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FREENESS WITH AMALGAMATION, LIMIT THEOREMS AND S-TRANSFORM IN NON-COMMUTATIVE PROBABILITY SPACES OF TYPE B

BY

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Abstract. The paper addresses several problems left open by P. Biane, F. Goodman and A. Nica [Trans. Amer. Math. Soc. 355 (2003)]. The main result is that a type B non-commutative probability space can be studied in the framework of freeness with amalgamation. This view allows easy ways of constructing a version of the S-transform as well as proving analogues to the Central Limit Theorem and Poisson Limit Theorem.

1. Introduction. The paper addresses several problems left open by P. Biane, F. Goodman and A. Nica [4].

The type A, B, C and D root systems determine corresponding lattices of non-crossing partitions (see [8], [2]). The type A_{n+1} corresponds to the lattice of non-crossing partitions on the ordered set $[n] = \{1 < \cdots < n\}$; the types B_n and C_n determine the same lattice of non-crossing partitions on $[\overline{n}] = \{1 < \cdots < n < -1 < \cdots < -n\}$, namely the partitions with the property that if V is a block, then $-V$ (the set containing the negatives of the elements from V) is also a block; the type D corresponds to a lattice of the symmetric non-crossing partitions with the property that if there exists a symmetric block, then it has more than two elements and contains $-n$ and n (see again [8], [2], [3]).

The lattices of type A and type B non-crossing partitions are self-dual with respect to the Kreweras complementary. In the type A case, the lattice structure was known to be connected to the combinatorics of Free Probability Theory (see [7]). For the type B case, the properties of the lattice also allow a construction, described in [4], of some associated non-commutative probability spaces, with a similar apparatus as in the type A case (such as R-transform and boxed convolution). The paper [4] leaves open some questions on these objects: possible connections to other types of independence, limit theorems, S-transform. These questions are addressed in the present material, which also makes the observation that a type B non-commutative

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probability space can be studied in the framework of freeness with amalgamation over a certain commutative algebra $\mathcal{C}$ (see Section 3).

The material is organized as follows: the second section reviews some results from [4]; the third section presents the connection with freeness with amalgamation; the fourth section outlines the construction of S-transform for type B non-commutative probability spaces, utilizing the commutativity of the matrix algebra $\mathcal{C}$; the fifth and sixth sections presents limit results: analogues of the Central Limit Theorem and Poisson Limit Theorem, respectively.

2. Preliminary definitions and results

DEFINITION 2.1. A *non-commutative probability space of type B* is a system $(\mathcal{A}, \varphi, \mathcal{X}, f, \Phi)$, where:

(i) $(\mathcal{A}, \varphi)$ is a non-commutative probability space (of type A), i.e. $\mathcal{A}$ is a complex unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$.
(ii) $\mathcal{X}$ is a complex vector space and $f : \mathcal{X} \to \mathbb{C}$ is a linear functional.
(iii) $\Phi : \mathcal{A} \times \mathcal{X} \times \mathcal{A} \to \mathcal{X}$ is a two-sided action of $\mathcal{A}$ on $\mathcal{X}$ (when there is no confusion, it will be written "$a\xi b$" instead of $\Phi(a\xi b)$, where $a, b \in \mathcal{A}$ and $\xi \in \mathcal{X}$).

On the vector space $\mathcal{A} \times \mathcal{X}$ we defined a structure of unital algebra with the multiplication

$$(a, \xi) \cdot (b, \eta) = (ab, a\eta + \xi b), \qquad a, b \in \mathcal{A},\ \xi, \eta \in \mathcal{X}.$$

The above algebra structure can be obtained when $(a, \xi) \in \mathcal{A} \times \mathcal{X}$ is identified with a 2×2 matrix,

$$(a, \xi) \leftrightarrow \begin{bmatrix} a & \xi \\ 0 & a \end{bmatrix}.$$

We will also consider the commutative unital algebra $\mathcal{C}$ by similarly endowing the vector space $\mathbb{C} \times \mathbb{C}$ with the multiplication

$$(x, t) \cdot (y, s) = (xy, xs + ty),$$

i.e. using the identification

$$\mathcal{C} \ni (x, t) \leftrightarrow \begin{bmatrix} x & t \\ 0 & x \end{bmatrix} \in M_2(\mathbb{C}).$$

