INERTIAL SUBRINGS OF A LOCALLY FINITE ALGEBRA

BY

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Abstract. I. S. Cohen proved that any commutative local noetherian ring \( R \) that is \( J(R) \)-adic complete admits a coefficient subring. Analogous to the concept of a coefficient subring is the concept of an inertial subring of an algebra \( A \) over a commutative ring \( K \). In case \( K \) is a Hensel ring and the module \( A_K \) is finitely generated, under some additional conditions, as proved by Azumaya, \( A \) admits an inertial subring. In this paper the question of existence of an inertial subring in a locally finite algebra is discussed.

Introduction. Let \( S \) be a commutative generalized local ring in the sense of Cohen [3]. Cohen introduced the concept of a coefficient subring of \( S \). If \( S \) is \( J(S) \)-adic complete, Cohen proved that \( S \) has a coefficient subring \( T \), which is unique to within isomorphisms. On the other hand, Azumaya [1] introduced the concept of an inertial subalgebra of an algebra \( A \) over a Hensel ring \( R \). In case \( A_R \) is finitely generated and \( \overline{A} = A/J(A) \) is separable over \( \overline{R} = R/J(R) \), \( A \) has an inertial subalgebra, which is unique to within conjugations [1, Theorem 33]; this result generalizes the Wedderburn–Mal’tsev Principal Theorem. A special case of Azumaya’s theorem, in case \( A \) is a finite ring, is given in Clark [2]. The concept of an inertial subring is analogous to that of a coefficient subring. Let \( A \) be a local ring which is an algebra over a Hensel ring \( R, J(R) = R \cap J(A) \) and \( \overline{A} = A/J(A) \) is a countably generated separable algebraic field extension of \( \overline{R} \). In case \( A \) is a locally finite \( R \)-algebra, the existence of a subalgebra \( T \) of \( A \) analogous to an inertial subring is shown in Theorem 2.5. This subalgebra is shown to be unique to within \( R \)-isomorphisms. In case \( A \) is commutative, this subalgebra is unique. In case \( A \) is not commutative, an example is given to show that unlike in Azumaya’s theorem, two such subalgebras need not be conjugate. In case \( A \) is an artinian duo ring and \( J(R) \) is nilpotent, in Theorem 2.7 the existence of a commutative local subring \( T \) analogous to an inertial subring is established. In Theorem 2.8 a locally finite algebra \( A \) over a Hensel ring \( R \) such that \( A \) is semi-perfect ring is studied. Some sufficient conditions for the existence of a subring \( T \) of \( A \) analogous to an inertial subring are given.

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1. Preliminaries. All rings considered here are with identity $1 \neq 0$, and all modules are unital right modules unless otherwise stated. For general concepts on rings and modules one may refer to Faith [4]. For any ring $A$, $J(A)$, $Z(A)$ will denote its Jacobson radical and center respectively. For any $c \in J(A)$ and any subset $Y$ of $A$, $Y^c$ denotes the conjugate $(1-c)^{-1}Y(1-c)$ of $Y$. For any module $M$ over $A$ of finite composition length, $d_A(M)$ denotes the composition length of $M$. Let $K$ be any commutative ring. Then a ring $A$ is called a $K$-algebra if $A$ is a $K$-module such that for any $a, b \in A$ and $x \in K$, $(ab)x = (ax)b = a(bx)$.

Let $A$ be a $K$-algebra and $A'$ be a $K$-algebra anti-isomorphic to $A$. If $A_K$ is finitely generated, then $A$ is called a finite $K$-algebra. If every finite subset $S$ of $A$ generates a finite subalgebra of $A$, then $A$ is called a locally finite $K$-algebra. $A$ is called a faithful $K$-algebra if for any $x \in K$, $Ax = 0$ implies $x = 0$. $A$ is called an unramified $K$-algebra if $J(A) = J(K)A$. A finite $K$-algebra $A$ is called a proper maximally central algebra if $A \otimes_K A'$ is isomorphic to the ring of endomorphisms of the module $A_K$ [1, p. 128]. A finite $K$-algebra $A$ is called maximally central if it is a finite direct sum of ideals $A_1, \ldots, A_k$ such that each $A_i$ is proper maximally central over $Z(A_i)$ [1, p. 132]. Proper maximally central algebras are also called Azumaya algebras [4, (13.7.6)]. For results on central simple algebras over a field one may consult Pierce [7]. Any proper maximally central algebra over a field is a central simple algebra [1, Theorem 14].

