ON A DECOMPOSITION OF POLYNOMIALS
IN SEVERAL VARIABLES, II

BY

A. SCHINZEL (Warszawa)

Abstract. One considers representation of cubic polynomials in several variables as the sum of values of univariate polynomials taken at linear combinations of the variables.

In the first paper of this series [6] we have defined for a field $K$ the number $M(n, d, K)$ as the least integer $M$ (provided it exists, otherwise $\infty$) such that for every polynomial $F \in K[x_1, \ldots, x_n]$ of degree $d$ there exist vectors $\alpha_\mu = [\alpha_{\mu 1}, \ldots, \alpha_{\mu n}] \in K^n$ and polynomials $f_\mu \in K[z]$ ($1 \leq \mu \leq M$) such that

\begin{equation}
F(x_1, \ldots, x_n) = \sum_{\mu=1}^{M} f_\mu \left( \sum_{\nu=1}^{n} \alpha_{\mu \nu} x_\nu \right).
\end{equation}

We have shown that $M(n, d, K) < \infty$ if $\text{char } K = 0$ or $\text{char } K > d$ and studied the cases $d = 2$ and $n = 2$. In this paper we study the next simplest case $d = 3$ and prove two theorems.

Theorem 1. For every field $K$ of characteristic different from 2, 3 we have

\[ M(n, 3, K) \leq \binom{n + 1}{2}. \]

Theorem 2. For every algebraically closed field $K$ of characteristic different from 2, 3 we have

\[ M(3, 3, K) = 5. \]

An analogue of Theorem 1 for forms and $K = \mathbb{C}$ was proved by B. Reznick [3]. Equality (1) for generic cubic forms $F$ over $\mathbb{C}$ has been studied by B. Reichstein [2], but his interesting results have no bearing on our theorems.

The proof of Theorem 1 is based on

Lemma 1. For every quadratic form $F \in K[x_1, \ldots, x_n] \setminus K[x_2, \ldots, x_n]$ there exist $n$ linearly independent vectors $\alpha_\mu \in K^n$ with $\alpha_{\mu 1} \neq 0$ and

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$a_\mu \in K \ (1 \leq \mu \leq n)$ such that

$$ F = \sum_{\mu=1}^{n} a_\mu (\alpha_\mu x)^2, \quad \text{where} \quad \alpha x = \sum_{i=1}^{n} \alpha_i x_i. $$

Proof. We proceed by induction on $n$. For $n = 1$ the lemma is obviously true. Assume that it is true for all quadratic forms over $K$ in less than $n$ variables.

If $\text{rank } F = r < n$, then there exist linearly independent vectors $\beta_1, \ldots, \beta_r \in K^n$ and a form $G \in K[y_1, \ldots, y_r]$ such that

$$ F = G(\beta_1 x, \beta_2 x, \ldots, \beta_r x). $$

Since $F \not\in K[x_2, \ldots, x_n]$ we may assume that $\beta_{11} \neq 0$. Let

$$ H = G\left(y_1, y_2 + \frac{\beta_{21}}{\beta_{11}} y_1, \ldots, y_r + \frac{\beta_{r1}}{\beta_{11}} y_1\right). $$

We have

$$ F = H\left(\beta_1 x, \beta_2 x - \frac{\beta_{21}}{\beta_{11}} \beta_1 x, \ldots, \beta_r x - \frac{\beta_{r1}}{\beta_{11}} \beta_1 x\right) $$

and $\beta_\varrho x - \frac{\beta_{\varrho 1}}{\beta_{11}} \beta_1 x \in K[x_2, \ldots, x_n] \ (1 \leq \varrho \leq r)$, hence $F \not\in K[x_2, \ldots, x_n]$ implies $H \not\in K[y_2, \ldots, y_r]$. By the inductive assumption

$$ H = \sum_{\mu=1}^{r} c_\mu (\gamma_\mu y)^2, \quad y = [y_1, \ldots, y_r], $$

where $c_\mu \in K, \ \gamma_\mu \in K^r \ (1 \leq \mu \leq r), \ \gamma_\mu$ are linearly independent and $\gamma_{11} \neq 0$. It follows that

$$ F = \sum_{\mu=1}^{r} c_\mu \left(\gamma_{1\mu} \beta_1 x + \sum_{\varrho=2}^{r} \gamma_{\mu \varrho} \left(\beta_\varrho x - \frac{\beta_{\varrho 1}}{\beta_{11}} \beta_1 x\right)\right)^2. $$

Let us put

$$ (\alpha_{\mu \nu})_{\mu \leq r, \ \nu \leq n} = (\gamma_{\mu \varrho})_{\mu \leq r, \ \varrho \leq r} \left(\frac{\beta_1}{\beta_{11}}, \frac{\beta_2 - \beta_1 \beta_{21}}{\beta_{11}}, \ldots, \frac{\beta_r - \beta_1 \beta_{r1}}{\beta_{11}}\right). $$