As in [4], we will denote by $\mathrm{NC}^{(\mathrm{A})}(n)$ the set of all non-crossing partitions (of type A) on an ordered set with n elements, and by $\mathrm{Kr}(\pi)$ the Kreweras complementary of a partition $\pi \in \mathrm{NC}^{(\mathrm{A})}(n)$. E will denote the linear map from $\mathcal{A} \times \mathcal{X}$ to $\mathcal{C}$ given by

$$E(a,\xi)) = (\varphi(a), f(\xi)), \qquad a \in \mathcal{A},\ \xi \in \mathcal{X}.$$

Definition 2.2. Let $(\mathcal{A}, \varphi, \mathcal{X}, f, \Phi)$ be a non-commutative probability space of type B. The *non-crossing cumulant functionals of type B* are the families of multilinear functionals $(\kappa_n : (\mathcal{A} \times \mathcal{X})^n \to \mathcal{C})_{n=1}^{\infty}$ defined by the following equations: for every $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$, $\xi_1, \ldots, \xi_n \in \mathcal{X}$, we have

$$\text{(1)} \qquad \sum_{\gamma \in \mathrm{NC}^{(\mathrm{A})}(n)} \prod_{B \in \gamma} \kappa_{\mathrm{card}(B)}((a_1,\xi_1)\cdots(a_n,\xi_n)|B) = E((a_1,\xi_1)\cdots(a_n,\xi_n))$$

where the product on the left-hand side is considered with respect to the multiplication on $\mathcal{C}$ and the product $(a_1,\xi_1)\cdots(a_n,\xi_n)$ on the right-hand side is considered with respect to the multiplication on $\mathcal{A} \times \mathcal{X}$ defined above.

Note that the first component of $\kappa_m((a_1,\xi_1)\cdots(a_n,\xi_m))$ equals the non-crossing cumulant $k_m(a_1,\ldots,a_m)$.

We will also use the notation $\kappa_n(a,\xi)$ for $\kappa_n((a,\xi)\cdots(a,\xi))$ and the notation M_n for $E((a,\xi)^n)$.

Definition 2.3. Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ and let $\mathcal{X}_1, \ldots, \mathcal{X}_k$ be linear subspaces of $\mathcal{X}$ such that each $\mathcal{X}_j$ is invariant under the action of $\mathcal{A}_j$. We say that $(\mathcal{A}_1, \mathcal{X}_1), \ldots, (\mathcal{A}_k, \mathcal{X}_k)$ are *type B free independent* if

$$\kappa_n((a_1,\xi_1),\ldots,(a_n,\xi_n)) = 0$$

whenever $a_l \in \mathcal{A}_{i_l}$, $\xi_l \in \mathcal{X}_{i_l}$ $(l = 1,\ldots,n)$ are such that there exist $1 \leq s < t \leq n$ with $i_s \neq i_t$.

For $(a,\xi) \in \mathcal{A} \times \mathcal{X}$ we consider the moment and cumulant, or R-transform, series:

$$M(a,\xi) = \sum_{n=1}^{\infty} (E((a,\xi)^n)) z^n, \qquad R(a,\xi) = \sum_{n=1}^{\infty} \kappa_n(a,\xi) z^n.$$

Definition 2.4. Let $\Theta^{(\mathrm{B})}$ be the set of power series of the form

$$f(z) = \sum_{n=1}^{\infty} (\alpha'_n, \alpha''_n) z^n,$$

where α'_n, α''_n are complex numbers. For $p \in \mathrm{NC}^{(\mathrm{A})}(n)$ and $f \in \Theta^{(\mathrm{B})}$, consider

$$\mathrm{Cf}_p(f) = \prod_{D\,\text{block in}\,p} (\alpha'_{\mathrm{card}(D)}, \alpha''_{\mathrm{card}(D)})$$