A ring $R$ is called a local ring if $R/J(R)$ is a division ring. Let $R$ be a local, commutative ring. $R$ is called a Hensel ring if for any monic polynomial $f(x) \in R[x]$, any factorization of $f(x)$ modulo $J(R)$ into two co-prime monic polynomials can be lifted to a factorization into co-prime monic polynomials in $R[x]$. Any local commutative ring $R$ which is noetherian and $J(R)$-adic complete, is a Hensel ring [3, Theorem 3].

Let $A$ be a finite $K$-algebra. A finite subalgebra $T$ of $A$ is called an inertial subalgebra if $A = T + J(A)$, $T \cap J(A) = J(T)$ and $T$ is unramified over $K$.

The following fundamental theorem is due to Azumaya [1, Theorem 33].

**Theorem 1.1** (Generalized Wedderburn–Mal’tsev Theorem). Let $A$ be a finite $K$-algebra, where $K$ is a Hensel ring, such that $A/J(A)$ is separable over $K/J(K)$. Then there exists a maximally central inertial subalgebra of $A$, and such an inertial algebra is uniquely determined up to inner automorphisms of $A$ generated by the elements of $J(A)$, in the sense that given any two inertial subalgebras $T$ and $T'$ of $A$, we have $T' = T^c$ for some $c \in J(A)$.

A local ring $R$ with maximal ideal $M$ will also be denoted by $(R, M)$. Consider any commutative local ring $(R, M)$, and any monic polynomial $f(x) \in R[x]$ such that for some monic polynomial $g(x) \in R[x]$ irreducible modulo $M$, $f(x) \equiv g(x)^t \pmod{M[x]}$. It follows, by using the fact that
$R[x]/(f(x))$ is a finite $R$-module, that $R[x]/(f(x))$ is a local ring with radical $(M, g(x))//(f(x))$. In particular if $t = 1$, then the radical of $R[x]/(f(x))$ is $M[x] = (M, f(x))//(f(x))$, so that this ring is unramified over $R$.

2. Inertial subrings. Let $R$ be a commutative local ring. Then a ring $S$ is called an $R$-separable algebra if it is a commutative, local, faithful, finite, unramified $R$-algebra such that $S/J(S)$ is a finite separable field extension of $\bar{R} = R/J(R)$. If a local ring $S$ is $R$-separable, where $R$ is a special primary ring (i.e., $R$ is a local artinian principal ideal ring [5, p. 200]), then $S$ is also a special primary ring and the indices of nilpotency of $J(S)$ and $J(R)$ are the same. Let $R$ be a Hensel ring and $S$ be an $R$-separable algebra. Then $S$ is a Hensel ring [1, Theorem 23]. If $\bar{S} = S/J(S)$ is generated by an $\bar{a}$ over $\bar{R}$, and $f(x) \in R[x]$ is a monic polynomial which, modulo $J(R)$, is the minimal polynomial of $\bar{a}$ over $\bar{R}$, then we can find a lifting $a \in S$ of $\bar{a}$ such that $f(a) = 0$.

**Lemma 2.1.** Let $A$ be a commutative local ring and $R$ be a local subring of $A$ such that $J(R) = R \cap J(A)$. Let some $\bar{a}$ in $\bar{A} = A/J(A)$ be separable over $\bar{R}$. If $f(x) \in R[x]$ is a monic polynomial which modulo $J(R)$ is the minimal polynomial of $\bar{a}$ over $\bar{R}$, then $\bar{a}$ has at most one lifting $a$ in $A$ satisfying $f(a) = 0$.

**Proof.** Let $b$ and $c$ be two liftings of $\bar{a}$ in $A$ such that $f(b) = f(c) = 0$. Now $c = b + h$ for some $h \in J(A)$. Let $f'(x)$ denote the derivative of $f(x)$. Then $0 = f(b + h) = f(b) + h(f'(b) + hd) = h(f'(b) + hd)$ for some $d \in A$. As $\bar{a}$ is separable over $\bar{R}$, $f'(b)$ is a unit. This gives $h = 0$, and hence $b = c$.