Since the matrix $(\gamma_{\mu \varrho})_{\mu \leq r, \ \varrho \leq r}$ is non-singular of order $r$ and the rank of the matrix

$$ \left(\frac{\beta_1}{\beta_{11}}, \frac{\beta_2 - \beta_1 \beta_{21}}{\beta_{11}}, \ldots, \frac{\beta_r - \beta_1 \beta_{r1}}{\beta_{11}}\right) $$

is $r$, the rank of the matrix $(\alpha_{\mu \nu})_{\mu \leq r, \ \nu \leq r}$ is $r$ and there exist $n - r$ vectors $\alpha_{r+1}, \ldots, \alpha_n \in K^n$ such that $A = (\alpha_{\mu \nu})_{\mu \leq r, \ \nu \leq n}$ is of rank $n$. We have, by (3), $\alpha_{11} = \gamma_{11} \neq 0$ for $\mu \leq r$. Adding, if necessary, the first row
of $A$ to rows $r+1, \ldots, n$ we achieve that $\alpha_{\mu_1} \neq 0$ for all $\mu \leq n$ and (2) is satisfied with $a_{\mu} = c_{\mu}$ for $\mu \leq r$, $a_{\mu} = 0$ for $\mu > r$.

If rank $F = n$, let

$$F = \sum_{i,j=1}^{n} a_{ij} x_i x_j, \quad \text{where } a_{ij} = a_{ji}.$$ 

If $a_{ii} \neq 0$ for at least one $i > 1$, then we consider the form

$$G = F - a_{ii} \left( \sum_{j=1}^{n} \frac{a_{ij}}{a_{ii}} x_j \right)^2.$$  

This form is of rank $n - 1$ and it does not depend on $x_i$, hence $G \not\in K[x_2, \ldots, x_n]$ and by the inductive assumption

$$G = \sum_{\mu=1}^{n-1} b_{\mu} (\beta_{\mu} x)^2,$$

where $b_{\mu} \in K$, $\beta_{\mu} \in K^n$ ($1 \leq \mu < n$), $\beta_{\mu}$ are linearly independent, $\beta_{\mu_1} \neq 0$ and $b_{\mu} \neq 0$ (otherwise $G$ would be of rank $n - 1$). If $a_{i1} \neq 0$ it suffices to take in (1)

$$a_{\mu} = b_{\mu}, \quad \alpha_{\mu} = \beta_{\mu} \quad \text{for } \mu < n,$$

$$a_n = a_{ii}, \quad \alpha_n = \left[ \frac{a_{i1}}{a_{ii}}, \ldots, \frac{a_{in}}{a_{ii}} \right].$$

If $a_{i1} = 0$, then we choose $c \in K$, $c^2 \neq 0, \pm a_{ii}/b_{n-1}$, which is possible unless $K = F_5$, $a_{ii} = \pm b_{n-1}$, and infer from (4), (5) that

$$F = \sum_{\mu=1}^{n-2} b_{\mu} (\beta_{\mu} x)^2 + b_{n-1} \left( \frac{b_{n-1} c^2 - a_{ii}}{b_{n-1} c^2 + a_{ii}} \beta_{n-1} x + \frac{2a_{ii} c}{b_{n-1} c^2 + a_{ii}} \sum_{j=1}^{n} \frac{a_{ij}}{a_{ii}} x_j \right)^2$$

$$+ a_{ii} \left( \frac{2b_{n-1} c}{b_{n-1} c^2 + a_{ii}} \beta_{n-1} x - \frac{b_{n-1} c^2 - a_{ii}}{b_{n-1} c^2 + a_{ii}} \sum_{j=1}^{n} \frac{a_{ij}}{a_{ii}} x_j \right)^2.$$ 

Hence (2) is satisfied with

$$a_{\mu} = b_{\mu}, \quad \alpha_{\mu} = \beta_{\mu} \quad \text{for } \mu < n - 1,$$

$$a_{n-1} = b_{n-1}, \quad \alpha_{n-1} = \frac{b_{n-1} c^2 - a_{ii}}{b_{n-1} c^2 + a_{ii}} \beta_{n-1} + \frac{2c}{b_{n-1} c^2 + a_{ii}} [a_{i1}, \ldots, a_{in}],$$

$$a_n = a_{ii}, \quad \alpha_n = \frac{2b_{n-1} c}{b_{n-1} c^2 + a_{ii}} \beta_{n-1} - \frac{b_{n-1} c^2 - a_{ii}}{b_{n-1} c^2 + a_{ii}} \left[ \frac{a_{i1}}{a_{ii}}, \ldots, \frac{a_{in}}{a_{ii}} \right].$$

By the choice of $c$ we have $\alpha_{\mu_1} \neq 0$. Also the $\alpha_{\mu}$ are linearly independent, since otherwise $F$ would be of rank less than $n$. 


In the remaining case $K = \mathbb{F}_5$, $a_{ii} = \varepsilon b_{n-1}$ ($\varepsilon = \pm 1$) it suffices to take
\[ a_{\mu} = b_{\mu}, \quad \alpha_{\mu} = \beta_{\mu} \quad \text{for} \quad \mu < n - 1, \]
\[ a_{n-1} = 3b_{n-1}, \quad \alpha_{n-1} = \beta_{n-1} + \frac{3 - \varepsilon}{2} \left[ \frac{a_{i1}}{a_{i1}}, \ldots, \frac{a_{in}}{a_{i1}} \right], \]
\[ a_n = 3b_{n-1}, \quad \alpha_n = \beta_{n-1} - \frac{3 - \varepsilon}{2} \left[ \frac{a_{i1}}{a_{i1}}, \ldots, \frac{a_{in}}{a_{i1}} \right]. \]
If $a_{ii} = 0$ for all $i > 1$, but $a_{ij} \neq 0$ for some $i > 1$, $j > 1$ then we make the linear transformation $x_i = x'_i + x'_j$, $x_j = x'_i - x'_j$ and reduce this case to the former.

