THE SET OF POINTS AT WHICH A MORPHISM OF AFFINE SCHEMES IS NOT FINITE

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Abstract. Assume that $X, Y$ are integral noetherian affine schemes. Let $f : X \to Y$ be a dominant, generically finite morphism of finite type. We show that the set of points at which the morphism $f$ is not finite is either empty or a hypersurface. An example is given to show that this is no longer true in the non-noetherian case.

1. Introduction. Let $f : X \to Y$ be a morphism of affine varieties over an algebraically closed field $k$. Let $y \in Y$. We say that $f$ is not finite at $y$ if there exists no open affine neighborhood $U$ of $y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. If $k = \mathbb{C}$, then $f$ is not finite at $y$ if there exists a sequence $x_n \to \infty$ such that $f(x_n) \to y$. The set of all points at which $f$ is not finite will be denoted by $S_f$.

In [2], [3] the first author proved that for a polynomial, generically finite dominant mapping $f : X \to Y$ of affine varieties, the set $S_f$ is either empty or a hypersurface.

The aim of this paper is to generalize this result to the case of a dominant, generically finite morphism of finite type of affine integral noetherian schemes. The main result is that even under such general assumptions the set $S_f$ is either empty or a hypersurface.

2. Preliminaries. We use the terminology and notation as in [4]. Let $A \subset B$ be arbitrary rings. We say that $B$ is a finite ring extension of $A$ if $B$ is a finitely generated $A$-module. A morphism $f : X \to Y$ of schemes is called finite if there exists a covering of $Y$ by open affine subsets $V_i = \text{Spec}(A_i)$ such that for each $i$, $f^{-1}(V_i)$ is affine, equal to $\text{Spec}(B_i)$, where $B_i$ is a finite ring extension of $A_i$. The morphism $f : X \to Y$ is finite if and only if for every covering of $Y$ by open affine subsets $V_i = \text{Spec}(A_i)$, the sets $f^{-1}(V_i)$ are affine, equal to $\text{Spec}(B_i)$, where $B_i$ is a finite ring extension of $A_i$ (see e.g. [4]). In particular if $f$ is finite, then for every open subset $V = \text{Spec}(A) \subset Y$

2000 Mathematics Subject Classification: 14E10, 14E22, 14E40.

Research of Z. Jelonek partially supported by the grant of KBN and by FNP (Polish Science Foundation).

Research of M. Karaś partially supported by FNP (Polish Science Foundation).
the set \( f^{-1}(V) \) is affine, equal to \( \text{Spec}(B) \), where \( B \) is a finite ring extension of \( A \).

**Definition 2.1.** Let \( f : X \to Y \) be a morphism of schemes and let \( y \in Y \). We say that \( f \) is finite at \( y \) if there exists an open affine neighborhood \( U \) of \( y \) such that \( f|_{f^{-1}(U)} : f^{-1}(U) \to U \) is finite.

If \( Y = \text{Spec}(A) \) is affine, then \( f \) is finite at \( y \in Y \) if and only if there exists \( h \in A \) such that \( y \in D(h) = \text{Spec}(A_h) \) and \( f|_{f^{-1}(D(h))} : f^{-1}(D(h)) \to D(h) \) is finite. For a morphism \( f : X \to Y \), we will denote by \( S_f \) the set of points at which \( f \) is not finite. Of course \( f \) is finite iff \( S_f \) is an empty set.

Let \( X = \text{Spec}(A) \) be an integral affine scheme. By a rational function on the scheme \( X \) we mean an element of the field \( A_0 \) (field of fractions of the ring \( A \)). We say that a rational function \( \xi \) has a pole at a point \( x \) iff \( \xi \notin A_x = \mathcal{O}_{X,x} \). For a rational function \( \xi \) on the scheme \( X \) we will denote by \( P_\xi \) the set of points at which \( \xi \) has a pole. This set is a closed proper subset of \( X \). Indeed, we have the following:

**Proposition 2.2.** Let \( \xi = b/a \) be a rational function on an integral scheme \( X = \text{Spec}(A) \). Then \( P_\xi = V(((a) : (b))) \).