Let $A$ be a ring and $R$ be a subring of $A$ contained in the center $Z(A)$. Any $a \in A$ is said to be algebraic over $R$ if $f(a) = 0$ for some monic $f(x) \in R[x]$. In case $A$ is a Hensel ring and $R$ is a local subring such that $R \cap J(A) = J(R)$, if an element $a \in A$ is algebraic over $R$ and $\bar{a}$ is separable over $\bar{R} = R/J(R)$, it follows from the definition of a Hensel ring that there exists a monic polynomial $f(x) \in R[x]$ which modulo $J(R)$ is irreducible over $\bar{R}$ and there exists a lifting $b$ of $\bar{a}$ such that $f(b) = 0$. In case $A$ is a local ring, an $a \in A$ is said to be lift algebraic over $R$ if there exists a monic polynomial $f(x) \in R[x]$ such that $f(a) = 0$ and $f(x)$ modulo $J(R)$ is irreducible over $\bar{R}$.

**Lemma 2.2.** Let $R$ be a Hensel ring and $S$ be a local finite unramified $R$-algebra such that $S$ is maximally central and $S/J(S)$ is commutative. Then:

(i) $S$ is a Hensel ring.

(ii) If $\bar{S} = S/J(S)$ is separable over $R/J(R)$, then $S = R[a]$ for some a lift algebraic over $R$.

(iii) If $R$ is a special primary ring, then $S$ is also a special primary ring.
Proof. Clearly $Z(S)$ is a local ring. As $S$ is proper maximally central over $Z(S)$, by [1, Theorem 13], there exists one-to-one correspondence between the ideals of $Z(S)$ and the ideals of $S$ given by $A \leftrightarrow AS$, where $A$ is an ideal of $Z(S)$. We get $J(S) = J(Z(S))S$. By the hypothesis $J(S) = J(R)S$. So $J(Z(S))S = J(R)Z(S)S$. Consequently, $J(Z(S)) = J(R)Z(S)$. By [1, Theorem 13], $S/J(S)$ is proper maximally central over $Z(S)/J(Z(S))$. But by [1, Theorem 14], any proper maximally central algebra over a field is central simple. Consequently, $S = Z(S) + J(R)S$. This gives a finite basis $B$ of the $R/J(R)$-module $S/J(S)$ that has a lifting $B$ in $Z(S)$. Then $S = R[B] = Z(S)$. By [1, Theorem 23], $S$ is a Hensel ring. This proves (i).

Let $S$ satisfy the hypothesis in (ii). There exists $a \in S$ such that $a$ is lift algebraic over $R$ and $\overline{a}$ generates $S$ over $\overline{R}$. Then $S = R[a]$. There exists a monic polynomial $f(x) \in R[x]$ irreducible modulo $J(R)$ satisfying $f(a) = 0$. As $R[x]/(f(x))$ is a local ring unramified over $R$, so is $R[a]$. This proves (ii).

Finally, let $R$ be a special primary ring. Then $J(R)$ is principal and nilpotent. This shows that $J(S)$ is principal and nilpotent, so $S$ is a special primary ring. This proves (iii).

Let $A$ be any locally finite algebra over a commutative ring $R$, and $S$ be any subalgebra of $A$. Consider any $a \in J(A) \cap S$ and let $b \in A$ be its quasi-inverse. As $R[a, b]$ is a finite $R$-algebra, by [1, Corollary to Theorem 9], $b \in R[a]$. Hence $J(A) \cap S \subseteq J(S)$.

Lemma 2.3. Let $A$ be a local, locally finite, faithful algebra over a local ring $R$ such that $R \cap J(A) = J(R)$ and $\overline{A} = A/J(A)$ is an algebraic field extension of $\overline{R}$. Then any $R$-subalgebra $S$ of $A$ is a local ring and $J(S) = S \cap J(A)$.

Proof. As remarked above, $J(A) \cap S \subseteq J(S)$. That $\overline{S} = S/J(A) \cap S$ is a field follows from the hypothesis that $\overline{A}$ is an algebraic field extension of $\overline{R}$. This proves the result.