There remains the case where $a_{ij} = 0$ for all $i, j > 1$. Then $1 < n = \text{rank} F \leq 2$, so $n = \text{rank} F = 2$ and we have
\[ F = a_{11}x_1^2 + 2a_{12}x_1x_2 \]
\[ = \frac{1}{4} \left( \left( c + \frac{a_{11}}{c} \right) x_1 + \frac{2a_{12}}{c} x_2 \right)^2 - \frac{1}{4} \left( \left( c - \frac{a_{11}}{c} \right) x_1 - \frac{2a_{12}}{c} x_2 \right)^2, \]
where $c$ in $K$ is chosen so that $c^2 \neq 0, \pm a_{11}$. Such a choice is possible unless $K = \mathbb{F}_5$, $a_{11} = \pm 1$. In that case
\[ F = 3a_{11}(x_1 - 2a_{11}a_{12}x_2)^2 + 3a_{11}(x_1 - a_{11}a_{12}x_2)^2. \]

Proof of Theorem 1. We proceed by induction on $n$. For $n = 1$ the theorem is obviously true. Assume that $M(n - 1, 3, K) \leq \binom{n}{2}$ and consider a polynomial $F \in K[x_1, \ldots, x_n]$ of degree 3. Let $F_0$ be its leading form.

Since $\text{card} K \geq 4$, by Lemma 1 of [6], or by Lemma 4.4.1 of [7], there exists $\beta_1 \in K^n$ such that $F_0(\beta_1) \neq 0$ and thus $\beta_1 \neq 0$. If vectors $\beta_2, \ldots, \beta_n$ are chosen in $K^n$ so that $\det(\beta_1, \ldots, \beta_n) \neq 0$, we can replace in our argument $F(x)$ by $F_1(y) := F(\beta_1 y, \ldots, \beta_n y)$, where $y = [y_1, \ldots, y_n]$ and where the coefficient of $y_i^3$ in $F_1(y)$ is $F_0(\beta_1) \neq 0$ (cf. [3]). Hence we may assume without loss of generality that
\[ \deg_{x_1} \frac{\partial F_0}{\partial x_1} > 0. \]
Then by Lemma 1,
\[ \frac{\partial F_0}{\partial x_1} = \sum_{\mu=1}^{n} a_{\mu}(\alpha_{\mu}x)^2, \]
where $a_{\mu} \in K$, $\alpha_{\mu} \in K^n$, $\alpha_{\mu1} \neq 0$ ($1 \leq \mu \leq n$) and the $\alpha_{\mu}$ are linearly independent. By the last condition there exist $c_{\mu} \in K$ ($1 \leq \mu \leq n$) and $d \in K$ such that
\[ \frac{\partial F}{\partial x_1} - \frac{\partial F_0}{\partial x_1} = \sum_{\mu=1}^{n} c_{\mu}(\alpha_{\mu}x) + d. \]
Consider now the polynomial

\begin{equation}
G = F - \sum_{\mu=1}^{n} \frac{a_{\mu}}{3\alpha_{\mu}} \left( \alpha_{\mu} x \right)^{3} - \sum_{\mu=1}^{n} \frac{c_{\mu}}{2\alpha_{\mu}} \left( \alpha_{\mu} x \right)^{2} - \frac{d}{\alpha_{11}} \left( \alpha_{1} x \right).
\end{equation}

By (6)–(8) we have

\[ \frac{\partial G}{\partial x_{1}} = 0, \]

hence \( G \in K[x_{2}, \ldots, x_{n}] \). By the inductive assumption there exist vectors \( \alpha_{\mu}^{*} \in K^{n-1} \) and polynomials \( g_{\mu} \in K[z] \) (\( n < \mu \leq \binom{n+1}{2} \)) such that

\[ G = \sum_{\mu=n+1}^{\binom{n+1}{2}} g_{\mu} \left( \sum_{\nu=1}^{n-1} \alpha_{\mu}^{*} x_{\nu+1} \right). \]

The decomposition (1) follows now from (8) with

\[
\begin{align*}
f_{1} &= \frac{a_{1}}{3\alpha_{11}} z^{3} + \frac{c_{1}}{2\alpha_{11}} z^{2} + \frac{d}{\alpha_{11}}, \\
f_{\mu} &= \frac{a_{\mu}}{3\alpha_{\mu}} z^{3} + \frac{c_{\mu}}{2\alpha_{\mu}} z^{2} \quad (1 < \mu \leq n), \\
f_{\mu} &= g_{\mu}, \quad \alpha_{\mu} = [0, \alpha_{\mu}^{*}] \quad \left( n < \mu \leq \binom{n+1}{2} \right).
\end{align*}
\]

The proof of Theorem 1 is complete. The idea this proof is taken from the paper of Rosanes [4], §6.

The proof of Theorem 2 is based on four lemmas.