**Proof.** Let \( p \in P_\xi \). Then \( \xi \notin A_p \), i.e., \( \xi \neq x/r \) for all \( x \in A \) and \( r \in A \setminus p \). Suppose \(((a) : (b)) \nsubseteq p \) and take \( r \in (((a) : (b)) \setminus p) \). Thus \( rb = xa \) for some \( x \in A \), but this is impossible if \( \xi \notin A_p \). Therefore \(((a) : (b)) \subset p \), i.e., \( p \in V(((a) : (b))) \).

Conversely, assume that \(((a) : (b)) \subset p \). If \( p \notin P_\xi \), then \( b/a \in A_p \). This means that there are \( x \in A \) and \( r \in A \setminus p \) such that \( rb = xa \). Consequently, \( r \in ((a) : (b)) \subset p \). This contradiction finishes the proof. ■

In what follows we need the following results:

**Proposition 2.3.** Let \( A \) and \( B = A[x_1, \ldots, x_n] \) be integral domains such that the field \( B_0 \) is a finite extension of \( A_0 \). Let \( f_i \in A_0[T], i = 1, \ldots, n \), be minimal (monic) polynomials of \( x_i \) over \( A_0 \). Assume that \( A \) is a normal ring. Then \( B \) is finite over \( A \) if and only if \( f_i \in A[t] \) for all \( i = 1, \ldots, n \).

**Proof.** Assume that \( B \) is finite over \( A \). Fix \( i \in \{1, \ldots, n\} \). Since \( x_i \) is integral over \( A \), there exists a monic polynomial \( g_i \in A[X] \) such that \( g_i(x_i) = 0 \).

Note that \( g_i = f_i h \in A_0[X] \). Indeed, \( g_i = f_i h + r \) in \( A_0[X] \) where \( \deg r < \deg f_i \). Moreover, \( r(x_i) = g_i(x_i) - f_i(x_i)h(x_i) = 0 \) and by the minimality of \( f_i \), we have \( r = 0 \). Since \( g_i \in A[X] \), we conclude by [1, Theorem 3.2.2, p. 114] that \( f_i \in A[X] \).

The converse implication is obvious. ■

**Proposition 2.4.** Let \( f : X \to Y \) be a morphism of finite type, and let \( V = \text{Spec}(A) \), \( U = \text{Spec}(B) \) be open affine subschemes of \( X \) and \( Y \), respectively. If \( U \subset f^{-1}(V) \), then \( B \) is a finitely generated \( A \)-algebra.
Proof. We start with the proof of the following:

Lemma 2.5. Let $X$ be a scheme and let $U = \text{Spec}(A)$, $V = \text{Spec}(B)$ be open affine subschemes of $X$. For every $x \in U \cap V$, there exists an open affine subscheme $W_x \subset U \cap V$ of $X$ such that $W_x = \text{Spec}(A_h) = \text{Spec}(B_g)$, where $h \in A$ and $g \in B$.

Proof. Let $x \in U \cap V$. Since $U \cap V$ is open in $U$, there exists $\tilde{h} \in A$ such that $x \in D_U(h) := \{x \in U : h \notin x\} \subset U \cap V$. Consequently, we can assume that $U \subset V$. Let $g \in B = \mathcal{O}_X(V)$ be such that $x \in D_V(g) \subset U$. Consider the restriction mapping $\tilde{g}_U^V : \mathcal{O}_X(V) = B \to A = \mathcal{O}_X(U)$ and take $h = \tilde{g}_U^V(g)$. It is easy to see that $\text{Spec}(A_h) = D_V(h) = D_U(g) = \text{Spec}(B_g)$. ■