Lemma 2.4. Let $A$ be a local, locally finite faithful algebra over a Hensel ring $R$ such that $R \cap J(A) = J(R)$ and $\overline{A} = A/J(A)$ is a separable algebraic field extension of $\overline{R}$. Let $a, b \in A$ be lift algebraic over $R$, $ab = ba$ and let $f(x) \in R[x]$ be a monic polynomial irreducible modulo $J(R)$ such that $f(a) = 0$. Then:

(i) $R[a]$ is a Hensel ring unramified over $R$.
(ii) If $\overline{R[a]} \subseteq \overline{R[b]}$, then $R[a] \subseteq R[b]$.
(iii) If $c, d \in A$ both lift $\overline{a}$ and $f(c) = f(d) = 0$, then they are conjugate in $A$.
(iv) Any finite, unramified $R$-subalgebra of $A$ is a Hensel ring and is of the form $R[d]$ for some $d$ lift algebraic over $R$.
Proof. By 2.3, $R[a]$ is a local ring, and it satisfies the hypothesis of 1.1. So $R[a]$ has an inertial subring $T$. By definition, $T$ is unramified over $R$. By 2.2(i), $T$ is a Hensel ring. As $\alpha \in \overline{T}$, there exists an $a' \in T$ lifting $\alpha$ such that $f(a') = 0$. By 2.1, $a = a'$. So $T = R[a]$. This proves (i).

As $R[b]$ is a Hensel ring, 2.1 gives (ii).

Consider $S = R[c, d]$. Now $\bar{c} = \bar{d} = \bar{a}$. By 2.3, $R[c] \cap J(S) = J(R[c])$. Also $R[c]$ is an unramified commutative $R$-algebra. This shows that $R[c]$ is an inertial subring of $S$; similarly $R[d]$ is also an inertial subring of $S$. By 1.1, there exists $g \in S$ such that $g^{-1}R[c]g = R[d]$. In $R[d]$, $d$ and $g^{-1}cg$ both lift $a$ and both are roots of $f(x)$. By 2.1, $d = g^{-1}cg$.

Let $S$ be a finite unramified $R$-subalgebra of $A$. As $\overline{S}$ is a simple extension of $\overline{R}$, there exists a $d \in S$ lift algebraic over $R$ such that $\overline{S} = \overline{R}[d]$. As $\overline{S} \cong S/J(R)S$, by [1, Corollary to Theorem 5], $S = R[d]$. Thus $S$ is commutative. By (i), $S$ is a Hensel ring.

Theorem 2.5. Let $A$ be a local, locally finite, faithful algebra over a Hensel ring $R$ such that $J(R) = R \cap J(A)$, $A = A/J(A)$ is a countably generated, separable algebraic field extension of $\overline{R}$. Then there exists a commutative local unramified $R$-subalgebra $T$ of $A$ such that

(i) $T$ is the union of a filter of unramified $R$-subalgebras of the form $R[a]$, where $a$ is lift algebraic over $R$,

(ii) $J(T) = T \cap J(A)$,

(iii) $A = T + J(A)$.

Further, any two such subrings are $R$-isomorphic. In case $A$ is commutative, $T$ is unique.

Proof. Let $K$ be a finite unramified $R$-subalgebra of $A$. By 2.4, $K = R[a]$, where $a$ is some lift algebraic element over $R$. Choose any lift algebraic element $b \in A$ such that $\overline{b} \notin \overline{K}$. Consider $L = R[a, b]$. By 2.3, $L$ is a local, finite $R$-subalgebra. By 1.1, $L$ has an inertial subalgebra $S$. As $S$ is maximally central, by 2.2, it is a Hensel ring. As $S$ is unramified over $R$, by 2.4, $S = R[c]$ for some lift algebraic element $c$ over $R$. Let $f(x) \in R[x]$ be a monic polynomial irreducible modulo $J(R)$ such that $f(a) = 0$. There exists $d \in S$ lifting $\alpha$ such that $f(d) = 0$. By 2.4(iii), $a = u^{-1}du$ for some $u \in A$. Then $K \subset u^{-1}Su$ and $\overline{b} \in u^{-1}Su$. Thus $K' = u^{-1}Su$ is a finite unramified $R$-subalgebra containing $K$, and $\overline{K'}$ contains $\overline{b}$. As $\overline{A}$ is countably generated over $\overline{R}$, the above construction gives an ascending sequence $\{S_n\}$, where each $S_n$ is a Hensel ring which is a finite $R$-subalgebra of $A$ unramified over $R$ and for $T = \bigcup S_n$, $\overline{T} = \overline{A}$. Then $A = T + J(A)$, $T$ is unramified over $R$, and 2.3 gives $J(T) = T \cap J(A)$.