(the product is in $\mathcal{C}$). On $\Theta^{(\mathrm{B})}$ we define the binary operation $\boxtimes$ by:

$$f \boxtimes g = \sum_{n=1}^{\infty} (\gamma_n', \gamma_n'') z^n \quad \text{where}$$

$$(\gamma_n', \gamma_n'') = \sum_{\pi \in \mathrm{NC}^{(\mathrm{A})}(n)} \mathrm{Cf}_\pi(f) \cdot \mathrm{Cf}_{\mathrm{Kr}(\pi)}(g) \quad \text{(the products are in } \mathcal{C}\text{)}.$$

THEOREM 2.5 (see [4, Theorem 5.3]). *The moment series M and R-transform R of (a, ξ) are related by the formula*

$$M = R \boxtimes \zeta'$$

where $\zeta' \in \Theta^{(B)}$ is the series $\sum_{n=1}^{\infty} (1,0) z^n$.

REMARK 2.6 (see [4, Theorem 6.4]). We denote by $k_{n,p}'$ or, for simplicity, by k_n', the multilinear functional from $\mathcal{A}^{p-1} \times \mathcal{X} \times \mathcal{A}^{n-p}$ to $\mathbb{C}$ which is defined by the same formula as for the (type A) free cumulants $k^n : \mathcal{A}^n \to \mathbb{C}$, but where the pth argument is a vector from $\mathcal{X}$ and φ is replaced by f in all the appropriate places. The connection between the type B cumulants κ_n and the functionals k_n, k_n' is given by

$$\begin{aligned} (2) \qquad & \kappa_n((a_1, \xi_1), \ldots, (a_n, \xi_n)) \\ & \quad = \Big(k_n(a_1, \ldots, a_n), \sum_{p=1}^{n} k_n'(a_1, \ldots, a_{p-1}, \xi_p, a_{p+1}, \ldots, a_n) \Big). \end{aligned}$$

THEOREM 2.7 (see [4, Theorem 7.3]). *If $(\mathcal{A}_1, \mathcal{X}_1), (\mathcal{A}_2, \mathcal{X}_2)$ are free independent, with $(a_1, \xi_1) \in (\mathcal{A}_1, \mathcal{X}_1)$, $(a_2, \xi_2) \in (\mathcal{A}_2, \mathcal{X}_2)$, and R_1, respectively R_2 denote the R-transforms of (a_1, ξ_1) and (a_2, ξ_2), then:*

(i) *the R-transform of $(a_1, \xi_1) + (a_2, \xi_2)$ is $R_1 + R_2$,*
(ii) *the R-transform of $(a_1, \xi_1) \cdot (a_2, \xi_2)$ is $R_1 \boxtimes R_2$.*

3. Connexion to "freeness with amalgamation". As shown in [4, Section 6.3, Remark 3], the definitions of type B cumulants are close to those from the framework of "operator-valued cumulants", yet some details are different—mainly the map E is not a conditional expectation and $\mathcal{A} \times \mathcal{X}$ is not a bimodule over C. Following a suggestion of Dimitri Shlyakhtenko, a modified algebra structure rather than $\mathcal{A} \times \mathcal{X}$ can be considered in order to overcome these difficulties.

Let $\mathfrak{C} = \mathcal{X} \oplus \mathcal{A}$. On $\mathcal{A} \times \mathfrak{C}$ we have a $\mathcal{C}$-bimodule structure given by

$$(x, t)(a, \xi + b) = (a, \xi + b)(x, t) = (ax, at + (\xi + b)x)$$

for any $x, t \in \mathbb{C}$, $a, b \in \mathcal{A}$, $\xi \in \mathcal{X}$. Since $\mathcal{A}$ is unital, $\mathcal{C}$ is a subspace of $\mathfrak{C}$.

The map E extends to $\mathfrak{C}$ via

$$\widetilde{E}(a, \xi + b) = (\varphi(a), f(\xi) + \varphi(b)),$$

and the extension is a conditional expectation, since

$$\begin{aligned}\widetilde{E}((x,t)(a,\xi+b)) &= \widetilde{E}(ax, at+(\xi+b)x) = (\varphi(ax), \varphi(ta)+f(\xi x)+\varphi(bx))\\ &= (x\varphi(a), t\varphi(a)+xf(\xi)+x\varphi(b)) = (x,t)(\varphi(a), f(\xi)+\varphi(b))\\ &= (x,t)\widetilde{E}(a,\xi+b).\end{aligned}$$

It follows that $(\mathcal{A}\times(\mathcal{A}\oplus\mathcal{X}),\widetilde{E})$ is a non-commutative probability space with amalgamation over $\mathcal{C}$. We will show that type B freeness in $(\mathcal{A},\varphi,\mathcal{X},f,\Phi)$ is equivalent to freeness with amalgamation over $\mathcal{C}$ in $(\mathcal{A}\times(\mathcal{A}\oplus\mathcal{X}),\widetilde{E})$.

