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INERTIAL SUBRINGS OF A LOCALLY FINITE ALGEBRA

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Abstract. I. S. Cohen proved that any commutative local noetherian ring R that is $J(R)$-adic complete admits a coefficient subring. Analogous to the concept of a coefficient subring is the concept of an inertial subring of an algebra A over a commutative ring K. In case K is a Hensel ring and the module A_K is finitely generated, under some additional conditions, as proved by Azumaya, A admits an inertial subring. In this paper the question of existence of an inertial subring in a locally finite algebra is discussed.

Introduction. Let S be a commutative generalized local ring in the sense of Cohen [3]. Cohen introduced the concept of a coefficient subring of S. If S is $J(S)$-adic complete, Cohen proved that S has a coefficient subring T, which is unique to within isomorphisms. On the other hand, Azumaya [1] introduced the concept of an inertial subalgebra of an algebra A over a Hensel ring R. In case A_R is finitely generated and $\overline{A} = A/J(A)$ is separable over $\overline{R} = R/J(R)$, A has an inertial subalgebra, which is unique to within conjugations [1, Theorem 33]; this result generalizes the Wedderburn–Mal'tsev Principal Theorem. A special case of Azumaya's theorem, in case A is a finite ring, is given in Clark [2]. The concept of an inertial subring is analogous to that of a coefficient subring. Let A be a local ring which is an algebra over a Hensel ring $R, J(R) = R \cap J(A)$ and $\overline{A} = A/J(A)$ is a countably generated separable algebraic field extension of $\overline{R}$. In case A is a locally finite R-algebra, the existence of a subalgebra T of A analogous to an inertial subring is shown in Theorem 2.5. This subalgebra is shown to be unique to within R-isomorphisms. In case A is commutative, this subalgebra is unique. In case A is not commutative, an example is given to show that unlike in Azumaya's theorem, two such subalgebras need not be conjugate. In case A is an artinian duo ring and $J(R)$ is nilpotent, in Theorem 2.7 the existence of a commutative local subring T analogous to an inertial subring is established. In Theorem 2.8 a locally finite algebra A over a Hensel ring R such that A is semi-perfect ring is studied. Some sufficient conditions for the existence of a subring T of A analogous to an inertial subring are given.

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1. Preliminaries. All rings considered here are with identity $1 \neq 0$, and all modules are unital right modules unless otherwise stated. For general concepts on rings and modules one may refer to Faith [4]. For any ring A, $J(A)$, $Z(A)$ will denote its *Jacobson radical* and *center* respectively. For any $c \in J(A)$ and any subset Y of A, Y^c denotes the conjugate $(1-c)^{-1}Y(1-c)$ of Y. For any module M over A of finite composition length, $d_A(M)$ denotes the *composition length* of M. Let K be any commutative ring. Then a ring A is called a *K-algebra* if A is a K-module such that for any $a, b \in A$ and $x \in K$, $(ab)x = (ax)b = a(bx)$.

Let A be a K-algebra and A' be a K-algebra anti-isomorphic to A. If A_K is finitely generated, then A is called a *finite K-algebra*. If every finite subset S of A generates a finite subalgebra of A, then A is called a *locally finite K-algebra*. A is called a *faithful K-algebra* if for any $x \in K$, $Ax = 0$ implies $x = 0$. A is called an *unramified K-algebra* if $J(A) = J(K)A$. A finite K-algebra A is called a *proper maximally central algebra* if $A \otimes_K A'$ is isomorphic to the ring of endomorphisms of the module A_K [1, p. 128]. A finite K-algebra A is called *maximally central* if it is a finite direct sum of ideals $A_1, \dots, A_k$ such that each A_i is proper maximally central over $Z(A_i)$ [1, p. 132]. Proper maximally central algebras are also called *Azumaya algebras* [4, (13.7.6)]. For results on central simple algebras over a field one may consult Pierce [7]. Any proper maximally central algebra over a field is a central simple algebra [1, Theorem 14].