Let $T'$ be another subalgebra of $A$ satisfying (i)–(iii). Thus $T'$ is an unramified $R$-algebra, and a Hensel ring. Consider any $a \in T$ which is lift
algebraic over \( R \). Let \( f(x) \in R[x] \) be a monic polynomial which is irreducible modulo \( J(R) \) and \( f(a) = 0 \). As \( \overline{a} \in \overline{T}' \), by (i) and 2.1 there exists a unique \( a' \in T' \) lifting \( \overline{a} \) and satisfying \( f(a') = 0 \). As \( a \) and \( a' \) are conjugate, we get an \( R \)-isomorphism \( \sigma : R[a] \to R[a'] \) such that \( \sigma(a) = a' \). Consider any other lift algebraic element \( b \in T \) such that \( R[a] \subseteq R[b] \). We get a unique \( b' \in T' \) for which we have an \( R \)-isomorphism \( \eta : R[b] \to R[b'] \). As \( R[b'] \) is a Hensel ring and \( \overline{a'} \in \overline{R[b']} \), by 2.4 \( a' \in R[b'] \). It is now obvious that \( \eta \) extends \( \sigma \). Thus (i) and the above construction of partial isomorphisms gives an \( R \)-monomorphism \( \lambda : T \to T' \). That \( \lambda \) is an isomorphism follows from condition (i). In case \( A \) is commutative, the above proof itself shows that \( T \) is unique.

Let us call such a \( T \) an inertial subring of \( A \).

Corollary 2.6. Let \( A \) be a finite local ring of characteristic \( p^k \), where \( p \) is a prime number. If \( A \) modulo \( J(A) \) is isomorphic to the Galois field \( GF(p^k) \), then \( A \) has a subring \( T \) isomorphic to \( GR(p^k,k) \) and \( A = T + J(A) \). This \( T \) is unique to within isomorphisms.

Proof. In the above theorem take \( R = \mathbb{Z}/(p^k) \).

Example. Consider fields \( K \subset F_1 \subset F_2 \subset F \) with \( F_1, F_2 \) different finite normal extensions of \( K \). Let there exist two commuting automorphisms \( \sigma, \eta \) of \( F \) such that the fixed fields of \( \sigma \) and \( \eta \) are \( F_1 \) and \( F_2 \) respectively. Consider the left skew polynomial ring \( F[x, \sigma] \) with \( xa = \sigma(a)x \) for \( a \in F \). Let \( R_1 = F[x, \sigma]/(x^3) \). For \( u = \overline{1 + x + x^2} \in R_1 \), \( u^{-1} = \overline{1 - x} \). For any \( b \in F \),

\[
 u^{-1}bu = b + (b - \sigma(b))x + (b - \sigma(b))x^2.
\]

Thus as \( F_2 \) is not contained in the fixed field of \( \sigma \), \( u^{-1}F_2u \not\subseteq F \). As \( \sigma\eta = \eta\sigma \), \( \eta \) induces an automorphism \( \lambda \) of \( F[x, \sigma] \) such that \( \lambda(ax^i) = \eta(a)x^i \); \( \lambda \) is identity over \( F_2[x, \sigma] \). We still denote \( \lambda \) by \( \eta \). We can form \( F[x, \sigma][y, \eta] = F[x, y, \sigma, \eta] \) with \( xy = yx \). Consider \( R_2 = F[x, y, \sigma, \eta]/(x^3, y^3) \). For \( v = \overline{1 + y + y^2} \in R_2 \), it is immediate that \( v^{-1}F_2v = F_2 \), and if \( F \neq F_2 \), then \( v^{-1}Fv \not\subseteq R_1 \).