Remark 3.1. The equation (1) can naturally be extended to the framework of $\mathfrak{E}$ and $\widetilde{E}$, which reduces the construction to freeness with amalgamation, namely defining the cumulants $\widetilde{\kappa}$ by

$$\sum_{\gamma\in\mathrm{NC}^{(\mathrm{A})}(n)}\prod_{B\in\gamma}\widetilde{\kappa}_{\mathrm{card}(B)}((a_1,\xi_1)\cdots(a_n,\xi_n)|B) = \widetilde{E}((a_1,\xi_1)\cdots(a_n,\xi_n)). \tag{3}$$

If $m:\mathcal{A}\times\mathcal{A}\ni(a,b)\mapsto m(a,b)=ab\in\mathcal{A}$ is the multiplication in $\mathcal{A}$, note that $(\mathcal{A},\varphi,\mathcal{X}\oplus\mathcal{A},f\oplus\varphi,\Phi\oplus m)$ is also a type B non-commutative probability space, therefore, according to Remark 2.6, the components of $\widetilde{\kappa}$ are given by

$$\begin{aligned}&\widetilde{\kappa}_n((a_1,\xi_1+b_1),\ldots,(a_n,\xi_n+b_n))\\ &\quad= \Big(k_n(a_1,\ldots,a_n),\sum_{p=1}^{n}k'_n(a_1,\ldots,a_{p-1},\xi_p+b_p,a_{p+1},\ldots,a_n)\Big)\\ &\quad= \Big(k_n(a_1,\ldots,a_n),\sum_{p=1}^{n}k'_n(a_1,\ldots,a_{p-1},\xi_p,a_{p+1},\ldots,a_n)\\ &\qquad+\sum_{r=1}^{n}k_n(a_1,\ldots,a_{p-1},b_r,a_{p+1},\ldots,a_n)\Big).\end{aligned}$$

Proposition 3.2. *Let $\mathcal{A}_1,\ldots,\mathcal{A}_n$ be unital subalgebras of $\mathcal{A}$ and let $\mathcal{X}_1,\ldots,\mathcal{X}_k$ be linear subspaces of $\mathcal{X}$ such that each $\mathcal{X}_i$ is invariant under the action of $\mathcal{A}_i$. Then $\{(\mathcal{A}_i,\mathcal{X}_i)\}_{i=1}^n$ are type B free if and only if the subalgebras $\{\mathcal{A}_i\times(\mathcal{X}_i\oplus\mathcal{A}_i)\}_{i=1}^n$ are free in $\mathcal{A}\times(\mathcal{A}\oplus\mathcal{X})$.*

Proof. First note that, since the first components of κ and $\widetilde{\kappa}$ are the free cumulants k in the non-commutative probability space $(\mathcal{A},f)$, both type B freeness of $\{(\mathcal{A}_i,\mathcal{X}_i)\}_{i=1}^n$ and freeness with amalgamation over $\mathcal{C}$ of $\{\mathcal{A}_i\times(\mathcal{X}_i\oplus\mathcal{A}_i)\}_{i=1}^n$ imply that $\mathcal{A}_1,\ldots,\mathcal{A}_n$ are free with respect to f.