A ring R is called a *local ring* if $R/J(R)$ is a division ring. Let R be a local, commutative ring. R is called a *Hensel ring* if for any monic polynomial $f(x) \in R[x]$, any factorization of $f(x)$ modulo $J(R)$ into two co-prime monic polynomials can be lifted to a factorization into co-prime monic polynomials in $R[x]$. Any local commutative ring R which is noetherian and $J(R)$-adic complete, is a Hensel ring [3, Theorem 3].

Let A be a finite K-algebra. A finite subalgebra T of A is called an *inertial subalgebra* if $A = T + J(A)$, $T \cap J(A) = J(T)$ and T is unramified over K.

The following fundamental theorem is due to Azumaya [1, Theorem 33].

THEOREM 1.1 (Generalized Wedderburn–Mal'tsev Theorem). *Let A be a finite K-algebra, where K is a Hensel ring, such that $A/J(A)$ is separable over $K/J(K)$. Then there exists a maximally central inertial subalgebra of A, and such an inertial algebra is uniquely determined up to inner automorphisms of A generated by the elements of $J(A)$, in the sense that given any two inertial subalgebras T and T' of A, we have $T' = T^c$ for some $c \in J(A)$.*

A local ring R with maximal ideal M will also be denoted by (R, M). Consider any commutative local ring (R, M), and any monic polynomial $f(x) \in R[x]$ such that for some monic polynomial $g(x) \in R[x]$ irreducible modulo M, $f(x) \equiv g(x)^t \pmod{M[x]}$. It follows, by using the fact that

$R[x]/\langle f(x)\rangle$ is a finite R-module, that $R[x]/\langle f(x)\rangle$ is a local ring with radical $\underline{\langle M, g(x)\rangle}/\langle f(x)\rangle$. In particular if $t = 1$, then the radical of $R[x]/\langle f(x)\rangle$ is $\overline{M[x]} = \langle M, f(x)\rangle/\langle f(x)\rangle$, so that this ring is unramified over R.

2. Inertial subrings. Let R be a commutative local ring. Then a ring S is called an *R-separable algebra* if it is a commutative, local, faithful, finite, unramified R-algebra such that $S/J(S)$ is a finite separable field extension of $\overline{R} = R/J(R)$. If a local ring S is R-separable, where R is a *special primary ring* (i.e., R is a local artinian principal ideal ring [5, p. 200]), then S is also a special primary ring and the indices of nilpotency of $J(S)$ and $J(R)$ are the same. Let R be a Hensel ring and S be an R-separable algebra. Then S is a Hensel ring [1, Theorem 23]. If $\overline{S} = S/J(S)$ is generated by an $\overline{a}$ over $\overline{R}$, and $f(x) \in R[x]$ is a monic polynomial which, modulo $J(R)$, is the minimal polynomial of $\overline{a}$ over $\overline{R}$, then we can find a lifting $a \in S$ of $\overline{a}$ such that $f(a) = 0$.

LEMMA 2.1. *Let A be a commutative local ring and R be a local subring of A such that $J(R) = R \cap J(A)$. Let some $\overline{a}$ in $\overline{A} = A/J(A)$ be separable over $\overline{R}$. If $f(x) \in R[x]$ is a monic polynomial which modulo $J(R)$ is the minimal polynomial of $\overline{a}$ over $\overline{R}$, then $\overline{a}$ has at most one lifting a in A satisfying $f(a) = 0$.*

Proof. Let b and c be two liftings of $\overline{a}$ in A such that $f(b) = f(c) = 0$. Now $c = b + h$ for some $h \in J(A)$. Let $f'(x)$ denote the derivative of $f(x)$. Then $0 = f(b+h) = f(b) + h(f'(b) + hd) = h(f'(b) + hd)$ for some $d \in A$. As $\overline{a}$ is separable over $\overline{R}$, $f'(b)$ is a unit. This gives $h = 0$, and hence $b = c$.