We now extend this construction. Consider a field \( F \) which admits an infinite properly ascending sequence \( \{F_n\} \) of subfields indexed by the set of natural numbers, with \( K = F_0 \), such that each \( F_n \) is a finite separable normal extension of \( K \). Further suppose that there exists a sequence \( \{\eta_n\} \) of pairwise commuting automorphisms of \( F \) such that the fixed field of any \( \eta_n \) is \( F_n \), and \( F = \bigcup_n F_n \). Consider a sequence of indeterminates \( x_j, j \geq 1 \). Set \( R_0 = F \), \( R_{n+1} = F[x_{n+1}, \eta_{n+1}]/(x_{n+1}^3) \) with \( x_ix_j = \eta_jx_i \), and \( R = \bigcup_n R_n \). Then \( R \) is a local, locally finite \( K \)-algebra such that \( R/J(R) \) is a countably generated, separable algebraic field extension of \( K \). Obviously \( F \) is an inertial subring of \( R \).
We now construct another inertial subring $F'$ of $R$ such that $F$ and $F'$ are not conjugate. Consider any $k \geq 1$. Set $v_k = \frac{1+x_k+x_k^2}{2} \in R_k$, $w_k = v_1 \ldots v_k$. Set $F'_1 = F_1$, $F'_{k+1} = w_k^{-1}F_{k+1}w_k$ for $k \geq 1$. As $\eta_k$ is identity on $F_k$ but not on $F_{k+1}$, it follows that $w_k^{-1}F_{k+1}w_k \subset R_k$, but $w_k^{-1}F_{k+1}w_k \not\subset R_{k-1}$. That means that $F'_1 \subset R_k$, but $F'_{k+1} \not\subset R_{k-1}$. Now $v_k^{-1}F_kw_k = F'_k$ gives $F'_k \subseteq F_{k+1}$. Then $F' = \bigcup_n F'_n$ is an inertial subring of $R$. As $F'$ is not contained in any $R_k$, it cannot be conjugate to $F$. To get a field $F$ of the above type, consider $K = \mathbb{Z}_2$. This gives rise to an ascending sequence of Galois fields $F_i$ of orders $2^n$ where $n_i = 2^i$.

A ring $R$ in which every one-sided ideal is two-sided, is called a duo ring.

**Theorem 2.7.** Let $A$ be a local artinian duo ring which is an algebra over a commutative local ring $(R,M)$ with $M$ nilpotent and $\bar{A} = A/J(A)$ a countably generated separable algebraic field extension of $\bar{R}$. Then $A$ has a commutative local subring $T$ unramified over $R$ such that $A = T + J(A)$. Further $T$ is unique to within $R$-isomorphisms.

**Proof.** Any $R$-subalgebra $S$ of $A$ is local with $J(S) = J(A) \cap S$, and $J(S)$ is nilpotent. Let $d_A(A) = n$. We apply induction on $n$. The result holds for $n = 1$. Let $n > 1$ and suppose that the result holds for $n - 1$. Let $L$ be a minimal ideal of $A$. Then for some $\pi \in L$, $L = \pi A = A\pi = \pi \bar{A}$ and there exists an $\bar{R}$-automorphism $\sigma$ of $\bar{A}$ such that $\bar{a}\pi = \pi \sigma(\bar{a})$ for any $\bar{a} \in \bar{A}$. By the induction hypothesis $B = A/L$ has a commutative local subring $T/L$ which satisfies the conclusion of the theorem. Then $A = T + J(A)$ and $L = \pi T$ is a minimal ideal of $T$. Let $K$ be any commutative unramified $R$-subalgebra of $T$ such that $\bar{K}$ is a finite extension of $\bar{R}$. Then $K = R[a]$ for some $a$ lift algebraic over $R$. Let $\bar{K} \neq \bar{T}$. Consider any $b \in T$ algebraic over $R$ modulo $L$. As $T/L$ is commutative, we can choose $b$ algebraic modulo $L$ over $R/C$, where $C = \text{ann}_R(T/L)$. So there exists a monic polynomial $g(x) \in R[x]$ such that $g(b) \in L$. Consequently, $g(b) = \pi c$ for some $c \in T$. As $T/L$ is commutative, $ba - ab = \pi d$ for some $d \in T$. Let $\deg g(x) = n$. There exists a finite field extension $\bar{G}$ of $\bar{R}$ in $\bar{T}$ such that $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are in $\bar{G}$ and for any $\bar{R}$-automorphism $\eta$ of $\bar{T}$, $\eta(\bar{G}) \subseteq \bar{G}$. Then

$$S = \sum_{i=1}^{n-1} R[a]b^i + \pi \bar{G}$$

is a finite $R$-subalgebra of $T$. Let $W$ be an inertial subring of $S$. Suppose first that $\bar{a} = \bar{b}$. Then $\bar{W} = \bar{R[a]} = \bar{S}$. So $R[a]$ is also an inertial subring of $S$, and $W$ is a conjugate of $R[a]$. So $W$ contains a conjugate of $a$. Now suppose that $\bar{b} \not\in \bar{K}$. As $\bar{a} \in \bar{W}$, we can find a lifting $a'$ of $\bar{a}$ in $W$. Then $a$ is a conjugate of $a'$. So, for some unit $u \in S$, $K \subseteq u^{-1}Wu$ and clearly $\bar{b} \in \bar{W}$. Now to conclude the proof we can follow the arguments of 2.5.
A part of the following result bears a similarity to the main result in Clark [2].