Suppose that $\mathcal{A}_1,\ldots,\mathcal{A}_n$ are free with respect to f, take m a positive integer and $i(1),\ldots,i(m)$ an m-uple from $\{1,\ldots,n\}$ such that there exist $1\leq l,s\leq n$ with $i(l)\neq i(s)$. We need to show that for any $a_j,b_j\in\mathcal{A}_{i(j)}$

and any $\xi_j \in \mathcal{X}_j$ we have

$$\kappa_m((a_1, \xi_1), \ldots, (a_m, \xi_m)) = 0$$

if and only if

$$\widetilde{\kappa}_m((a_1, \xi_1 + b_1), \ldots, (a_m, \xi_m + b_m)) = 0.$$

But Remark 3.1 gives

$$\begin{aligned}\widetilde{\kappa}_m&((a_1, \xi_1 + b_1), \ldots, (a_m, \xi_m + b_m)\\ &= \Big(k_m(a_1, \ldots, a_m), \sum_{p=1}^{m} k'_m(a_1, \ldots, a_{p-1}, \xi_p, a_{p+1}, \ldots, a_m)\\ &\quad + \sum_{r=1}^{m} k_m(a_1, \ldots, a_{r-1}, b_r, a_{r+1}, \ldots, a_m)\Big)\end{aligned}$$

Since $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are free and $i(l) \neq i(s)$ for some l and s, the cumulants $k_m(a_1, \ldots, a_{r-1}, b_r, a_{r+1}, \ldots, a_m)$ vanish, so the relation (2) implies that

$$\kappa_m((a_1, \xi_1), \ldots, (a_m, \xi_m)) = \widetilde{\kappa}_m((a_1, \xi_1 + b_1), \ldots, (a_m, \xi_m + b_m)). \blacksquare$$

4. The S-transform. Utilizing the commutativity of the algebra C, the construction of the S-transform is essentially a verbatim reproduction of the type A situation.

We will denote by

$$\mathcal{G} = \Big\{\sum_{n=1}^{\infty} \alpha_n z^n : \alpha_n \in \mathcal{C}\Big\}$$

the set of formal series without constant term with coefficients in $\mathcal{C}$, and

$$\mathcal{G}^{\langle -1\rangle} = \Big\{\sum_{n=1}^{\infty} \alpha_n z^n : \alpha_n \in \mathcal{C},\ \alpha_1 \text{ invertible}\Big\}$$

the set of all invertible series (with respect to substitutional composition) with coefficients in $\mathcal{C}$ (see [1]).

Definition 4.1. Let $(a, \xi) \in \mathcal{A} \times \mathcal{X}$ be such that $\varphi(a) \neq 0$, that is, $(\varphi(a), f(\xi))$ is invertible in $\mathcal{C}$. If $R_{(a,\xi)}(z)$ is the R-transform series of (a, ξ), then the *S-transform* of (a, ξ) is the series defined by

$$S_{(a,\xi)}(z) = \frac{1}{z} R^{\langle -1\rangle}_{(a,\xi)}(z).$$

Theorem 4.2. *If $(\mathcal{A}_1, \mathcal{X}_1), (\mathcal{A}_2, \mathcal{X}_2) \subset (\mathcal{A}, \mathcal{X})$ are free independent and $(x_j, \xi_j) \in (\mathcal{A}_j, \mathcal{X}_j)$, $j = 1, 2$, are such that $\varphi(x_j) \neq 0$, then*

$$S_{(a_1,\xi_1)(x_2,\xi_2)}(z) = S_{(a_1,\xi_1)}(z) S_{(a_2,\xi_2)}(z).$$

Proof. The proof in [7] for the type A case works also for the freeness with amalgamation over a commutative algebra. Yet, for the convenience of the reader, we will outline the main steps.

Since, for $(a_1,\xi_1),(a_2,\xi_2)$ free, $R_{(a_1,\xi_1)\cdot(a_2,\xi_2)} = R_{(a_1,\xi_1)} \boxtimes R_{(a_2,\xi_2)}$, it suffices to prove that the mapping

$$\mathcal{F} : \mathcal{G}^{\langle -1\rangle} \ni f \mapsto \frac{1}{z} f^{\langle -1\rangle} \in \mathcal{G}$$

has the property

$$\mathcal{F}(f \boxtimes g) = \mathcal{F}(f)\mathcal{F}(g). \tag{4}$$

Indeed, (4) is equivalent to

$$z(f \boxtimes g) = f^{\langle -1\rangle}(f \boxtimes g) \cdot g^{\langle -1\rangle}(g \boxtimes f). \tag{5}$$