Let A be a ring and R be a subring of A contained in the center $Z(A)$. Any $a \in A$ is said to be *algebraic* over R if $f(a) = 0$ for some monic $f(x) \in R[x]$. In case A is a Hensel ring and R is a local subring such that $R \cap J(A) = J(R)$, if an element $a \in A$ is algebraic over R and $\overline{a}$ is separable over $\overline{R} = R/J(R)$, it follows from the definition of a Hensel ring that there exists a monic polynomial $f(x) \in R[x]$ which modulo $J(R)$ is irreducible over $\overline{R}$ and there exists a lifting b of $\overline{a}$ such that $f(b) = 0$. In case A is a local ring, an $a \in A$ is said to be *lift algebraic* over R if there exists a monic polynomial $f(x) \in R[x]$ such that $f(a) = 0$ and $f(x)$ modulo $J(R)$ is irreducible over $\overline{R}$.

LEMMA 2.2. *Let R be a Hensel ring and S be a local finite unramified R-algebra such that S is maximally central and $S/J(S)$ is commutative. Then*:

(i) *S is a Hensel ring.*

(ii) *If $\overline{S} = S/J(S)$ is separable over $R/J(R)$, then $S = R[a]$ for some a lift algebraic over R.*

(iii) *If R is a special primary ring, then S is also a special primary ring.*

Proof. Clearly $Z(S)$ is a local ring. As S is proper maximally central over $Z(S)$, by [1, Theorem 13], there exists one-to-one correspondence between the ideals of $Z(S)$ and the ideals of S given by $A \leftrightarrow AS$, where A is an ideal of $Z(S)$. We get $J(S) = J(Z(S))S$. By the hypothesis $J(S) = J(R)S$. So $J(Z(S))S = J(R)Z(S)S$. Consequently, $J(Z(S)) = J(R)Z(S)$. By [1, Theorem 13], $S/J(S)$ is proper maximally central over $Z(S)/J(Z(S))$. But by [1, Theorem 14], any proper maximally central algebra over a field is central simple. Consequently, $S = Z(S) + J(R)S$. This gives a finite basis $\overline{B}$ of the $R/J(R)$-module $S/J(S)$ that has a lifting B in $Z(S)$. Then $S = R[B] = Z(S)$. By [1, Theorem 23], S is a Hensel ring. This proves (i).

Let S satisfy the hypothesis in (ii). There exists $a \in S$ such that a is lift algebraic over R and $\overline{a}$ generates $\overline{S}$ over $\overline{R}$. Then $S = R[a]$. There exists a monic polynomial $f(x) \in R[x]$ irreducible modulo $J(R)$ satisfying $f(a) = 0$. As $R[x]/\langle f(x)\rangle$ is a local ring unramified over R, so is $R[a]$. This proves (ii).

Finally, let R be a special primary ring. Then $J(R)$ is principal and nilpotent. This shows that $J(S)$ is principal and nilpotent, so S is a special primary ring. This proves (iii).

Let A be any locally finite algebra over a commutative ring R, and S be any subalgebra of A. Consider any $a \in J(A) \cap S$ and let $b \in A$ be its quasi-inverse. As $R[a, b]$ is a finite R-algebra, by [1, Corollary to Theorem 9], $b \in R[a]$. Hence $J(A) \cap S \subseteq J(S)$.

LEMMA 2.3. *Let A be a local, locally finite, faithful algebra over a local ring R such that $R \cap J(A) = J(R)$ and $\overline{A} = A/J(A)$ is an algebraic field extension of $\overline{R}$. Then any R-subalgebra S of A is a local ring and $J(S) = S \cap J(A)$.*

Proof. As remarked above, $J(A) \cap S \subseteq J(S)$. That $\overline{S} = S/J(A) \cap S$ is a field follows from the hypothesis that $\overline{A}$ is an algebraic field extension of $\overline{R}$. This proves the result.

