COLLOQUIUM MATHEMATICUM

VOL. 92 2002 NO. 1

ON SOME PROBLEMS OF MĄKOWSKI–SCHINZEL AND ERDŐS CONCERNING THE ARITHMETICAL FUNCTIONS ϕ AND σ

BY

FLORIAN LUCA (Morelia) and CARL POMERANCE (Murray Hill, NJ)

Abstract. Let $\sigma(n)$ denote the sum of positive divisors of the integer n, and let ϕ denote Euler's function, that is, $\phi(n)$ is the number of integers in the interval $[1,n]$ that are relatively prime to n. It has been conjectured by Mąkowski and Schinzel that $\sigma(\phi(n))/n \geq 1/2$ for all n. We show that $\sigma(\phi(n))/n \to \infty$ on a set of numbers n of asymptotic density 1. In addition, we study the average order of $\sigma(\phi(n))/n$ as well as its range. We use similar methods to prove a conjecture of Erdős that $\phi(n-\phi(n)) < \phi(n)$ on a set of asymptotic density 1.

1. Introduction. In this paper, we investigate a couple of conjectures concerning inequalities involving the arithmetical functions ϕ, σ, and compositions of these. As usual, for a positive integer n we write $\phi(n)$ for the Euler function of n, and $\sigma(n)$ for the sum-of-divisors function of n. For any positive integer k and any positive number x we define $\log_k(x)$ recursively as the maximum of 1 and $\log(\log_{k-1}(x))$, where $\log_1 = \log$ is the natural logarithm. Throughout this paper, we use p, q and P to denote prime numbers, and $c_1, c_2, \ldots$ will denote positive computable constants.

The first conjecture we are looking at is due to Mąkowski and Schinzel (see [18]) and asserts that the inequality

$$\frac{\sigma(\phi(n))}{n} \geq \frac{1}{2} \tag{1}$$

holds for all positive integers n. It is known that

$$\limsup_n \frac{\sigma(\phi(n))}{n} = \infty \quad \text{and} \quad 0 < \liminf_n \frac{\sigma(\phi(n))}{n} \leq \frac{1}{2} + \frac{1}{2^{34}-1}. \tag{2}$$

The first limit in (2) is due to Alaoglu and Erdős (see [1]). The positive lower bound for the second limit in (2) is due to the second author (see [20]) and the upper bound for the same limit appears in the original paper of Mąkowski and Schinzel [18]. It is known that (1) holds for positive integers n of various shapes (see, for example, [2], [6], [11], [18]), and in fact in [1] it

2000 *Mathematics Subject Classification*: 11A25, 11N37, 11N56.

Key words and phrases: sum-of-divisors function, Euler's function.

is asserted that $\sigma(\phi(n))/n \to \infty$ on a set of asymptotic density 1, and that $\sigma(\phi(n))/\phi(n) \sim e^\gamma \log_3(n)$ on a set of asymptotic density 1.

Especially since the proofs of these results stated in [1] were not given, and since later researchers have labored to give somewhat weaker results (in [15] it is shown that (1) holds on a set of lower density at least 0.74), we think it is worthwhile to prove these assertions from [1], in a somewhat stronger form. For every positive integer n we write $S(n) := \sigma(\phi(n))/n$. Our first result in this paper gives the maximal, normal, and average orders of the function $S(n)$.

THEOREM 1. (i) *We have*

$$\limsup_n \frac{S(n)}{\log_2(n)} = e^\gamma. \tag{3}$$

(ii) *For each number u, $0 \le u \le 1$, the asymptotic density of the set of numbers n with*

$$S(n) > ue^\gamma \log_3(n) \tag{4}$$

exists, and this density function is strictly decreasing, varies continuously with u, and is 0 when $u = 1$.

(iii) *For every positive number x,*

$$\frac{1}{x} \sum_{1 \le n \le x} S(n) = \frac{6e^\gamma}{\pi^2} \cdot \log_3(x) + O((\log_3(x))^{1/2}). \tag{5}$$

We remark that Warlimont (see [22]) has recently shown a result similar to part (iii) of Theorem 1 for the function $\phi(n)/\phi(\phi(n))$.