For $f, g \in \mathcal{G}$, we denote

$$(f \check{\boxtimes} g)(z) = \sum_{n\geq 1} \lambda_n z^n$$

where, with the notations from Definition 2.4,

$$\lambda_n = \sum_{\substack{\sigma\in \mathrm{NC}^{(\mathrm{A})}(n)\\ (1)\ \text{block in } \sigma}} \mathrm{Cf}_\sigma(f) \cdot \mathrm{Cf}_{\mathrm{Kr}(\sigma)}(g).$$

For $f = \sum_{n\geq 1} \alpha_n z^n \in \mathcal{G}^{\langle -1\rangle}$ we have

$$f^{\langle -1\rangle} \circ (f \boxtimes g) = \alpha_1^{-1}(f \check{\boxtimes} g)$$

since, with the above notations, the coefficient of z^m on the right-hand side is

$$\sum_{n\geq 1} \sum_{\substack{i_1,\dots,i_n\geq 1\\ i_1+\cdots+i_n=m}} \alpha_n \alpha_1^{-n} \lambda_{i_1}\cdots\lambda_{i_n}$$

while the coefficient of z^m on the left-hand side is

$$\sum_{n\geq 1} \sum_{1=b_1<\cdots<b_n\leq m} \sum_{\substack{\pi\in \mathrm{NC}(m)\\ (b_1,\dots,b_n)\in\pi}} \mathrm{Cf}_\pi(f)\cdot \mathrm{Cf}_{\mathrm{Kr}(\pi)}(g)$$

and the equality follows by setting $\pi_k = \pi|\{b_k,\dots,b_{k+1}-1\}$ (notationally $b_{n+1} = m$) and remarking that $\mathrm{Kr}(\pi)$ is the juxtaposition of $\mathrm{Kr}(\pi_1),\dots$ $\dots,\mathrm{Kr}(\pi_n)$.

It follows that if $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are respectively the coefficients of f and g, the relation (5) is equivalent to

$$(f \check{\boxtimes} g)(z) \cdot (f \check{\boxtimes} g)(z) = \alpha_1\beta_1 z \cdot (f \boxtimes g)(z).$$

The coefficient of z^{m+1} on the left-hand side is

$$\sum_{n=1}^{m} \sum_{\substack{\pi \in \mathrm{NC}(n) \\ (1) \in \pi}} \sum_{\substack{\rho \in \mathrm{NC}(m+n-1) \\ (1) \in \rho}} \mathrm{Cf}_{\pi}(f) \cdot \mathrm{Cf}_{\mathrm{Kr}(\pi)}(g) \cdot \mathrm{Cf}_{\rho}(g) \cdot \mathrm{Cf}_{\mathrm{Kr}(\rho)}(f),$$

while the coefficient of z^{m+1} on the right-hand side is

$$\sum_{\sigma \in \mathrm{NC}^{(\mathrm{A})}(m)} \alpha_1 \beta_1 \cdot \mathrm{Cf}_{\sigma}(f) \cdot \mathrm{Cf}_{\mathrm{Kr}(\sigma)}(g).$$

As shown in [7], the conclusion follows from the bijection between the index sets of the above sums. More precisely, if $1 \leq n \leq m$, to the pair consisting of $\pi \in \mathrm{NC}(n)$ and $\rho \in \mathrm{NC}^{(\mathrm{A})}(m+1-n)$ both containing the block (1), we associate the partition from $\mathrm{NC}^{(\mathrm{A})}(n+m-1)$ obtained by juxtaposing $\pi \setminus (1)$ and $\mathrm{Kr}(\rho)$. ∎

5. Central limit theorem

THEOREM 5.1. *Let* $\{(\mathcal{A}_k, \mathcal{X}_k)\}_{k \geq 1} \subset (\mathcal{A}, \mathcal{X})$ *be type B free independent and* $(x_k, \xi_k) \in (\mathcal{A}_k, \mathcal{X}_k)$ *identically distributed such that* $E((x_k, \xi_k)) = (0,0)$ *and* $E((x_k, \xi_k)^2) = (1,1)$. *The limit distribution moments of*

$$\frac{(a_1, \xi_1) + \cdots + (a_N, \xi_N)}{\sqrt{N}}$$

are $\{m_n, \mathfrak{m}_n\}_n$, *where* $\{m_n\}_n$ *are the moments of the semicircular distribution and*

$$\mathfrak{m}_n = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ \binom{2k}{k+1} & \text{if } n = 2k \text{ is even}. \end{cases}$$