LEMMA 2.4. *Let A be a local, locally finite faithful algebra over a Hensel ring R such that $R \cap J(A) = J(R)$ and $\overline{A} = A/J(A)$ is a separable algebraic field extension of $\overline{R}$. Let $a, b \in A$ be lift algebraic over R, $ab = ba$ and let $f(x) \in R[x]$ be a monic polynomial irreducible modulo $J(R)$ such that $f(a) = 0$. Then:*

(i) *$R[a]$ is a Hensel ring unramified over R.*

(ii) *If $\overline{R[a]} \subseteq \overline{R[b]}$, then $R[a] \subseteq R[b]$.*

(iii) *If $c, d \in A$ both lift $\overline{a}$ and $f(c) = f(d) = 0$, then they are conjugate in A.*

(iv) *Any finite, unramified R-subalgebra of A is a Hensel ring and is of the form $R[d]$ for some d lift algebraic over R.*

Proof. By 2.3, $R[a]$ is a local ring, and it satisfies the hypothesis of 1.1. So $R[a]$ has an inertial subring T. By definition, T is unramified over R. By 2.2(i), T is a Hensel ring. As $\overline{a} \in \overline{T}$, there exists an $a' \in T$ lifting $\overline{a}$ such that $f(a') = 0$. By 2.1, $a = a'$. So $T = R[a]$. This proves (i).

As $R[b]$ is a Hensel ring, 2.1 gives (ii).

Consider $S = R[c, d]$. Now $\overline{c} = \overline{d} = \overline{a}$. By 2.3, $R[c] \cap J(S) = J(R[c])$. Also $R[c]$ is an unramified commutative R-algebra. This shows that $R[c]$ is an inertial subring of S; similarly $R[d]$ is also an inertial subring of S. By 1.1, there exists $g \in S$ such that $g^{-1}R[c]g = R[d]$. In $R[d]$, d and $g^{-1}cg$ both lift a and both are roots of $f(x)$. By 2.1, $d = g^{-1}cg$.

Let S be a finite unramified R-subalgebra of A. As $\overline{S}$ is a simple extension of $\overline{R}$, there exists a $d \in S$ lift algebraic over R such that $\overline{S} = \overline{R[d]}$. As $\overline{S} \cong S/J(R)S$, by [1, Corollary to Theorem 5], $S = R[d]$. Thus S is commutative. By (i), S is a Hensel ring.

THEOREM 2.5. *Let A be a local, locally finite, faithful algebra over a Hensel ring R such that $J(R) = R \cap J(A)$, $\overline{A} = A/J(A)$ is a countably generated, separable algebraic field extension of $\overline{R}$. Then there exists a commutative local unramified R-subalgebra T of A such that*

(i) *T is the union of a filter of unramified R-subalgebras of the form $R[a]$, where a is lift algebraic over R,*

(ii) $J(T) = T \cap J(A)$,

(iii) $A = T + J(A)$.

Further, any two such subrings are R-isomorphic. In case A is commutative, T is unique.

Proof. Let K be a finite unramified R-subalgebra of A. By 2.4, $K = R[a]$, where a is some lift algebraic element over R. Choose any lift algebraic element $b \in A$ such that $\overline{b} \notin \overline{K}$. Consider $L = R[a, b]$. By 2.3, L is a local, finite R-subalgebra. By 1.1, L has an inertial subalgebra S. As S is maximally central, by 2.2, it is a Hensel ring. As S is unramified over R, by 2.4, $S = R[c]$ for some lift algebraic element c over R. Let $f(x) \in R[x]$ be a monic polynomial irreducible modulo $J(R)$ such that $f(a) = 0$. There exists $d \in S$ lifting $\overline{a}$ such that $f(d) = 0$. By 2.4(iii), $a = u^{-1}du$ for some $u \in A$. Then $K \subset u^{-1}Su$ and $\overline{b} \in \overline{u^{-1}Su}$. Thus $K' = u^{-1}Su$ is a finite unramified R-subalgebra containing K, and $\overline{K'}$ contains $\overline{b}$. As $\overline{A}$ is countably generated over $\overline{R}$, the above construction gives an ascending sequence $\{S_n\}$, where each S_n is a Hensel ring which is a finite R-subalgebra of A unramified over R and for $T = \bigcup S_n$, $\overline{T} = \overline{A}$. Then $A = T + J(A)$, T is unramified over R, and 2.3 gives $J(T) = T \cap J(A)$.