**Lemma 2.** For

\[ \phi(x) = ax_{1}^{3} + bx_{2}^{3} + cx_{3}^{3} + 3a_{2}x_{1}^{2}x_{2} + 3a_{3}x_{1}^{2}x_{3} + 3b_{1}x_{2}^{2}x_{1} + 3b_{3}x_{2}^{2}x_{3} + 3c_{1}x_{2}^{2}x_{1} + 3c_{2}x_{3}^{2}x_{2} + 6mx_{1}x_{2}x_{3} \in K[x_{1}, x_{2}, x_{3}], \]

let

\[
\begin{align*}
S(\phi) &= abcm - (bca_{2}a_{3} + cab_{1}b_{3} + abc_{1}c_{2}) - m(ab_{3}c_{2} + bc_{1}a_{3} + ca_{2}b_{1}) \\
&\quad + (ab_{1}c_{2}^{2} + bc_{2}a_{3}^{2} + ac_{1}b_{3}^{2} + ba_{2}c_{1}^{2} + cb_{3}a_{2}^{2} + ca_{3}b_{1}^{2}) - m^{4} \\
&\quad + 2m^{2}(b_{1}c_{1} + c_{2}a_{2} + a_{3}b_{3}) - 3m(a_{2}b_{3}c_{1} + a_{3}b_{1}c_{2}) \\
&\quad - (b_{1}^{2}c_{1} + c_{2}^{2}a_{2} + a_{3}^{2}b_{3}) + (c_{2}a_{2}a_{3}b_{3} + a_{3}b_{3}c_{1} + b_{1}c_{1}a_{2}).
\end{align*}
\]

Then for every matrix \( A \in K^{3 \times 3} \),

\[ S(\phi(Ax)) = S(\phi)(\det A)^{4}. \]

Moreover, if \( \phi = \sum_{i=1}^{3} (\alpha_{i} x)^{3} \) for some \( \alpha_{i} \in K^{3} \), then \( S(\phi) = 0 \).

**Proof.** See Salmon [5], Section 221.
Lemma 3. Every non-zero ternary cubic form $\phi$ over an algebraically closed field $K$ of characteristic different from 2, 3 can be transformed by a non-singular linear transformation over $K$ into one of the forms

\begin{align*}
F_1 &= \eta x_2^3 + x_3^3, \quad \eta = 0 \text{ or } 1, \\
F_2 &= x_1^3 + x_2^3 + x_3^3 + 6m x_1 x_2 x_3, \quad m \in K, \\
F_3 &= x_2^3 + x_3^3 + 6x_1 x_2 x_3, \\
F_4 &= \varepsilon x_1^3 + 3x_2^2 x_3, \quad \varepsilon = 0 \text{ or } 1, \\
F_5 &= 6x_1 x_2 x_3 + x_3^3, \\
F_6 &= 3x_1^2 x_2 + 3x_1 x_3^2.
\end{align*}

Proof. See Gordan [1]. Gordan gives 10 types of forms, but two of them, $x_1^3 + x_2^3 + x_3^3$ and $6x_1 x_2 x_3$, are obtained from $F_2$ for $m = 0$ and $m = -\frac{1}{2}$ respectively. Indeed,

$$
x_1^3 + x_2^3 + x_3^3 - 3x_1 x_2 x_3 = (x_1 + x_2 + x_3)(x_1 + g x_2 + g^2 x_3)(x_1 + g^2 x_2 + g x_3),$$

where $g^3 = 1$, $g \neq 1$.

Lemma 4. For $\nu = 2, 3, 4$ we have for some $\alpha_\mu \in K^3$, $a_\mu \in K$ ($1 \leq \mu \leq 4$),

\begin{equation}
F_\nu = \sum_{\mu=1}^{4} a_\mu (\alpha_\mu \cdot x)^3,
\end{equation}

where $\alpha_1, \alpha_2, \alpha_3$ are linearly independent, $\text{rank } A_\nu < 4$ and, for any $\alpha_1, \alpha_2, \alpha_3 \in K$, the conditions $\alpha_1 \neq 0$ and $\text{rank } B_\nu < 5$ imply that $[\alpha_1, \alpha_2, \alpha_3]$ is a scalar multiple of one of the $\alpha_\mu$.

Here

\begin{equation}
A_\nu = \begin{pmatrix}
\alpha_{11} \alpha_{12} & \ldots & \alpha_{11} \alpha_{14} \\
\alpha_{21} \ldots & \ldots & \alpha_{42} \\
\alpha_{11} \alpha_{13} & \ldots & \alpha_{41} \alpha_{43} \\
\alpha_{12} \alpha_{13} & \ldots & \alpha_{42} \alpha_{43} \\
\alpha_{13} \ldots & \ldots & \alpha_{43}
\end{pmatrix}, \quad B_\nu = \begin{pmatrix}
\alpha_{11}^2 & \ldots & \alpha_{14}^2 \\
\alpha_{21} & \ldots & \alpha_{24} \\
\alpha_{11} \alpha_{13} & \ldots & \alpha_{41} \alpha_{43} \\
\alpha_{12} \alpha_{13} & \ldots & \alpha_{42} \alpha_{43} \\
\alpha_{13} & \ldots & \alpha_{43}
\end{pmatrix}.
\end{equation}

Proof. We have, with $g^3 = 1$, $g \neq 1$,

\[
F_2(x) = (1 - m^3)x_1^3 + \frac{1}{3}(mx_1 + x_2 + x_3)^3 \\
+ \frac{1}{3}(mx_1 + g x_2 + g^2 x_3)^3 + \frac{1}{3}(mx_1 + g^2 x_2 + g x_3)^3,
\]

hence
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A_2 = A_2(m) = \begin{pmatrix}
0 & m & m\rho & m\rho^2 \\
0 & 1 & \rho^2 & \rho \\
0 & m & m\rho^2 & m\rho \\
0 & 1 & 1 & 1 \\
0 & 1 & \rho & \rho^2
\end{pmatrix}.