**Theorem 2.8.** Let $A$ be a semi-perfect ring which is a locally finite faithful algebra over a Hensel ring $R$ such that $J(R) = R \cap J(A)$ and $\overline{A} = A/J(A)$ is a direct sum of matrix rings over fields which are countably generated separable algebraic extensions of $\overline{R}$. Then $A$ has a subalgebra $T$ such that $A = T + J(A)$ and $T \cap J(A) = J(T)$. Moreover $T$ is a direct sum of full matrix rings over commutative local rings $T'$ such that if $R'$ is the homomorphic image of $R$ in $T'$, then $T'$ is the union of a filter of unramified local $R'$-subalgebras of the form $R'[a]$. Further, $T$ is unique to within $R$-isomorphisms.

**Proof.** Since the proof is similar to that of [1, Theorem 33], we only outline it. Observe that for any idempotent $e \in A$, $eAe$ is a locally finite $R$-algebra. As in the proof of [1, Theorem 33], we first consider the case when $A/J(A)$ is simple. As idempotents can be lifted modulo $J(A)$, $A$ is a full $n \times n$-matrix ring over a local ring $B$. Let $D = \{e_{ij} : 1 \leq i, j \leq n\}$ be the corresponding system of matrix units in $A$. The hypothesis on $A$ gives that $B$ is an $R$-algebra satisfying the hypothesis of 2.5. Consequently, $B$ has an inertial subring $S$. Then $T = M_n(S)$ is the desired subring of $A$.

Let $T'$ be another such subring of $A$. As $T'$ is a full matrix ring over a local ring and $T'/J(T') \cong A/J(A)$, we can find a system $L = \{f_{ij} : 1 \leq i, j \leq n\}$ of matrix units of $T'$ that is also a system of matrix units of $A$. Now $T' = M_n(S')$, where $S'$ is the centralizer of $L$ in $T'$. If $A'$ is the centralizer of $L$ in $A$, then $A'$ is a local ring and $S'$ is an inertial subring of $A'$. By [1, Theorem 4], there exists a $c \in J(A)$ such that $f_{ij}^c = e_{ij}$. As in the proof of [1, Theorem 33] we see that $(S')^c$ is an inertial subring of $B$. By 2.5, $S$ and $(S')^c$ are $R$-isomorphic. This proves that $T$ and $T'$ are $R$-isomorphic. Now, the general case can be proved along similar lines to [1, Theorem 33].

Let $A$ be any ring, and $P$ be the smallest subring of $A$ such that any $a \in P$ is a unit in $P$ if and only if $a$ is a unit in $A$. If $I$ is the identity element of $A$, then $P$ is the set of elements $nI/mI$, where $n$, $m$ are integers and $mI$ is a unit in $A$. We call $P$ the total prime subring of $A$. Let $A$ be a local ring. If the characteristic of $\overline{A} = A/J(A)$ is zero, then $P$ is isomorphic to the field $\mathbb{Q}$ of rational numbers, and if the characteristic of $\overline{A}$ is a prime number $p$, then $P$ is a homomorphic image of the localization $\mathbb{Z}_{(p)}$. By 2.5 and 2.7 we get the following.

**Theorem 2.9.** Let $A$ be a local ring and $P$ be its total prime subring such that $P$ is isomorphic either to $\mathbb{Q}$ or to $\mathbb{Z}/(p^n)$. Let $A$ be either a locally finite $P$-algebra or an artinian duo ring. If $\overline{A} = A/J(A)$ is an absolutely algebraic field, then it has a local subring $T$ such that $A = T + J(R)$, $J(T) = T \cap J(A)$
and the following hold:

(i) if the characteristic of $A$ is zero, then $T$ is a field isomorphic to $\overline{A}$,

(ii) if the characteristic of $A$ is $p^n$ for some prime number $p$ and an $n \geq 1$, then $T$ is the union of an ascending sequence of subrings which are Galois rings of the type $GR(p^n, r)$.

Further, $T$ is unique to within isomorphisms; in any case $J(T) = qT$, where $q$ is the characteristic of $\overline{A}$. ($T$ is called a coefficient ring of $A$.)

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REFERENCES


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