There are several results in the literature which assert that if $f(n)$ is, for example, either of the functions $\sigma(n)/n$ or $\phi(n)/n$, then the closure of the set $\{f(n)\}_{n\ge1}$ is an interval. Our next result shows that the same is true for the function $f(n) = S(n)$.

THEOREM 2. *Let $\alpha = \liminf_n S(n)$. Then $\{S(n) \mid n \ge 1\}$ is dense in the interval $[\alpha, \infty]$.*

We now leave the function $S(n)$ and the next question we address is a conjecture of Erdős (see [8]) which asserts that the inequality

$$\phi(n - \phi(n)) < \phi(n) \tag{6}$$

holds on a set of positive integers n of asymptotic density 1 (i.e., for almost all positive integers n), but the inequality

$$\phi(n - \phi(n)) > \phi(n) \tag{7}$$

holds for infinitely many positive integers n. Some infinite families of positive integers n for which inequality (7) holds were pointed out in [16]. In that paper, it was also shown that (6) holds for a set of positive integers n of lower density at least 0.54. In this note, we prove that (6) holds indeed for almost all positive integers n. In fact, we prove a stronger statement, namely:

THEOREM 3. (i) *Let $\varepsilon(x)$ be any positive function of the positive variable x which tends to zero when x tends to infinity. The set of integers $n > 1$ for which the inequality*

$$\phi(n - \phi(n)) < \phi(n) - n \cdot \varepsilon(n) \tag{8}$$

fails has asymptotic density 0.

(ii) *The set of positive integers n for which the inequality*

$$\left|\frac{\phi(n)}{n} - \frac{\phi(n-\phi(n))}{n-\phi(n)}\right| < \frac{2\log_3(n)}{\log_2(n)} \tag{9}$$

fails has asymptotic density 0.

For example, (9) implies that for any fixed $\varepsilon > 0$ the set of n for which

$$\left|\frac{\phi(n)}{n} - \frac{\phi(n-\phi(n))}{n-\phi(n)}\right| < \varepsilon$$

fails has asymptotic density 0. In particular, the two functions $\phi(n)/n$ and $\phi(n - \phi(n))/(n - \phi(n))$ are asymptotically equal on a set of n of asymptotic density 1.

For $n > 1$, let $f(n) = \phi(n - \phi(n))/\phi(n)$. It can be shown, using the method of proof of Theorem 2, that the set of numbers $f(n)$ is dense in the interval $[0, \infty]$. This result shows, in particular, that if c is any positive number, then the inequality

$$\phi(n - \phi(n)) > c\phi(n) \tag{10}$$

holds for infinitely many positive integers n, which is a statement much stronger than the fact that (7) holds for infinitely many positive integers n. We do not give further details here.

2. Preliminary results. In 1928, Schoenberg (see [21]) proved that the function $\phi(n)/n$ has a distribution. That is, $D(u)$ defined as the asymptotic density of the set of n with $\phi(n)/n \geq u$ exists for every u. In addition, $D(u)$ is continuous and strictly decreasing on $[0, 1]$. Clearly, $D(0) = 1$ and $D(1) = 0$. From these considerations, we immediately derive the following:

LEMMA 1. *Let $\varepsilon : (0, \infty) \to (0, 1)$ be any function such that $\varepsilon(x)$ tends to zero as x tends to infinity. Then, for a set of n of asymptotic density* 1,

$$\frac{\phi(n)}{n} > \varepsilon(n). \tag{11}$$

Proof. This is almost obvious. Indeed, let ε be an arbitrarily small positive number. For large x we have $\varepsilon(x) < \varepsilon$. Thus, the set of positive integers n for which inequality (11) holds contains a set of n of asymptotic density at least $D(\varepsilon)$. Since this holds for every $\varepsilon > 0$, it follows that inequality (11) holds for a set of n of asymptotic density $D(0) = 1$.