Let T' be another subalgebra of A satisfying (i)–(iii). Thus T' is an unramified R-algebra, and a Hensel ring. Consider any $a \in T$ which is lift algebraic over R. Let $f(x) \in R[x]$ be a monic polynomial which is irreducible modulo $J(R)$ and $f(a) = 0$. As $\overline{a} \in \overline{T'}$, by (i) and 2.1 there exists a unique $a' \in T'$ lifting $\overline{a}$ and satisfying $f(a') = 0$. As a and a' are conjugate, we get an R-isomorphism $\sigma : R[a] \to R[a']$ such that $\sigma(a) = a'$. Consider any other lift algebraic element $b \in T$ such that $R[a] \subseteq R[b]$. We get a unique $b' \in T'$ for which we have an R-isomorphism $\eta : R[b] \to R[b']$. As $R[b']$ is a Hensel ring and $\overline{a'} \in \overline{R[b']}$, by 2.4 $a' \in R[b']$. It is now obvious that η extends σ. Thus (i) and the above construction of partial isomorphisms gives an R-monomorphism $\lambda : T \to T'$. That λ is an isomorphism follows from condition (i). In case A is commutative, the above proof itself shows that T is unique.

Let us call such a T an *inertial subring* of A.

COROLLARY 2.6. *Let A be a finite local ring of characteristic p^k, where p is a prime number. If A modulo $J(A)$ is isomorphic to the Galois field* $\mathrm{GF}(p^r)$, *then A has a subring T isomorphic to* $\mathrm{GR}(p^k, k)$ *and $A = T + J(A)$. This T is unique to within isomorphisms.*

Proof. In the above theorem take $R = \mathbb{Z}/\langle p^k\rangle$.

EXAMPLE. Consider fields $K \subset F_1 \subset F_2 \subset F$ with F_1, F_2 different finite normal extensions of K. Let there exist two commuting automorphisms σ, η of F such that the fixed fields of σ and η are F_1 and F_2 respectively. Consider the left skew polynomial ring $F[x, \sigma]$ with $xa = \sigma(a)x$ for $a \in F$. Let $R_1 = F[x, \sigma]/\langle x^3\rangle$. For $u = \overline{1 + x + x^2} \in R_1$, $u^{-1} = \overline{1 - x}$. For any $b \in F$,

$$u^{-1}bu = \overline{b + (b - \sigma(b))x + (b - \sigma(b))x^2}.$$

Thus as F_2 is not contained in the fixed field of σ, $u^{-1}F_2u \nsubseteq F$. As $\sigma\eta = \eta\sigma$, η induces an automorphism λ of $F[x, \sigma]$ such that $\lambda(ax^i) = \eta(a)x^i$; λ is identity over $F_2[x, \sigma]$. We still denote λ by η. We can form $F[x, \sigma][y, \eta] = F[x, y, \sigma, \eta]$ with $xy = yx$. Consider $R_2 = F[x, y, \sigma, \eta]/\langle x^3, y^3\rangle$. For $v = \overline{1 + y + y^2} \in R_2$, it is immediate that $v^{-1}F_2v = F_2$, and if $F \neq F_2$, then $v^{-1}Fv \nsubseteq R_1$.

We now extend this construction. Consider a field F which admits an infinite properly ascending sequence $\{F_n\}$ of subfields indexed by the set of natural numbers, with $K = F_0$, such that each F_n is a finite separable normal extension of K. Further suppose that there exists a sequence $\{\eta_n\}$ of pairwise commuting automorphisms of F such that the fixed field of any η_n is F_n, and $F = \bigcup_n F_n$. Consider a sequence of indeterminates x_j, $j \geq 1$. Set $R_0 = F$, $R_{n+1} = R_n[x_{n+1}, \eta_{n+1}]/\langle x_{n+1}^3\rangle$ with $x_ix_j = x_jx_i$, and $R = \bigcup_n R_n$. Then R is a local, locally finite K-algebra such that $R/J(R)$ is a countably generated, separable algebraic field extension of K. Obviously F is an inertial subring of R.