Now we can continue the proof of Proposition 2.4. Since $f$ is of finite type, there exist open affine subschemes $U_1, \ldots, U_s$ of $X$ such that $U_1 \cup \ldots \cup U_s = f^{-1}(V)$. Moreover, $U_i = \text{Spec}(B_i)$, where $B_i$ is a finitely generated $A$-algebra. For every $x \in U$, there is an index $i_x$ such that $x \in U \cap U_{i_x}$. By the lemma above, there exists an open affine subscheme $W_x \subset X$ such that $x \in W_x \subset U \cap U_{i_x}$ and $W_x$ is of the form $W_x = \text{Spec}((B_i)_{h_x}) = \text{Spec}(B_{g_x})$. Since $(B_i)_{h_i}$ is a finitely generated $A$-algebra, so is $B_{g_x}$. The family $A = \{\text{Spec}(B_{g_x}) \} x \in U$ is an open covering of $U = \text{Spec}(B)$. Since every affine scheme is quasi-compact, we can choose a finite subcovering $\{\text{Spec}(B_{g_1}), \ldots, \text{Spec}(B_{g_r})\} \subset A$. Note that $(g_1, \ldots, g_r) = 1$. Let $h_1, \ldots, h_r \in B$ be such that $h_1g_1 + \ldots + h_rg_r = 1$. Since $B_{g_i}$ is a finitely generated $A$-algebra, we can write $B_{g_i} = A[s_{i,1}/g_i^{k_1}, \ldots, s_{i,n_i}/g_i^{k_{n_i}}]$.

Take an $s \in B$. Observe that $s/1 \in B_{g_i} = A[s_{i,1}/g_i^{k_1}, \ldots, s_{i,n_i}/g_i^{k_{n_i}}]$ for $i = 1, \ldots, r$. This implies that there is a natural number $k_i$ such that $g_i^{k_i}s \in A[g_i, s_{i,1}, \ldots, s_{i,n_i}]$. Set $k = \max_{i=1,\ldots,r} k_i$. Now we can write

$$s = (h_1g_1 + \ldots + h_rg_r)^k s = \sum_{\alpha \in \mathbb{N}^r, \left|\alpha\right| = k} \frac{\left|\alpha\right|!}{\alpha!} (h_1g_1, \ldots, h_rg_r)^\alpha s.$$

Thus $s \in A[g_1, \ldots, g_r, h_1, \ldots, h_r, s_{1,1}, \ldots, s_{r,n_r}]$. ■

3. Basic definitions. Let us recall some basic definitions.

Definition 3.1. Let $A$ be a ring and let $p$ be a prime ideal in $A$. The height $ht(p)$ is the upper bound of the lengths of chains of distinct prime ideals $p_0 \subset p_1 \subset \ldots \subset p_d = p$ of $A$. The height $ht(a)$ of any ideal $a$ is the number

$$ht(a) = \inf_{p \supset a} ht(p),$$

where $p$ ranges over all the prime ideals containing $a$. 
Now let us recall the notion of a Krull ring.

**Definition 3.2.** A ring $A$ is called a *Krull ring* if it is integral and
1) for every prime ideal $p$ of height 1 the ring $A_p$ is a discrete valuation ring,
2) $A = \bigcap_{\text{ht}(p)=1} A_p$,
3) for any non-zero $r \in A$, there exist only finitely many prime ideals $p$ of height 1 such that $(r) \subseteq p$.

We have the following fundamental theorem of Nagata (see [5]):

**Theorem 3.3.** The normalization of an integral noetherian ring is a Krull ring.

Now we pass to the definition of a hypersurface.

**Definition 3.4.** Let $X$ be a topological space and let $Z \subset X$ be an irreducible closed subset. The *codimension* of $Z$ in $X$, denoted by $\text{codim}_X Z$, is the upper bound of the lengths of chains of distinct closed irreducible subsets of $X$:

$$Z = Z_0 \subset Z_1 \subset \ldots \subset Z_n.$$  

If $Z \subset X$ and all irreducible components of $Z$ have codimension one, we say that $Z$ is a *hypersurface*.

For an ideal $a$ of a ring $A$ we denote by $\text{Ass}(a)$ the set of all associated prime ideals of $a$. It is easy to see that the following proposition holds:

**Proposition 3.5.** Let $A$ be a ring and let $a$ be an ideal in $A$. The subset $V(a) \subset \text{Spec}(A)$ is a hypersurface if and only if for every ideal $p \in \text{Ass}(a)$ we have $\text{ht}(p) = 1$.

**Example 3.6.** If $A$ is a Krull ring and $r \in A$ is a non-zero element, then the subset $V((r)) \subset \text{Spec}(A)$ is a hypersurface (see [1]).

4. **Krull schemes.** In this section we prove our main theorem in the Krull case.