Clearly rank A_2 < 4 and by adding the first three columns of B_2 = B_2(m) to the fourth we obtain

\text{rank } B_2(m) = \text{rank } C_2(m) + 2,

where

C_2(m) = \begin{pmatrix}
m & m\rho & \alpha_1\alpha_2 \\
1 & \rho^2 & \alpha_2 \\
m & m\rho^2 & \alpha_1\alpha_3 \\
1 & \rho & \alpha_3
\end{pmatrix}.

If rank C_2(m) < 3, then \alpha_1\alpha_3 - m\rho_2 = m\rho_3 - \alpha_1\alpha_2 = 0, hence if \alpha_1 \neq 0 we have either m \neq 0 and \frac{m}{\alpha_1}[\alpha_1, \alpha_2, \alpha_3] = [m, 1, 1], [m, \rho, \rho^2] or [m, \rho^2, \rho], or m = 0 and \frac{1}{\alpha_1}[\alpha_1, \alpha_2, \alpha_3] = [1, 0, 0].

Further, we have

F_3(x) = -x_1^3 + \frac{1}{3}(x_1 + x_2 + x_3)^3 + \frac{1}{3}(x_1 + \rho x_2 + \rho^2 x_3)^3 + \frac{1}{3}(x_1 + \rho^2 x_2 + \rho x_3)^3,

hence

A_3 = A_2(1), \quad B_3 = B_2(1)

and the assertion follows.

Finally, we have

F_4(x) = \varepsilon x_1^3 + \frac{1}{3}(x_3 + 2x_2)^3 + \frac{1}{3}(x_3 - 2x_2)^3 - \frac{1}{4}x_3^3,

hence

A_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 4 & 4 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 \\
0 & 1 & 1 & 1
\end{pmatrix}, \quad B_4 = \begin{pmatrix}
1 & 0 & 0 & \alpha_1^2 \\
\alpha_1\alpha_2 \\
\alpha_2 \alpha_3 \\
\alpha_3
\end{pmatrix}.

Clearly rank A_4 < 4 and rank B_4 = rank C_4 + 2, where

C_4 = \begin{pmatrix}
0 & 0 & \alpha_1\alpha_2 \\
4 & 4 & \alpha_2^2 \\
0 & 0 & \alpha_1\alpha_3 \\
2 & -2 & \alpha_2\alpha_3
\end{pmatrix}.

If rank C_4 < 3, then \alpha_1\alpha_2 = \alpha_1\alpha_3 = 0, hence if \alpha_1 \neq 0, then \frac{1}{\alpha_1}[\alpha_1, \alpha_2, \alpha_3] = [1, 0, 0].
Lemma 5. For \( \nu = 5 \) we have (9) for some \( \alpha_\mu \in K^3, \alpha_1, \alpha_2, \alpha_3 \) are linearly independent and the condition \( \text{rank} \, B_5 < 5 \), where \( B_5 \) is given by the formula (10), implies that \([\alpha_1, \alpha_2, \alpha_3]\) is a scalar multiple of one of the \( \alpha_\mu \).

Proof. We have

\[
F_5(x) = \frac{1}{48}(3x_1 - 2x_2 - 2x_3)^3 - \frac{1}{48}(3x_1 - 2x_2 - 2x_3)^3 \\
- \frac{1}{96}(3x_1 - 2x_2 - 4x_3)^3 + \frac{1}{96}(3x_1 - 2x_2 - 4x_3)^3,
\]

hence

\[
B_5 = \begin{pmatrix}
9 & 9 & 9 & \alpha_1^2 \\
-6 & -6 & 6 & 6 \\
4 & 4 & 4 & 4 \\
-6 & 6 & -12 & 12 \\
4 & -4 & -8 & 8 \\
4 & 4 & 16 & 16 \\
\end{pmatrix}.
\]

It is easily seen that

\( \text{rank} \, B_5 = \text{rank} \, C_5 + 2, \)

where

\[
C_5 = \begin{pmatrix}
9 & 9 & \alpha_1^2 \\
-6 & 6 & \alpha_1\alpha_2 \\
4 & 4 & \alpha_2^2 \\
4 & 16 & \alpha_3^2 \\
\end{pmatrix}.
\]

If \( \text{rank} \, C_5 < 3 \) then \(-48\alpha_1^2 + 108\alpha_2^2 = 0, 48\alpha_1\alpha_2 + 120\alpha_2^2 - 48\alpha_3^2 = 0, \) hence \( 3[\alpha_1, \alpha_2, \alpha_3] = \alpha_1[3, -2, -2], \alpha_1[3, -2, 2], \alpha_1[3, 2, -4] \) or \( \alpha_1[3, 2, 4] \).