Lemma 1 may also be proved by using the average order of $\phi(n)/n$, namely,

$$\sum_{n\le x} \frac{\phi(n)}{n} = \frac{6}{\pi^2}x + O(\log(x)), \tag{12}$$

but we shall have other uses for the distribution function $D(u)$ later.

The next result plays a key role in the proofs of both Theorems 1 and 3 and is essentially due to Erdős, Granville, Pomerance and Spiro (see [9]). However, since this result was not explicitly stated in [9], we state it below and provide a short proof.

Lemma 2. *There exists a computable positive constant* c_1 *such that on a set of* n *of asymptotic density* 1, $\phi(n)$ *is divisible by all prime powers* p^a *with* $p^a < c_1 \log_2(n)/\log_3(n)$.

Proof. We use the notations from [9]. For any positive integer m and any positive number x we let

$$S(x,m) = \sum_{\substack{q\le x\\ m\mid(q-1)}} \frac{1}{q}. \tag{13}$$

From Theorem 3.4 in [9], we know that there exist computable positive numbers c_1, x_0 such that the inequality

$$S(x,m) > \frac{c_1 \log_2(x)}{\phi(m)} \tag{14}$$

holds for all $x > x_0$ and all $m \le \log(x)$. Let $g(x) = c_1 \log_2(x)/\log_3(x)$. From the proof of Theorem 4.1 in [9], we have, uniformly for all m,

$$\sum_{\substack{n\le x\\ m\nmid\phi(n)}} 1 < \frac{c_2 x}{\exp(S(x,m))}. \tag{15}$$

Assume now that p^a is any prime power with $p^a < g(x)$. By increasing x_0 if necessary, we assume that the inequality $\log(x) > g(x)$ holds for $x > x_0$. By inequalities (14) and (15), it follows that for such p^a,

$$\sum_{\substack{n\le x\\ p^a\nmid\phi(n)}} 1 < \frac{c_2 x}{\exp(S(x,p^a))} < \frac{c_2 x}{\exp\left(\log_3(x)\right)} = \frac{c_2 x}{\log_2(x)}. \tag{16}$$

Now let $M(x)$ be the least common multiple of all the prime powers $p^a < g(x)$. Inequality (16) shows that the number of $n \le x$ for which $\phi(n)$ is not a multiple of $M(x)$ is at most

$$\sum_{p^a<g(x)} \sum_{\substack{n\le x\\ p^a\nmid\phi(n)}} 1 < \frac{c_2x}{\log_2(x)}\pi_0(g(x)) < \frac{2c_2xg(x)}{\log_2(x)\log(g(x))} < \frac{c_3x}{(\log_3(x))^2}, \tag{17}$$

where $\pi_0(y)$ denotes the number of prime powers $p^a \le y$ with $a \ge 1$. Inequality (17) shows that but for $O(x/(\log_3(x))^2) = o(x)$ positive integers $n \le x$, $\phi(n)$ is a multiple of $M(x)$. Lemma 2 is therefore proved.

For the remainder of this paper, we let $g(n) = c_1 \log_2(n)/\log_3(n)$ and denote by $M(n)$ the least common multiple of all prime powers $p^a < g(n)$. Here, c_1 is the constant appearing in the statement of Lemma 2.

We shall also make use of the following result:

LEMMA 3. *On a set of positive integers n of asymptotic density* 1 *the following inequalities hold*:

$$h_1(n) := \sum_{\substack{p>g(n)\\ p\mid n}} \frac{1}{p} < \frac{\log_3(n)}{\log_2(n)}, \tag{18}$$

$$h_2(n) := \sum_{\substack{p>g(n)\\ p\mid(n-\phi(n)),\, p\nmid n}} \frac{1}{p} < \frac{\log_3(n)}{\log_2(n)}, \tag{19}$$

$$h_3(n) := \sum_{\substack{p>\log_2(n)\\ p\mid\phi(n)}} \frac{1}{p} < \frac{\log_4(n)}{\log_3(n)}. \tag{20}$$

Proof. The fact that inequality (18) holds for a set of n of asymptotic density 1 follows from an averaging argument. Indeed, if T is any slowly increasing function of x (like $T(x) = g(x)$, for example), then

$$\sum_{1\le n\le x} \sum_{\substack{T(n)<p\\ p\mid n}} \frac{1}{p} = O\left(\frac{x}{T(x)\log(T(x))}\right) \tag{21}$$

(see also [9], page 199). Taking $T(x) = g(x)$ in formula (21), we get

$$\sum_{n\le x} h_1(n) \le \frac{c_4x}{\log_2(x)}.$$

In particular, but for a set of $n \le x$ of cardinality $O(x/\log_3(x)) = o(x)$, inequality (18) holds.