**Proposition 4.1.** Let $X,Y$ be affine integral schemes, and let $f : X \to Y$ be a dominant, generically finite morphism of finite type. If $Y$ is normal, then $S_f$ is the union of the sets of poles of finitely many rational functions on $Y$.

**Proof.** Let $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$. The rings $A$ and $B$ are integral domains, and $A$ is also a normal ring. The morphism $f : X \to Y$ is given by a morphism of rings $\varphi : A \to B$. By Proposition 2.4 the morphism $\varphi$ makes $B$ a finitely generated $A$-algebra. Since $\varphi$ is a monomorphism, we can identify $A$ with $\varphi(A)$ and consequently we can write $B = A[x_1, \ldots, x_n]$. 
Moreover, $f : X \to Y$ is a dominant, generically finite morphism of finite type, so the field $B_0$ is a finite extension of $A_0$. Let

$$P_i = T^{n_i} + a_1^iT^{n_i-1} + \ldots + a_{n_i}^i \in A_0[T],$$

for $i = 1, \ldots, n$, be the minimal polynomial of $x_i$ over $A_0$.

We will show that the set

$$S = \bigcup_{i,j} \{ y \in Y : a_i^j \not\in A_y = \mathcal{O}_{Y,y} \}$$

is equal to $S_f$. By Proposition 2.2 and irreducibility of $Y$ the set $S$ is a closed proper subset of $Y$. Let $y \in Y \setminus S$. Then $a_i^j \in \mathcal{O}_{Y,y}$. Thus there exists a neighborhood $D(h)$, where $h \in A$, such that $a_i^j|_{D(h)} \in A_h = \mathcal{O}_Y(D(h))$. It is easy to see that $P_i \in A_0[T] \subset (A_h)_0[T]$, where $i = 1, \ldots, n$, is a minimal polynomial for $x_i|_{D(h)}$ over $(A_h)_0$. Since $a_i^j|_{D(h)} \in \mathcal{O}_Y(D(h))$, we see that $P_i \in A_h[T]$. By Lemma 2.3, the mapping $f|_{f^{-1}(D(h))} : f^{-1}(D(h)) \to D(h)$ is finite. This implies that $f$ is finite at $y$.

Conversely, let $f$ be finite at $y \in Y$. This means that there exists $h \in A$ such that $y \in D(h)$ and the mapping $f|_{f^{-1}(D(h))} : f^{-1}(D(h)) \to D(h)$ is finite. Now Lemma 2.3 yields that $P_i \in A_h[T]$. This implies that $a_i^j|_{D(h)} \in A_h = \mathcal{O}_Y(D(h))$ and consequently $D(h) \subset Y \setminus S$. ■

The converse statement is also true:

**Proposition 4.2.** Let $Y$ be a normal integral affine scheme, and let $\xi_1, \ldots, \xi_r$ be rational functions on $Y$. There exist an affine integral scheme $X$ and a dominant, generically finite morphism of finite type $f : X \to Y$ such that $S_f = \bigcup_{i=1}^r P(\xi_i)$.

**Proof.** It is sufficient to take $X = \text{Spec}(A[\xi_1, \ldots, \xi_r])$ and $f : X \to Y$ given by the inclusion $A \hookrightarrow A[\xi_1, \ldots, \xi_r]$. ■

Let us recall that if $A$ is a Krull ring, we say that a scheme $\text{Spec}(A)$ is a Krull scheme. Let $X$ be a Krull scheme and let $f$ be a rational function on $X$. Then the set $P_f$ of poles of $f$ is either empty or it has pure codimension one. Indeed, we have the following:

**Proposition 4.3.** Let $X = \text{Spec}(A)$ be a Krull scheme and $f \in A_0$. Then the set of poles of $f$, i.e. the set $P_f = \{ p \in X : f \not\in A_p \}$, is either empty or of the form

$$P_f = \bigcup_{i=1}^r V(p_i),$$

where the $p_i$ are prime ideals with $\text{ht}(p_i) = 1$. In particular, $P_f$ is either empty or a hypersurface.
Proof. Let \( f = b/a \). In virtue of Proposition 2.2, \( P_f = V(((a) : (b))) \). Since \( A \) is a Krull ring, the principal ideal \((a)\) has a primary decomposition, say \((a) = \bigcap_{i=1}^{r} q_i \). Moreover, every associated prime ideal \( \sqrt{q_i} \) is minimal and has height one.