Proof of Theorem 2. We shall show first that

\( M(3, 3, K) \geq 5. \)

Indeed, suppose that

\[
3x_1^2x_2 + 3x_1x_2^3 = \sum_{\mu=1}^4 f_\mu(\alpha_\mu x), \quad \text{where} \; \alpha_\mu \in K^3, \; f_\mu \in K[z].
\]

Since the left hand side is homogeneous we may assume that the \( f_\mu \) are monomials, and since \( K \) is algebraically closed, that \( f_\mu = z^3 \) (\( 1 \leq \mu \leq 4 \)). Then for each \( \mu \leq 4, \)

\[
G_\mu := 3x_1^2x_2 + 3x_1x_2^3 - (\alpha_\mu x)^3
\]

is the sum of three cubes of linear forms and by Lemma 2 we have \( S(G_\mu) = 0. \) However by the same lemma and a tedious computation

\[
S(G_\mu) = -\alpha_{\mu_2}^3.
\]
hence \( \alpha_{\mu 2} = 0 \) for all \( \mu \leq 4 \). This contradicts (11), since the left hand side depends on \( x_2 \), while the right does not.

We shall now show that
\[
M(3, 3, K) \leq 5.
\]
Let \( F \in K[x] \) be a polynomial of degree 3 with the highest homogeneous part \( F_0 \). Since the statement of the theorem is invariant with respect to non-singular linear transformations we may assume by virtue of Lemma 3 that \( F_0(x) = F_\nu(x) \), where \( 1 \leq \nu \leq 6 \). Also we may assume that \( F(0) = 0 \), since a constant can be added to any polynomial \( f_\mu \). If \( \nu = 1 \) we have
\[
F_0(x) = F_1(x) = \eta x_2^3 + x_3^3.
\]
On the other hand, by Theorem 3 of [6],
\[
F(x) - F_1(x) = \sum_{\mu=1}^{3} f_\mu(\alpha_\mu x), \quad \alpha_\mu \in K^3, \quad f_\mu \in K[z],
\]
hence (1) holds with \( M = 5 \) and
\[
\alpha_4 = [0, 1, 0], \quad f_4 = \eta z^3,
\]
\[
\alpha_5 = [0, 0, 1], \quad f_5 = z^3.
\]
If \( 1 < \nu \leq 5 \) we have, by Lemmas 4 and 5,
\[
F_0(x) = F_\nu(x) = \sum_{\mu=1}^{4} a_\mu(\alpha_\mu x)^3.
\]
Now, let
\[
F(x) - F_0(x) = \sum_{i,j=1}^{3} b_{ij}x_ix_j + \sum_{i=1}^{3} c_i x_i, \quad b_{ij} = b_{ji},
\]
and let
\[
D = \det \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \sum_{1 \leq i \leq j \leq 3} c_{ij} \alpha_i \alpha_j.
\]
If \( \nu \leq 4 \), in view of the condition \( \text{rank } A_\nu = 0 \) we have \( c_{11} = 0 \), hence the equation
\[
\sum_{1 \leq i \leq j \leq 3} c_{ij} \alpha_i \alpha_j = 0
\]
has infinitely many pairwise linearly independent solutions with \( \alpha_1 \neq 0 \). Thus there exists a vector \([\alpha_{51}, \alpha_{52}, \alpha_{53}]\) with \( \alpha_{51} \neq 0 \) that is not a scalar multiple of any \( \alpha_\mu \) \((1 \leq \mu \leq 4)\) and satisfies \( D = 0 \). According to Lemma 4,
for $\alpha_i = \alpha_{5i}$ $(1 \leq i \leq 3)$ we have rank $B_\nu = 5$. It follows by Kronecker–Capelli’s theorem that the system of linear equations

$$\sum_{\mu=1}^{5} b_\mu \alpha_{\mu i} \alpha_{\mu j} = b_{ij} \quad (1 \leq i \leq j \leq 3)$$

is solvable for $b_\mu$ in $K$ $(1 \leq \mu \leq 5)$. Also the system

$$\sum_{\mu=1}^{3} d_\mu \alpha_{\mu i} = c_i \quad (1 \leq i \leq 3)$$

is solvable for $d_\mu$ in $K$ $(1 \leq \mu \leq 3)$, because of the linear independence of $\alpha_1, \alpha_2, \alpha_3$. Therefore, with $d_4 = d_5 = 0$,

$$F(x) - F_0(x) = \sum_{\mu=1}^{5} b_\mu (\alpha_\mu x)^2 + \sum_{\mu=1}^{5} d_\mu (\alpha_\mu x),$$

which together with (12) gives (1) with

$$f_\mu = \alpha_\mu z^3 + b_\mu z^2 + d_\mu z, \quad a_5 = 0.$$

If $\nu = 5$, then the equation (14) has infinitely many pairwise linearly independent solutions. Thus there exists a vector $[\alpha_{51}, \alpha_{52}, \alpha_{53}]$ that is not a scalar multiple of any $\alpha_\mu$ $(1 \leq \mu \leq 4)$ and satisfies $D = 0$. According to Lemma 5, for $\alpha_i = \alpha_{5i}$ $(1 \leq i \leq 3)$ we have rank $B_5 = 5$. The remainder of the proof is identical with the one given above.