We now turn to the second part of the lemma. For $n \le x$,

$$h_2(n) \le \sum_{p\le x} \frac{1}{p} \le \log_2(x) + c_5, \tag{22}$$

though it is not hard to get a sharper inequality here. Let

$$\eta = \eta(x) = \frac{\log_4(x)}{3\log_3(x)}.$$

For any positive integer $n > 1$, we let $P(n)$ be the largest prime divisor of n. By de Bruijn [4], the number of $n \le x$ with $P(n) \le x^\eta$ is $x/(\log_2(x))^{3+o(1)}$, and so is at most $x/(\log_2(x))^{2.5}$ for all sufficiently large values of x. It is easy to see that the number of $n \le x$ with $P(n) > x^\eta$ and $P(n)^2 \mid n$ is at most $x^{1-\eta}$. Let A be the set of numbers n with

$$x^{1/2} < n \le x, \qquad P(n) > x^\eta, \qquad P(n)^2 \nmid n. \tag{23}$$

It follows from (22) and the above estimates that

$$\sum_{n\le x,\, n\notin A} h_2(n) \le \frac{x}{\log_2(x)} \tag{24}$$

for all sufficiently large values of x.

Next, for $n \in A$, let $h_2(n) = h_{2,0}(n) + h_{2,1}(n)$, where

$$h_{2,0}(n) = \sum_{\substack{x^{\eta/2} \ge p > g(n)\\ p\mid(n-\phi(n)),\, p\nmid n}} \frac{1}{p}, \qquad h_{2,1}(n) = \sum_{\substack{p > x^{\eta/2}\\ p\mid(n-\phi(n)),\, p\nmid n}} \frac{1}{p}. \tag{25}$$

For $m \le x$, the number of primes $p \mid m$ with $p > x^{\eta/2}$ is at most $2/\eta$. Thus, for $n \in A$, $h_{2,1}(n) \le (2/\eta)x^{-\eta/2}$, and so

$$\sum_{n\in A} h_{2,1}(n) \le \frac{x}{\log(x)} \tag{26}$$

for x sufficiently large.

For an estimation of $h_{2,0}(n)$ we again use an averaging argument. We have

$$\sum_{n\in A} h_{2,0}(n) \le \sum_{g(x^{1/2}) < p \le x^{\eta/2}} \frac{1}{p} \sum_{\substack{n\not\equiv 0 \pmod p\\ n\equiv\phi(n) \pmod p\\ n\in A}} 1. \tag{27}$$

We now fix a prime number p in the interval $(g(x^{1/2}), x^{\eta/2}]$ and we find an upper bound for the inner sum appearing in (27). Assume that $n \in A$ is such that $p \mid (n - \phi(n))$. Write $n = Pm$ where $P = P(n)$. Then $n - \phi(n) = Pm - (P-1)\phi(m) = P(m - \phi(m)) + \phi(m)$. Thus,

$$P(m - \phi(m)) \equiv -\phi(m) \pmod p. \tag{28}$$

Notice that p does not divide $m - \phi(m)$. Indeed, if $p \mid (m - \phi(m))$, then congruence (28) implies that $p \mid \phi(m)$, therefore $p \mid m$, which contradicts the fact that $p \nmid n$. Let a_m be the integer in the interval $[0, p-1]$ given by $a_m \equiv -\phi(m)(m-\phi(m))^{-1} \pmod p$. Congruence (28) implies that $P \equiv a_m$