Proof. Note $S_N = \frac{1}{\sqrt{N}}[(a_1, \xi_1) + \cdots + (a_N, \xi_N)]$ and $R_N = R(S_N)$. Theorem 2.7 implies

$$\lim_{N \to \infty} R_N = (1,1) z^2.$$

The first component of the limit distribution is Voiculescu's semicircular distribution. To compute the second component of the moments, we will use equation (1), which becomes

$$E((a_1, \xi_1)^n) = \sum_{\gamma \in \mathrm{NC}_2^{(\mathrm{A})}(n)} (\kappa_2(a_1, \xi_1))^{n/2}$$

where $\mathrm{NC}_2^{(\mathrm{A})}(n)$ is the set of all non-crossing partitions with n elements such that each of their blocks contains exactly two elements.

It follows that all the odd moments are zero, and, since in $\mathcal{C}$ we have $(a,b)^n = (a^n, n a^{n-1} b)$, the even moments are given by

$$\begin{aligned}\mathfrak{m}_{2n} &= nC_n, \quad \text{where } C_n \text{ stands for the } n\text{th Catalan number,}\\ &= n\,\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n+1}. \ \blacksquare\end{aligned}$$

REMARK 5.2. The second components of the above limit moments are not the moments of a positive Borel measure on $\mathbb{R}$. For example, a necessary condition for $\{\mathfrak{m}_k\}_{k\geq 1}$ to be the moments of a measure on $\mathbb{R}$ (see [9], [7]) is that

$$\mathfrak{m}_2\mathfrak{m}_4 \geq \mathfrak{m}_3^2, \tag{6}$$

but $\mathfrak{m}_3^2 = 225$ while $\mathfrak{m}_2\mathfrak{m}_4 = 4\cdot 56 = 224$. Yet, the sequence $\{\mathfrak{m}_n\}_n$ is connected with the moments of another remarkable distribution appearing in non-commutative probability—the central limit distribution for monotonic independence.

For variables that are monotonically independent (see [5], [6]), the limit moments in the Central Limit Theorem are given by the "arcsine law", i.e. the nth moment μ_n is given by

$$\mu_n = \begin{cases} 0 & \text{if } n \text{ is odd},\\ \binom{2k}{k} = (k+1)C_k & \text{if } n = 2k \text{ is even}.\end{cases}$$

Hence $\mu_n = m_n + \mathfrak{m}_n$, which implies the following:

COROLLARY 5.3. *On $\mathcal{A}\oplus\mathcal{X}$ consider the algebra structure given by*

$$(a+\xi)(b+\eta) = ab + \xi b + a\eta$$

and $\Psi : \mathcal{A}\oplus\mathcal{X} \ni a+\xi \mapsto \varphi(a)+f(\xi) \in \mathbb{C}$. Let $(a_j,\xi_j)_{j=1}^{\infty}$ be a family of type B free identically distributed elements from $\mathcal{A}\oplus\mathcal{X}$ such that $E((a_j,\xi_j)) = (0,0)$ and $E((a_j,\xi_j)^2) = (1,1)$. Then the limit in distribution of

$$\frac{a_1+\xi_1+\cdots+a_N+\xi_N}{\sqrt{N}}$$

is the "arcsine law".

6. Poisson limit theorem. We will consider an analogue of the classical Bernoulli distribution in a type B probability space.

Let $A = (\alpha_1,\alpha_2) \in \mathbb{R}^2 \subset \mathcal{C}$. We call an element $(a,\xi) \in \mathcal{A}\times\mathcal{X}$ *type B Bernoulli* with rate Λ and jump size A if

$$E((a,\xi)^n) = \Lambda A^n \quad \text{for some } \Lambda = (\lambda_1,\lambda_2) \in \mathcal{C}.$$

THEOREM 6.1. *Let $\Lambda \in \mathcal{C}$ and $A \in \mathbb{R}^2$. Then the limit law for $N\to\infty$ of the sum of N free independent type B Bernoulli variables with rate Λ/N, and jump size A has cumulants which are given by $\kappa_n = \Lambda A^n$.*