There remains the most difficult case $\nu = 6$. Here, as the proof of $M(3,3,K) \geq 5$ shows, the above approach is impossible and we argue as follows.

Let again (13) hold and consider first the case where $b_{22} \neq 0$, or $b_{22} = 0$, $b_{23} = 0$. We choose $c \in K$ as a solution to the equation

(15) \[ cb_{22} = b_{23} \]

and then choose $\alpha_{43}, \alpha_{53}$ in $K$ satisfying the conditions

$$ca_{43}\alpha_{53} + \alpha_{43} + \alpha_{53} = 0, \quad \alpha_{43}\alpha_{53}(\alpha_{43} - \alpha_{53}) \neq 0.$$  

This gives

$$\begin{vmatrix} \alpha_{43} & \alpha_{53} & -c \\ \alpha_{43}^2 & \alpha_{53}^2 & 1 \\ \alpha_{43}^3 & \alpha_{53}^3 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} \alpha_{43} & \alpha_{53} \\ \alpha_{43}^2 & \alpha_{53}^2 \end{vmatrix} \neq 0$$

and the systems of linear equations

(16) \[
\begin{align*}
a_4\alpha_{43} + a_5\alpha_{53} &= -c, \quad a_4\alpha_{43}^2 + a_5\alpha_{53}^2 = 1, \quad a_4\alpha_{43}^3 + a_5\alpha_{53}^3 = 0; \\
b_4\alpha_{43} + b_5\alpha_{53} &= b_{13} - cb_{12}, \quad b_4\alpha_{43}^2 + b_5\alpha_{53}^2 = b_{33} - c^2b_{22}
\end{align*}
\]

are solvable for $a_4, a_5, b_4, b_5$ in $K$. 

Then we choose \( \alpha_{12}, \alpha_{22}, \alpha_{32} \) in \( K \) such that
\[
(a_4 + a_5)\alpha_{12}\alpha_{22}\alpha_{32} + \alpha_{12}\alpha_{22} + \alpha_{12}\alpha_{32} + \alpha_{22}\alpha_{32} = 0,
\]
\[
(\alpha_{22} - \alpha_{12})(\alpha_{32} - \alpha_{12})(\alpha_{32} - \alpha_{22}) \neq 0.
\]
This gives
\[
\begin{vmatrix}
1 & 1 & 1 & -a_4 - a_5 \\
\alpha_{12} & \alpha_{22} & \alpha_{32} & 1 \\
\alpha_{12}^2 & \alpha_{22}^2 & \alpha_{32}^2 & 1 \\
\alpha_{12}^3 & \alpha_{22}^3 & \alpha_{32}^3 & 0
\end{vmatrix} = 0,
\]
and the systems of linear equations
\[
\sum_{\mu=1}^{3} a_\mu = -a_4 - a_5, \quad \sum_{\mu=1}^{3} a_\mu \alpha_\mu = 1, \quad \sum_{\mu=1}^{3} a_\mu \alpha_\mu^2 = 0, \quad \sum_{\mu=1}^{3} a_\mu \alpha_\mu^3 = 0;
\]
\[
\sum_{\mu=1}^{3} b_\mu = b_{11} - b_4 - b_5, \quad \sum_{\mu=1}^{3} b_\mu \alpha_\mu = b_{12}, \quad \sum_{\mu=1}^{3} b_\mu \alpha_\mu^2 = b_{22};
\]
\[
\sum_{\mu=1}^{2} d_\mu + d_4 = c_1, \quad \sum_{\mu=1}^{2} d_\mu \alpha_\mu = c_2, \quad \sum_{\mu=1}^{2} d_\mu \alpha_\mu^2 + d_4 \alpha_{43} = c_3
\]
are solvable for \( a_1, a_2, a_3, b_1, b_2, b_3, d_1, d_2, d_4 \) in \( K \). Then we put \( d_3 = d_5 = 0, \)
\[
\alpha_\mu = \begin{cases} 
[1, \alpha_\mu^2, c_\alpha_\mu] & \text{for } \mu \leq 3, \\
[1, 0, \alpha_\mu^3] & \text{for } \mu = 4, 5,
\end{cases}
\]
\[
f_\mu = a_\mu z^3 + b_\mu z^2 + d_\mu z \quad (1 \leq \mu \leq 5),
\]
and verify that (12)–(13) and (15)–(17) imply (1).

Consider now the case where \( b_{22} = 0, b_{23} \neq 0 \). We take \( c \) as the solution to the equation
\[
c b_{23} + b_{12} = b_{33}.
\]
Then we choose \( \alpha_{\mu 3} \) as 5 distinct roots of the equation
\[
\alpha_{3}^5 = d(c\alpha_{3} + 1)^3,
\]
where \( d \) is chosen so that the equation has distinct roots. It follows that \( c\alpha_{3} + 1 \neq 0 \) and we take
\[
\alpha_{\mu 2} = \frac{\alpha_{\mu 3}^2}{c\alpha_{\mu 3} + 1} \quad (1 \leq \mu \leq 5).
\]
Let us consider the matrix