(mod p). From (23) we deduce that $m \le x^{1-\eta}$. Thus, summing up first over all the possible values of $P(n)$ when m is fixed, we get

$$\sum_{\substack{n \not\equiv 0 \pmod p \\ n \equiv \phi(n) \pmod p \\ n \in A}} 1 \le \sum_{1 \le m \le x^{1-\eta}} \sum_{\substack{1 \le P \le x/m \\ P \equiv a_m \pmod{p}}} 1 = \sum_{1 \le m \le x^{1-\eta}} \pi\left(\frac{x}{m}, a_m, p\right). \tag{29}$$

By a well known result of Montgomery and Vaughan (see [19]),

$$\pi\left(\frac{x}{m}, a_m, p\right) \le \frac{2x}{m(p-1)\log(x/(mp))}. \tag{30}$$

Since $p \le x^{\eta/2}$, we have

$$\sum_{1 \le m \le x^{1-\eta}} \frac{2x}{m(p-1)\log(x/(mp))} \le \frac{2x}{(p-1)(\eta/2)\log(x)} \sum_{1 \le m \le x^{1-\eta}} \frac{1}{m} < \frac{5x}{\eta p} \tag{31}$$

for x sufficiently large. Hence (29)–(31) imply that

$$\sum_{\substack{n \not\equiv 0 \pmod p \\ n \equiv \phi(n) \pmod p \\ n \in A}} 1 < \frac{5x}{\eta p},$$

so that from (27), we have

$$\sum_{n \in A} h_{2,0}(n) \le \sum_{p > g(x^{1/2})} \frac{5x}{p^2 \eta} \le \frac{6x}{\eta g(x^{1/2}) \log(g(x^{1/2}))} \le \frac{20x \log_3(x)}{c_1 \log_2(x) \log_4(x)} \tag{32}$$

for all sufficiently large values of x. With (24) and (26), (32) implies that

$$\sum_{n \le x} h_2(n) \le \frac{c_6 x \log_3(x)}{\log_2(x) \log_4(x)}.$$

In particular, but for $O(x/\log_4(x)) = o(x)$ values of $n \le x$, inequality (19) holds.

The third part of Lemma 3 follows immediately from the inequality

$$\sum_{\substack{n \le x \\ p \mid \phi(n)}} 1 = O\left(\frac{x \log_2 x}{p}\right)$$

(uniformly for every prime p and x larger than some x_1 that is independent of p; see Theorem 3.5 in [9]). It follows that the average of $h_3(n)$ for $n \le x$ is $O(1/\log_3(x))$, and thus, we get (20) for a set of asymptotic density 1.

Lemma 3 is therefore proved.

3. Proofs of the theorems

Proof of Theorem 1. (i) We first observe that since

$$\frac{c_7 n}{\log_2(n)} < \phi(n) \le n$$

for all integers n (see [17], Theorem 328), it follows that

$$\lim_n \frac{\log_2(\phi(n))}{\log_2(n)} = 1.$$

Thus,

$$\limsup_n \frac{\sigma(\phi(n))}{n\log_2(n)} \le \limsup_n \frac{\sigma(\phi(n))}{\phi(n)\log_2(\phi(n))} \le \limsup_n \frac{\sigma(n)}{n\log_2(n)} = e^{\gamma} \tag{33}$$

(see Theorem 323 in [17]). For the reverse inequality, let n_k be a sequence of integers which attains the last lim sup, that is,

$$\lim_k \frac{\sigma(n_k)}{n_k\log_2(n_k)} = e^{\gamma}. \tag{34}$$

Let p_k be the least prime with $p_k \equiv 1 \pmod{n_k}$. By Linnik's theorem we have $p_k < n_k^{c_8}$, so that $\log_2(p_k) \sim \log_2(n_k)$. Thus,

$$\frac{S(p_k)}{\log_2(p_k)} = \frac{\sigma(p_k-1)}{p_k\log_2(p_k)} \sim \frac{\sigma(p_k-1)}{(p_k-1)\log_2(n_k)} \ge \frac{\sigma(n_k)}{n_k\log_2(n_k)}.$$

With (33) and (34), we thus have part (i) of Theorem 1.

