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WEAK-TYPE ESTIMATES FOR THE MODIFIED HANKEL TRANSFORM

BY

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Abstract. We use the Galé and Pytlik [3] representation of Riesz functions to prove the Hörmander multiplier theorem for the modified Hankel transform.

1. Introduction. Fix $\alpha > -1/2$ and for $1 < p < \infty$ consider the space $L^p_{(\alpha)}$ of measurable functions f on $\mathbb{R}_+$ for which

$$\|f\|_p = \Big(\int_0^\infty |f(x)|^p x^{2\alpha+1}\,dx\Big)^{1/p} < \infty.$$

As in Hirschman [7], for $f \in L^1$ we define the modified Hankel transform of order α by the formula

$$H_\alpha f(y) = \int_0^\infty \frac{J_\alpha(xy)}{(yx)^\alpha} f(x) x^{2\alpha+1}\,dx,$$

where J_α is the Bessel function of the first kind of order α.

For any bounded function m on $\mathbb{R}_+$ the multiplier operator T_m is defined by $H_\alpha(T_m f) = mH_\alpha f$.

We will obtain a multiplier theorem in terms of the spaces $WBV_{s,\lambda}$ of functions of weak bounded variation which were introduced by Gasper and Trebels [4]. The condition which defines these spaces is an extension of the Hörmander condition [8] for fractional derivatives.

For $1 \le s < \infty$, $\lambda > 0$ a bounded continuous function m on $\mathbb{R}_+$ belongs to $\mathrm{WBV}_{s,\lambda}$ if it satisfies

$$\|m\|_{s,\lambda} = \|m\|_\infty + \sup_{j\in\mathbb{Z}} \Big(\int_{2^{j-1}}^{2^j} |x^\lambda m^{(\lambda)}(x)|^s \,\frac{dx}{x}\Big)^{1/s} < \infty,$$

where $m^{(\lambda)}$ denotes the fractional derivative of order λ in the sense of Gasper and Trebels.

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As usual, we use C with or without subscripts for a constant which is not necessarily the same at each occurrence.

THEOREM. *Fix* $\alpha > -1/2$, $1 \le s \le 2$. *Assume that a function* m *on* $\mathbb{R}_+$ *belongs to* $\mathrm{WBV}_{s,\lambda}$ *for some* $\lambda > \alpha + 1/2 + 1/s$. *Then the operator* T_m *is of weak type* $(1,1)$, *and consequently is bounded on every* $L^p_{(\alpha)}$, $1 < p < \infty$.

One can verify using $\mathrm{WBV}_{s,\lambda}$ embeddings (see [4, Theorem 4 b), c)]) that the conclusion our Theorem with $s = 2$ implies the conclusion for $1 \le s < 2$. For $s = 2$ we get an optimal λ ($\lambda > \alpha + 1$), which improves the results from earlier papers [2], [6] and [9]. The optimal multiplier theorem of strong type (p, p) was proved by Gasper and Trebels [5], using the transference method.

2. Proof of Theorem. The main idea of the proof is based on Galé and Pytlik's [3] representation of the modified Hankel transform of the function

$$R_u(x) = \frac{1}{\Gamma(\lambda)}(u-x)_+^{\lambda-1}, \qquad x > 0,$$

in a very special form:

$$H_\alpha R_u(y) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, z = t} \frac{p_z(y)}{z^\lambda} \exp(uz)\, dz, \tag{1}$$

where $t > 0$ is arbitrarily fixed, dz is the Lebesgue measure on the line $\mathrm{Re}\, z = t$ and $p_z(y)$ is the Poisson kernel,

$$p_z(y) = D_\alpha \frac{z}{(z^2 + y^2)^{\alpha+3/2}}.$$

We also strongly exploit the inequality

$$\int\limits_0^\infty \frac{r^{\alpha-1}}{|z^2 + r^2|^\beta}\, dr \le C_{\alpha,\beta} (\mathrm{Re}\, z)^{1-\beta} |z|^{\alpha-\beta-1} \tag{2}$$

valid for any $\mathrm{Re}\, z > 0$, $\alpha > 0$, $\beta > \max\{1, \alpha/2\}$ (see [3, p. 348]).

As usual we decompose the function m into small dyadic pieces by using a fixed bump function. Let Ψ be a C^∞ function on $(0, \infty)$ supported in $(1/2, 2)$ such that $\sum_{j=-\infty}^{\infty} \Psi(2^{-j}x) = 1$, and set $m_j(x) = m(x)\Psi(2^{-j}x)$.