Proof. We will introduce first several new notations in order to simplify the writing. β_N will stand for a type B Bernoulli variable with rate Λ/N and s_N for a sum of N such free independent variables. μ will denote the

Möbius function of the lattice $\mathrm{NC}^{(\mathrm{A})}(n)$ (see [7, Lecture 10]) and, for $\pi \in \mathrm{NC}^{(\mathrm{A})}(n)$ and $\beta \in \mathcal{A} \times \mathcal{X}$, we will use the notation

$$M_\pi(\beta) = \prod_{B \text{ block of } \pi} M_{\mathrm{card}(B)}(\beta)$$

where $M_n(\beta) = E(\beta^n)$ is the nth moment of β.

With the above notations, equation (1) gives

$$\begin{aligned}\kappa_n(\beta_N) &= \sum_{\pi \in \mathrm{NC}^{(\mathrm{A})}(n)} M_\pi(\beta_N)\mu(\pi, 1_n) = \frac{\Lambda}{N} A^n + \sum_{\substack{\pi \in \mathrm{NC}^{(\mathrm{A})}(n) \\ 1_n \neq \pi}} M_\pi(\beta_N)\mu(\pi, 1_n) \\ &= \frac{\Lambda}{N} A^n + O\left(\frac{1}{N^2}\right)\end{aligned}$$

Therefore $\lim_{N\to\infty} \kappa_n(s_N) = \lim_{N\to\infty} N \cdot \kappa_n(\beta_N) = \Lambda A^n$. ∎

Definition 6.2. An element of a type B probability space with cumulants $\kappa_n = \Lambda A^n$ for some $\Lambda \in \mathcal{C}$ and $A \in \mathbb{R}^2$ will be called a *type B free Poisson element* of rate A and jump size Λ.

As in the type A case, we have the following:

Corollary 6.3. *The square of a type B random variable* (a, ξ) *with distribution given by the central limit theorem such that* $E\left((a,\xi)^2\right) = \sigma \in \mathcal{C}$ *is a type B free Poisson element of rate* $(1, 0)$ *and jump size* σ.

Remark 6.4. The first component of the moments of a type B free Poisson variable coincides with the type A case, therefore are given by a probability measure on $\mathbb{R}$. In general, the second component of the moments of a type B free Poisson random variable are not the moments of a real measure.

The first part of the assertion is clear. For the second part, we will consider the particular case when $\lambda_2 = 0$ and $\lambda_1 = \lambda$ is close to 0 and $\alpha_1 = \alpha_2 = \alpha$. It follows that

$$\kappa_n = \Lambda A^n = ((\lambda, 0)(\alpha^n, n\alpha^n)).$$

Since equation (1) implies

$$\begin{aligned}&M_2 = \kappa_2 + \kappa_1^2 = (\lambda + \lambda^2)A, \\ &M_3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3 = (\lambda + 3\lambda^2 + \lambda^3)A^3, \\ &M_4 = \kappa_4 + 4\kappa_1\kappa_3 + 2\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4 = (\lambda + 6\lambda^2 + 6\lambda^3 + \lambda^4)A^4,\end{aligned}$$

their second components are given by

$$\begin{aligned}\mathfrak{m}_2 &= 2(\lambda + \lambda^2)\alpha^2, \\ \mathfrak{m}_3 &= 3(\lambda + 3\lambda^2 + \lambda^3)\alpha^3, \\ \mathfrak{m}_4 &= 4(\lambda + 6\lambda^2 + 6\lambda^3 + \lambda^4)\alpha^4.\end{aligned}$$

The condition (6) amounts to

$$8(\lambda+\lambda^2)(\lambda+6\lambda^2+6\lambda^3+\lambda^4)\alpha^6 \geq 9(\lambda+3\lambda^2+\lambda^3)^2\alpha^6,$$

that is, $8(1+\lambda)(1+6\lambda+6\lambda^2+\lambda^3) \geq 9(1+3\lambda+\lambda^2)^2$, or, equivalently,

$$8+O(\lambda) \geq 9+O(\lambda),$$

which does not hold true for λ small.

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