(ii) Here we use the notations from the proof of Lemma 2. By Lemma 2, for a set of positive integers n of asymptotic density 1, $\phi(n)$ is a multiple of $M(n)$. For each prime $p < g(n)$, let p^{a_p} be the power of p in the prime factorization of $M(n)$. For such n, we have the inequality

$$\frac{\sigma(\phi(n))}{\phi(n)} \ge \prod_{p<g(n)} \left(1+\frac{1}{p}+\ldots+\frac{1}{p^{a_p}}\right) \sim \prod_{p<g(n)} \frac{p}{p-1} \sim e^{\gamma}\log_3(n) \tag{35}$$

as $n \to \infty$. In addition, we have

$$\begin{aligned}\frac{\sigma(\phi(n))}{\phi(n)} &< \prod_{p\mid\phi(n)} \frac{p}{p-1} \le \Bigg(\prod_{p\le\log_2(n)} \frac{p}{p-1}\Bigg)\Bigg(\prod_{\substack{p\mid\phi(n)\\ p>\log_2(n)}} \frac{p}{p-1}\Bigg)\\ &\sim e^{\gamma}\log_3(n)\cdot\exp\Bigg(\sum_{\substack{p\mid\phi(n)\\ p>\log_2(n)}} \frac{1}{p}\Bigg).\end{aligned} \tag{36}$$

We now use (20). On a set of asymptotic density 1, the argument of exp in (36) is $o(1)$, so that with (35), which also holds on a set of asymptotic density 1, we have

$$\frac{\sigma(\phi(n))}{\phi(n)} \sim e^{\gamma} \log_3(n).$$

To see that (ii) of Theorem 1 holds, we write

$$S(n) = \frac{\sigma(\phi(n))}{\phi(n)} \cdot \frac{\phi(n)}{n}.$$

The density function referred to in (ii) of Theorem 1 is then exactly the function $D(u)$ in Schoenberg's theorem for $\phi(n)/n$.

(iii) We first show that the mean value of $S(n)$ is at least what is claimed. Let c_1 be the constant appearing in Lemma 2, and for large x let $A_0(x)$ be the set of those n with $\sqrt{x} \le n < x$ such that $\phi(n)$ is a multiple of $M(n)$. By the arguments from the proof of Lemma 2, we know that the cardinality of the complement of $A_0(x)$ in the interval $[1, x]$ is at most $O(x/(\log_3(x))^2)$. For $n \in A_0(x)$ and a_p as in (35), we have

$$\begin{aligned}
\frac{\sigma(\phi(n))}{\phi(n)} &\ge \frac{\sigma(M(n))}{M(n)} = \prod_{p<g(n)} \left(1 + \frac{1}{p} + \ldots + \frac{1}{p^{a_p}}\right) \\
&= \prod_{p<g(n)} \frac{p}{p-1}\left(1 - \frac{1}{p^{a_p+1}}\right) \\
&\ge \left(\prod_{p<g(n)} \frac{p}{p-1}\right)\left(1 - \frac{1}{g(n)}\right)^{\pi(g(n))} \\
&= (e^{\gamma} \log(g(n)) + O(1))\left(1 + O\left(\frac{1}{\log(g(n))}\right)\right) \\
&\ge e^{\gamma} \log_3(x) - c_9.
\end{aligned} \tag{37}$$

Thus,

$$\begin{aligned}
\frac{1}{x}\sum_{1\le n\le x} S(n) &\ge \frac{1}{x}\sum_{n\in A_0(x)} \frac{\sigma(\phi(n))}{\phi(n)} \cdot \frac{\phi(n)}{n} \\
&\ge (e^{\gamma}\log_3(x) - c_9) \cdot \frac{1}{x}\sum_{n\in A_0(x)} \frac{\phi(n)}{n}.
\end{aligned} \tag{38}$$

Using estimate (12) and the fact that

$$x - |A_0(x)| = O\left(\frac{x}{(\log_3(x))^2}\right),$$

we get

$$\frac{1}{x}\sum_{n\in A_0(x)} \frac{\phi(n)}{n} > \frac{6}{\pi^2} - \frac{c_{10}}{(\log_3(x))^2}. \tag{39}$$

Combining (38) with (39), we further get

$$\frac{1}{x}\sum_{1\le n<x} S(n) > \frac{6e^{\gamma}}{\pi^2}\log_3(x) - c_{11}, \tag{40}$$

which is a lower bound better than the one asserted.