We have \(((a) : (b)) = \bigcap (q_i : (b))\). We also know that
\[
\sqrt{(q_i : (b))} = \begin{cases} 
A, & b \notin p_i = \sqrt{q_i} \\
\sqrt{q_i}, & b \in p_i = \sqrt{q_i}.
\end{cases}
\]

Thus if \(((a) : (b)) \subset p\), then there exists \( i \) such that \( \sqrt{q_i} \subset p \).

Since a Krull ring is a normal ring (see [1], 4.2.5), by Proposition 4.3 we have our first main result:

**Theorem 4.4.** Let \( X, Y \) be affine integral schemes and let \( f : X \to Y \) be a dominant, generically finite morphism of finite type. If \( Y \) is a Krull scheme, then \( S_f \) is either empty or a hypersurface.

5. **Noetherian schemes.** In this section we prove our theorem in the noetherian case.

**Theorem 5.1.** Let \( X, Y \) be affine integral schemes and let \( f : X \to Y \) be a dominant, generically finite morphism of finite type. If \( Y \) is noetherian, then \( S_f \) is either empty or a hypersurface.

**Proof.** Let \( \tilde{Y} \) be a normalization of the scheme \( Y \), say \( \tilde{Y} = \text{Spec}(\tilde{A}) \), where \( \tilde{A} \) is the integral closure of \( A \) in the field \( A_0 \). Let \( \pi_Y : \tilde{Y} \to Y \) be the morphism given by the inclusion \( A \hookrightarrow \tilde{A} \). Set \( \tilde{X} = \text{Spec}(\tilde{A}[x_1, \ldots, x_n]) \) and let \( \pi_X : \tilde{X} \to X \) be given by the inclusion \( A[x_1, \ldots, x_n] \hookrightarrow \tilde{A}[x_1, \ldots, x_n] \).

We have the natural morphism \( \tilde{f} : \tilde{X} \to \tilde{Y} \) such that the diagram
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]
commutes. The morphism \( \tilde{f} : \tilde{X} \to \tilde{Y} \) is a dominant, generically finite morphism of finite type. Since \( \tilde{A} \) is a Krull ring (see Theorem 3.3), by Proposition 4.3, the set \( S_{\tilde{f}} \) is either empty or a hypersurface.

Now we prove that \( \pi_Y(S_{\tilde{f}}) \subset S_f \). For \( h \in A \), we have \( \pi_{Y}^{-1}(D_Y(h)) = D_{\tilde{Y}}(h) \) and \( D_Y(h) = \text{Spec}(A_h) \), \( D_{\tilde{Y}}(h) = \text{Spec}(\tilde{A}_h) \). Of course \( \tilde{A}_h \) is an integral extension of \( A_h \). Moreover, \( f^{-1}(D_Y(h)) = D_X(h) \) and \( \pi_X^{-1}(D_X(h)) = D_{\tilde{X}}(h) \). The ring \( (\tilde{A}[x_1, \ldots, x_n])_h = \tilde{A}_h[x_1, \ldots, x_n] \) is an integral extension of \( (A[x_1, \ldots, x_n])_h = A_h[x_1, \ldots, x_n] \). Thus if \( f|_{D_X(h)} : D_X(h) \to D_Y(h) \) is a finite morphism, then \( \tilde{A}_h[x_1, \ldots, x_n] \) is an integral extension of \( A \). It follows
that $\tilde{A}_h[x_1, \ldots, x_n]$ is an integral extension of $\tilde{A}_h$. Since $\tilde{A}_h[x_1, \ldots, x_n]$ is a finitely generated $\tilde{A}_h$-algebra, $\tilde{A}_h[x_1, \ldots, x_n]$ is a finite ring extension of $\tilde{A}_h$. Thus $\tilde{f}|_{D\tilde{X}(h)} : D\tilde{X}(h) \rightarrow D\tilde{Y}(h)$ is a finite morphism, which proves the inclusion $\pi_Y(S_f) \subseteq S_f$.