We now construct another inertial subring F' of R such that F and F' are not conjugate. Consider any $k \geq 1$. Set $v_k = \overline{1 + x_k + x_k^2} \in R_k$, $w_k = v_1 \dots v_k$. Set $F'_1 = F_1$, $F'_{k+1} = w_k^{-1} F_{k+1} w_k$ for $k \geq 1$. As η_k is identity on F_k but not on F_{k+1}, it follows that $w_k^{-1} F_{k+1} w_k \subset R_k$, but $w_k^{-1} F_{k+1} w_k \nsubseteq R_{k-1}$. That means that $F'_{k+1} \subset R_k$, but $F'_{k+1} \nsubseteq R_{k-1}$. Now $v_k^{-1} F'_k v_k = F'_k$ gives $F'_k \subseteq F'_{k+1}$. Then $F' = \bigcup_n F'_n$ is an inertial subring of R. As F' is not contained in any R_k, it cannot be conjugate to F. To get a field F of the above type, consider $K = \mathbb{Z}_2$. This gives rise to an ascending sequence of Galois fields F_i of orders 2^{n_i} where $n_i = 2^i$.

A ring R in which every one-sided ideal is two-sided, is called a *duo ring.*

THEOREM 2.7. *Let A be a local artinian duo ring which is an algebra over a commutative local ring (R, M) with M nilpotent and $\bar{A} = A/J(A)$ a countably generated separable algebraic field extension of $\bar{R}$. Then A has a commutative local subring T unramified over R such that $A = T + J(A)$. Further T is unique to within R-isomorphisms.*

Proof. Any R-subalgebra S of A is local with $J(S) = J(A) \cap S$, and $J(S)$ is nilpotent. Let $d_A(A) = n$. We apply induction on n. The result holds for $n = 1$. Let $n > 1$ and suppose that the result holds for $n - 1$. Let L be a minimal ideal of A. Then for some $\pi \in L$, $L = \pi A = A\pi = \pi \bar{A}$ and there exists an $\bar{R}$-automorphism σ of $\bar{A}$ such that $\bar{a}\pi = \pi\sigma(\bar{a})$ for any $\bar{a} \in \bar{A}$. By the induction hypothesis $B = A/L$ has a commutative local subring T/L which satisfies the conclusion of the theorem. Then $A = T + J(A)$ and $L = \pi T$ is a minimal ideal of T. Let K be any commutative unramified R-subalgebra of T such that $\bar{K}$ is a finite extension of $\bar{R}$. Then $K = R[a]$ for some a lift algebraic over R. Let $\bar{K} \neq \bar{T}$. Consider any $b \in T$ algebraic over R modulo L. As T/L is commutative, we can choose b algebraic modulo L over R/C, where $C = \operatorname{ann}_R(T/L)$. So there exists a monic polynomial $g(x) \in R[x]$ such that $g(b) \in L$. Consequently, $g(b) = \pi\bar{c}$ for some $c \in T$. As T/L is commutative, $ba - ab = \pi\bar{d}$ for some $d \in T$. Let $\deg g(x) = n$. There exists a finite field extension $\bar{G}$ of $\bar{R}$ in $\bar{T}$ such that $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are in $\bar{G}$ and for any $\bar{R}$-automorphism η of $\bar{T}$, $\eta(\bar{G}) \subseteq \bar{G}$. Then

$$S = \sum_{i=1}^{n-1} R[a]b^i + \pi\bar{G}$$

is a finite R-subalgebra of T. Let W be an inertial subring of S. Suppose first that $\bar{a} = \bar{b}$. Then $\bar{W} = \overline{R[a]} = \bar{S}$. So $R[a]$ is also an inertial subring of S, and W is a conjugate of $R[a]$. So W contains a conjugate of a. Now suppose that $\bar{b} \notin \bar{K}$. As $\bar{a} \in \bar{W}$, we can find a lifting a' of $\bar{a}$ in W. Then a is a conjugate of a'. So, for some unit $u \in S$, $K \subseteq u^{-1}Wu$ and clearly $\bar{b} \in \bar{W}$. Now to conclude the proof we can follow the arguments of 2.5.