\[
A = \begin{pmatrix}
1 & \ldots & 1 \\
\alpha_{12} & \ldots & \alpha_{52} \\
\alpha_{12}^2 & \ldots & \alpha_{52}^2 \\
\alpha_{13} & \ldots & \alpha_{53} \\
\alpha_{12}\alpha_{13} & \ldots & \alpha_{52}\alpha_{53}
\end{pmatrix}
\]

and suppose that a linear combination of its successive rows with coefficients \(e_1, \ldots, e_5\), respectively, equals 0. From (20) we obtain

\[
e_1 + e_2\frac{\alpha_{\mu 3}^2}{c\alpha_{\mu 3} + 1} + e_3\frac{\alpha_{\mu 3}^4}{(c\alpha_{\mu 3} + 1)^2} + e_4\alpha_{\mu 3} + e_5\frac{\alpha_{\mu 3}^3}{c\alpha_{\mu 3} + 1} = 0,
\]

hence

\[
(e_3 + e_5c)e_{\mu 3}^4 + (e_2c + e_4c^2 + e_5)e_{\mu 3}^3 + (e_1c^2 + e_2 + 2e_4c)e_{\mu 3}^2 + (2e_1c + e_4)e_{\mu 3} + e_1 = 0
\]

and since the left hand side is a polynomial of degree at most 4 in \(\alpha_{\mu 3}\), and \(\alpha_{13}, \ldots, \alpha_{53}\) are distinct, we have

\[
e_3 + e_5c = 0, \quad e_2c + e_4c^2 + e_5 = 0, \quad e_1c^2 + e_2 + 2e_4c = 0,
\]

\[
2e_1c + e_4 = 0, \quad e_1 = 0,
\]

which implies \(e_{\mu} = 0\) (1 \(\leq\) \(\mu\) \(\leq\) 5). Thus the rows of \(A\) are linearly independent and the systems of linear equations

(21)

\[
\sum_{\mu=1}^{5} a_\mu = 0, \quad \sum_{\mu=1}^{5} a_\mu a_{\mu 2} = 1, \quad \sum_{\mu=1}^{5} a_\mu a_{\mu 2}^2 = 0,
\]

(22)

\[
\sum_{\mu=1}^{5} a_\mu a_{\mu 3} = 0, \quad \sum_{\mu=1}^{5} a_\mu a_{\mu 2} a_{\mu 3} = 0;
\]

(23)

\[
\sum_{\mu=1}^{5} b_\mu = b_{11}, \quad \sum_{\mu=1}^{5} b_\mu a_{\mu 2} = b_{12}, \quad \sum_{\mu=1}^{5} b_\mu a_{\mu 2}^2 = 0,
\]

\[
\sum_{\mu=1}^{5} b_\mu a_{\mu 3} = b_{13}, \quad \sum_{\mu=1}^{5} a_\mu a_{\mu 2} a_{\mu 3} = b_{23},
\]

\[
\sum_{\mu=1}^{5} d_\mu = c_1, \quad \sum_{\mu=1}^{5} d_\mu a_{\mu 2} = c_2, \quad \sum_{\mu=1}^{5} d_\mu a_{\mu 3} = c_3
\]

are solvable for \(a_\mu, b_\mu, d_\mu\) in \(K\).

In view of (18) and of the identities

\[
\alpha_{\mu 2}^3 = d\alpha_{\mu 3}, \quad \alpha_{\mu 3}^2 = c\alpha_{\mu 2} a_{\mu 3} + \alpha_{\mu 2}, \quad \alpha_{\mu 2}^2 a_{\mu 3} = d(c\alpha_{\mu 3} + 1),
\]

\[
\alpha_{\mu 2} a_{\mu 3}^2 = c\alpha_{\mu 2}^2 a_{\mu 3} + \alpha_{\mu 2}^2, \quad \alpha_{\mu 3}^3 = c\alpha_{\mu 2} a_{\mu 3}^2 + \alpha_{\mu 2} a_{\mu 3},
\]

\[
\alpha_{\mu 3}^2 = c\alpha_{\mu 2} a_{\mu 3} + \alpha_{\mu 2}, \quad \alpha_{\mu 3}^3 = c\alpha_{\mu 2} a_{\mu 3}^2 + \alpha_{\mu 2} a_{\mu 3},
\]
which follow from (19)–(20), the resulting $a_\mu, b_\mu, d_\mu$ satisfy also the equations

\[\sum_{\mu=1}^{5} a_\mu \alpha_{\mu}^3 = 0, \quad \sum_{\mu=1}^{5} a_\mu \alpha_{\mu}^2 \alpha_{\mu}^3 = 0;\]

\[\sum_{\mu=1}^{5} a_\mu \alpha_{\mu} \alpha_{\mu}^2 \alpha_{\mu}^3 = 0, \quad \sum_{\mu=1}^{3} a_\mu \alpha_{\mu}^3 = 0;\]

\[\sum_{\mu=1}^{3} b_\mu \alpha_{\mu}^2 \alpha_{\mu}^3 = c b_{23} + b_{12} = b_{33},\]

hence, by (12) and (13), (1) holds with

\[\alpha_\mu = [1, \alpha_{\mu 2}, \alpha_{\mu 3}], \quad f_\mu = a_\mu z^3 + b_\mu z^2 + d_\mu z \quad (1 \leq \mu \leq 5).\]