Conversely, we have $S_f \subseteq \pi_Y(S_f)$. It is enough to show that if $y \notin \pi_Y(S_f)$, then $y \notin S_f$. Let $y \notin \pi_Y(S_f)$. Then there is an open neighborhood $U$ of $y$ disjoint from $\pi_Y(S_f)$. The morphism $g := \pi_Y \circ \tilde{f} : g^{-1}(U) \rightarrow U$ is integral, as a composition of integral morphisms. Consequently, so is $f \circ \pi_X : g^{-1}(U) \rightarrow U$ and hence also $f : f^{-1}(U) \rightarrow U$. Since $f$ is of finite type it is finite, and consequently $y \notin S_f$.

It remains to prove that $\pi_Y(S_f)$ is either empty or a hypersurface. Note that $S_f$ is the union of irreducible hypersurfaces $V(P_i)$, where $P_i \in \text{Ass}(a_i)$ for some non-zero and non-invertible element $a_i \in \tilde{A}$ (see Propositions 2.2 and 3.5).

In fact, we can assume that $a_i \in A$. Indeed, the irreducible equations of $x_i$ over $\tilde{A}$ are the same as over $A$, in particular, the coefficients of these equations are of the type $a/b$, where $a, b \in A$. Since the morphism $\pi_Y : \tilde{Y} \rightarrow Y$ is integral, we have $\pi_Y(V(P_i)) = V(P_i \cap A)$.

Hence (by the Krull theorem) it is enough to prove that if $a$ is a non-zero and non-invertible element of $A$, and $P_i \in \text{Ass}_A((a))$, then $P_i \cap A$ is a minimal ideal in the set $\text{Ass}_A((a))$. But this can be done exactly as in the proof of Theorem 4.7.2 of [1], pp. 199–200.

To end this paper, we show that our results can be generalized neither to the non-noetherian nor to the normal non-Krull case.

**Example 5.2.** For $k \in \mathbb{N}$ we construct a ring $R_k$ such that:

1) $R_k$ is a normal domain,

2) there exists $p \in R_k$ such that the ideal $(p)$ is prime and $\text{ht}(p) \geq k$.

We proceed by induction. Let $R_1 = \mathbb{Z}$. Having defined a ring $R_k$, put $R_{k+1} = R_k + (R_k)_0 X + (R_k)_0 X^2 + \ldots \subseteq (R_k)_0[X]$. The ring $R_{k+1}$ is a normal domain.

Indeed, if $\xi \in (R_{k+1})_0 = (R_k)_0(X)$ is an integral element over $R_{k+1}$, then it is also integral over $(R_k)_0[X]$. Thus $\xi \in (R_k)_0[X]$, say $\xi = \xi_0 + \xi_1 X + \ldots + \xi_d X^d$. Consequently, if $\xi^n + a_{n-1}\xi^{n-1} + \ldots + a_0 = 0$, where $a_i = a_{i,0} + a_{i,1} X + \ldots + a_{i,d_i} X^{d_i} \in R_k$, then $\xi_0^n + a_{n-1,0}\xi_0^{n-1} + \ldots + a_{0,0} = 0$. Hence $\xi_0 \in R_k$ and $\xi \in R_{k+1}$.

Let $(p) = p_k \supseteq p_{k-1} \supseteq \ldots \supseteq p_0 = 0$ be a sequence of distinct prime ideals in $R_k$. Now let $\tilde{p}_i = \{f = f_0 + f_1 X_{k+1} + \ldots + f_d X_{k+1}^d \in R_{k+1} : f_0 \in p_i\}$. It is easy to see that $(p) = \tilde{p}_k \supseteq \tilde{p}_{k-1} \supseteq \ldots \supseteq \tilde{p}_0 \supseteq 0$ is a sequence of distinct prime ideals in $R_{k+1}$. Consequently, $\text{ht}(p) \geq k + 1$. 


Let \( f : \text{Spec}(R_k[1/p]) \rightarrow \text{Spec}(R_k) \) be the morphism given by the inclusion \( R_k \hookrightarrow R_k[1/p] \). By the proof of Proposition 4.1 and Proposition 2.2, applied to \( f \), we see that \( S_f = V(p) \) and consequently \( \text{codim} S_f \geq k \).

REFERENCES