A part of the following result bears a similarity to the main result in Clark [2].

THEOREM 2.8. *Let A be a semi-perfect ring which is a locally finite faithful algebra over a Hensel ring R such that $J(R) = R \cap J(A)$ and $\bar{A} = A/J(A)$ is a direct sum of matrix rings over fields which are countably generated separable algebraic extensions of $\bar{R}$. Then A has a subalgebra T such that $A = T + J(A)$ and $T \cap J(A) = J(T)$. Moreover T is a direct sum of full matrix rings over commutative local rings T' such that if R' is the homomorphic image of R in T', then T' is the union of a filter of unramified local R'-subalgebras of the form $R'[a]$. Further, T is unique to within R-isomorphisms.*

Proof. Since the proof is similar to that of [1, Theorem 33], we only outline it. Observe that for any idempotent $e \in A$, eAe is a locally finite R-algebra. As in the proof of [1, Theorem 33], we first consider the case when $A/J(A)$ is simple. As idempotents can be lifted modulo $J(A)$, $A = M_n(B)$, a full $n \times n$-matrix ring over a local ring B. Let $D = \{e_{ij} : 1 \leq i, j \leq n\}$ be the corresponding system of matrix units in A. The hypothesis on A gives that B is an R-algebra satisfying the hypothesis of 2.5. Consequently, B has an inertial subring S. Then $T = M_n(S)$ is the desired subring of A.

Let T' be another such subring of A. As T' is a full matrix ring over a local ring and $T'/J(T') \cong A/J(A)$, we can find a system $L = \{f_{ij} : 1 \leq i, j \leq n\}$ of matrix units of T' that is also a system of matrix units of A. Now $T' = M_n(S')$, where S' is the centralizer of L in T'. If A' is the centralizer of L in A, then A' is a local ring and S' is an inertial subring of A'. By [1, Theorem 4], there exists a $c \in J(A)$ such that $f_{ij}^c = e_{ij}$. As in the proof of [1, Theorem 33] we see that $(S')^c$ is an inertial subring of B. By 2.5, S and $(S')^c$ are R-isomorphic. This proves that T and T' are R-isomorphic. Now, the general case can be proved along similar lines to [1, Theorem 33].

Let A be any ring, and P be the smallest subring of A such that any $a \in P$ is a unit in P if and only if a is a unit in A. If I is the identity element of A, then P is the set of elements nI/mI, where n, m are integers and mI is a unit in A. We call P the *total prime subring* of A. Let A be a local ring. If the characteristic of $\bar{A} = A/J(A)$ is zero, then P is isomorphic to the field $\mathbb{Q}$ of rational numbers, and if the characteristic of $\bar{A}$ is a prime number p, then P is a homomorphic image of the localization $\mathbb{Z}_{(p)}$. By 2.5 and 2.7 we get the following.

THEOREM 2.9. *Let A be a local ring and P be its total prime subring such that P is isomorphic either to $\mathbb{Q}$ or to $\mathbb{Z}/(p^n)$. Let A be either a locally finite P-algebra or an artinian duo ring. If $\bar{A} = A/J(A)$ is an absolutely algebraic field, then it has a local subring T such that $A = T + J(R)$, $J(T) = T \cap J(A)$*

and the following hold:

(i) *if the characteristic of* A *is zero*, *then* T *is a field isomorphic to* $\bar{A}$,

(ii) *if the characteristic of* A *is* p^n *for some prime number* p *and an* $n \geq 1$, *then* T *is the union of an ascending sequence of subrings which are Galois rings of the type* $\mathrm{GR}(p^n, r)$.

Further, T *is unique to within isomorphisms*; *in any case* $J(T) = qT$, *where* q *is the characteristic of* $\bar{A}$. (T *is called a coefficient ring of* A.)

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