The following characterization of $\mathrm{WBV}_{s,\lambda}$ spaces was proved in [1, Theorem 2]. Assume that $1 \le s \le \infty$, $\lambda \ge 1/s$ and $\Psi_{1/t}(x) = \Psi(x/t)$. Then

$$\sup_{t>0} t^{\lambda - 1/s} \Big(\int\limits_{-\infty}^{\infty} |(\Psi_{1/t} m)^{(\lambda)}(x)|^s\, dx \Big)^{1/s} \le C \|m\|_{s,\lambda}, \tag{3}$$

with $C > 0$ independent of m and $\Psi_{1/t}$. Moreover for the function m_j with compact support in $(2^{j-1}, 2^{j+1})$ we have $\operatorname{supp} m_j^{(\lambda)} \subset (0, 2^{j+1})$ and we can write the reproducing formula

$$m_j(x) = \frac{1}{\Gamma(\lambda)} \int_0^{2^{j+1}} m_j^{(\lambda)}(u)(u-x)_+^{\lambda-1}\,du.$$

By (1) we get

$$H_\alpha m_j(y) = \int_0^{2^{j+1}} m_j^{(\lambda)}(u) H_\alpha R_u(y)\,du.$$

Substitution $t = 2^j$ in (3) leads to

$$\Big(\int_0^{2^{j+1}} |m_j^{(\lambda)}(x)|^s\,dx\Big)^{1/s} \le C(2^j)^{1/s-\lambda}\|m\|_{s,\lambda}, \tag{4}$$

where C does not depend on j. As in [6] in order to prove that T_m is of weak type $(1,1)$ it is sufficient to establish the following estimates:

$$\int_0^\infty |H_\alpha m_j(y)| y^{2\alpha+1}\,dy \le C\|m\|_{s,\lambda}, \tag{5}$$

$$\int_d^\infty |H_\alpha m_j(y)| y^{2\alpha+1}\,dy \le C\|m\|_{s,\lambda}(2^j d)^{-\varepsilon} \tag{6}$$

for some $0 < \varepsilon < 1$. Fix $t > 0$, write $z = t + iw$ and observe that

$$|H_\alpha m_j(y)| \le C \int_{-\infty}^\infty \frac{|p_{t+iw}(y)|}{|t+iw|^\lambda} |\mathcal{F}h_t(w)|\,dw, \tag{7}$$

where h_t denotes the function $h_t(u) = m_j^{(\lambda)}(u)\exp(ut)$ and $\mathcal{F}$ is the Fourier transform on the real line.

To prove (5) observe that by (2) and (7) we get

$$\int_0^\infty |H_\alpha m_j(y)| y^{2\alpha+1}\,dy \le C \int_{-\infty}^\infty \frac{|t+iw|^{\alpha+1/2-\lambda}}{t^{\alpha+1/2}} |\mathcal{F}h_t(w)|\,dw.$$

The Hölder inequality, the Hausdorff–Young inequality applied to h_t and (4) give

$$\begin{aligned}\int_0^\infty & |H_\alpha m_j(y)| y^{2\alpha+1}\,dy \\ &\le C t^{1/s-\lambda} \Big(\int_{-\infty}^\infty \Big(\frac{|1+iw|^{\alpha+1/2}}{|1+iw|^\lambda}\Big)^s dw\Big)^{1/s} \Big(\int_{-\infty}^\infty |h_t(u)|^s\,du\Big)^{1/s} \\ &\le C\|m\|_{s,\lambda}(2^j t)^{1/s-\lambda}\exp(2^j t).\end{aligned}$$

Choosing $t = 2^{-j}$ gives (5).

To prove (6) first choose q such that $\lambda > \alpha + 1/2 + 1/s + \varepsilon + 1/q$, and then use Hölder's inequality to obtain

$$\int_d^\infty |H_\alpha m_j(y)| y^{2\alpha+1}\, dy \le d^{-\varepsilon}\Big(\int_0^\infty |H_\alpha m_j(y)|^{q'} y^{(2\alpha+1+\varepsilon+1/q)q'}\, dy\Big)^{1/q'}.$$

Applying Minkowski's inequality, (7) and (2) we get

$$\Big(\int_0^\infty |H_\alpha m_j(y)|^{q'} y^{(2\alpha+1+\varepsilon+1/q)q'}\, dy\Big)^{1/q'} \le C \int_{-\infty}^{\infty} \frac{|t+iw|^{\alpha+1/2+1/q+\varepsilon-\lambda}}{t^{\alpha+3/2-1/q'}} |\mathcal{F}h_t(w)|\, dw.$$

Now the Hölder inequality, the Hausdorff–Young inequality applied to h_t and (4) give

$$\begin{aligned}\int_d^\infty & |H_\alpha m_j(y)| y^{2\alpha+1}\, dy \\ &\le C\frac{d^{-\varepsilon}}{t^{-\varepsilon-1/s+\lambda}}\Big(\int_{-\infty}^{\infty}\Big(\frac{|1+iw|^{\alpha+1/2+\varepsilon+1/q}}{|1+iw|^\lambda}\Big)^s dw\Big)^{1/s}\Big(\int_{-\infty}^{\infty}|h_t(u)|^s\, du\Big)^{1/s} \\ &\le C\|m\|_{s,\lambda}(d/t)^{-\varepsilon}(2^j t)^{1/s-\lambda}\exp(2^j t).\end{aligned}$$

Choosing $t = 2^{-j}$ gives (6). This finishes the proof of the Theorem. ∎

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