\alpha\big]}.$$

Since

$$\frac{b}{\prod_{r=1}^{n_0}\big[1-\frac{k}{\eta}\big[\frac{2\delta}{(\eta-\varepsilon_0)^r}\big]^\alpha\big]} \overset{\text{Rem.2(3)}}{<} b\sqrt[3]{1+\varepsilon} \overset{(6)}{=} \lambda_{n_{j_0}},$$

we have

$$|T_{n_0}| < \lambda_{n_{j_0}}.$$

As

$$\frac{a}{\prod_{r=1}^{n_0}\big[1+\frac{k}{\eta}\big[\frac{2\delta}{(\eta-\varepsilon_0)^r}\big]^\alpha\big]} \overset{\text{Rem.2(4)}}{>} \frac{a}{\sqrt[3]{1+\varepsilon}} \overset{(6)}{\ge} \lambda_{n_{j_0+1}},$$

we conclude that

$$|T_{n_0}| > \lambda_{n_{j_0+1}}. \ \blacksquare$$

LEMMA 3. *There exists an ε_1-covering $\{[\alpha_i,\beta_i]\}$ of E_{K_0} such that β_i/α_i is constant.*

Proof. By Lemma 2 with $\lambda = 1/n$ there exist a_n and b_n such that

$$(11) \quad 0 < a_n < b_n < 1/n, \qquad \frac{b_n}{a_n} = \sqrt[3]{1+\varepsilon}, \qquad \Big[\bigcup_{i=0}^{+\infty}(a_n\eta^i, b_n\eta^i)\Big] \cap E_{K_0} = \emptyset.$$

Let $r > 0$ and consider $[r, r\eta^2]$. For each $n \in \mathbb{N}$, let

$$m_n = \min\{k \in \mathbb{Z} : a_n\eta^k \in [r, r\eta^2]\}.$$

As $\eta > 1+\varepsilon > \sqrt[3]{1+\varepsilon}$ and $b_n/a_n = \sqrt[3]{1+\varepsilon}$, we have

$$[a_n\eta^{m_n}, b_n\eta^{m_n}] \subset [r, r\eta^2].$$

Let z be an accumulation point of $a_n\eta^{m_n}$ and $\varepsilon_1 = \sqrt[6]{1+\varepsilon} - 1$. We will show that

$$E_{K_0} \cap \bigcup_{i\in\mathbb{Z}}(z\eta^i, z(\varepsilon_1+1)\eta^i) = \emptyset.$$

Suppose that there exists $l \in E_{K_0} \cap \bigcup_{i\in\mathbb{Z}}(z\eta^i, z(\varepsilon_1+1)\eta^i)$. Then there exists $k \in \mathbb{Z}$ such that $l\eta^k \in (z, z(\varepsilon_1+1))$. As z is an accumulation point of $a_n\eta^{m_n}$, there exists $m_n > k$ such that

$$l\eta^k \in (a_n\eta^{m_n}, b_n\eta^{m_n});$$

therefore $l \in (a_n\eta^{m_n-k}, b_n\eta^{m_n-k})$ and this contradicts (11).

Taking $\alpha_i = z(\varepsilon_1+1)\eta^{-i-1}$ and $\beta_i = z\eta^{-i}$ we obtain an ε_1-covering of E_{K_0}. ∎

Proof of Theorem A. Suppose that K is C^1-minimal. By Proposition 1 there exists an open arc $A \subset [x_0-\delta, x_0+\delta]$ such that $K_0 = K \cap A$ is C^1-minimal. By Lemma 3, there exists an ε_1-covering of E_{K_0} with β_i/α_i constant. By Theorem 1 we deduce that K_0 is not C^1-minimal, and this is a contradiction. ∎

REMARK 3. As the Cantor set K equals C_S for $S \in \mathcal{S}^0(I_1, \ldots, I_k, L)$, we see that S has a dense set of repelling periodic points in K. Therefore, if in the statement of Theorem A one replaces the point x_0 by a periodic repelling point p of period n, and $S|_J \in C^{1+\alpha}$ by $S^n|_J \in C^{1+\alpha}$ for some $\alpha > 0$, the arguments of the proof are still valid.

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