It remains to get an upper bound for the mean value of $S(n)$. Let x be a large real number. We split the positive integers in the interval $[1,x]$ into the following four subsets:

$$A(x) = \{n \mid 1 \le n < \sqrt{x}\}, \tag{41}$$

$$B(x) = \left\{n \,\middle|\, \sqrt{x} \le n \le x,\ h_3(n) < \frac{1}{\sqrt{\log_3(x)}}\right\}, \tag{42}$$

$$C(x) = \left\{n \,\middle|\, \sqrt{x} \le n \le x,\ h_3(n) \ge \frac{1}{\sqrt{\log_3(x)}} \text{ and } \omega(\phi(n)) < (\log_2(x))^3\right\}, \tag{43}$$

and finally let $D(x)$ be the complement of $A(x)\cup B(x)\cup C(x)$ in the interval $[1,x]$. Here, $h_3(n)$ is as in Lemma 3 and $\omega(n)$ denotes the number of different primes that divide n.

We first comment on the sizes of the cardinalities of the four sets defined above. By the proof of Lemma 3, the cardinality of $C(x)$ is at most $O(x/\sqrt{\log_3(x)})$. Clearly, the cardinality of $A(x)$ is exactly $\lfloor\sqrt{x}\rfloor$. We now show that $D(x)$ is small as well. More precisely, we show that the cardinality of $D(x)$ is at most $O(x/\log_2(x))$. To see why this is so, notice that

$$\omega(\phi(n)) \le \Omega(\phi(n)) \le \Omega(n) + \sum_{p\mid n}\Omega(p-1), \tag{44}$$

where $\Omega(m)$ is the number of prime factors of m, counted with multiplicity. Thus,

$$\sum_{1\le n\le x}\omega(\phi(n)) \le \sum_{1\le n\le x}\Omega(n) + \sum_{1\le n\le x}\sum_{p\mid n}\Omega(p-1) = \sum\nolimits_1 + \sum\nolimits_2. \tag{45}$$

It is well known that

$$\sum\nolimits_1 = \sum_{1\le n\le x}\Omega(n) \sim x\log_2(x). \tag{46}$$

For the second sum appearing in formula (45), we interchange the order of summation to get

$$\sum\nolimits_2 = \sum_{1\le p\le x}\Omega(p-1)\cdot\left\lfloor\frac{x}{p}\right\rfloor < x\sum_{1\le p\le x}\frac{\Omega(p-1)}{p} \sim \frac{1}{2}x(\log_2(x))^2. \tag{47}$$

The rightmost approximation appearing in formula (47) is a result due to Erdős and Pomerance (see Lemma 2.3 in [10]). Formulae (46) and (47) imply that the cardinality of $D(x)$ is indeed at most $O(x/\log_2(x))$.

The above considerations show that $B(x)$ contains all integers up to x, except for $O(x/\sqrt{\log_3(x)})$ of them.

If E is any one of the letters $A,\ B,\ C,\ D$, we define

$$S_E = \frac{1}{x}\sum_{n\in E(x)} S(n). \tag{48}$$

It suffices to bound each one of the numbers S_E for $E \in \{A, B, C, D\}$. Notice that since

$$\frac{\sigma(\phi(n))}{n} < c_{12}\log_2(x) \tag{49}$$

for all $n \le x$ (see (34)), it follows that

$$S_A < c_{12}\frac{|A(x)|}{x}\log_2(x) = O\left(\frac{\log_2(x)}{\sqrt{x}}\right) = o(1), \tag{50}$$

and similarly

$$S_D < c_{12}\frac{|D(x)|}{x}\log_2(x) = O(1). \tag{51}$$

For the remaining numbers, write

$$\phi(n) = n_1 \cdot n_2, \tag{52}$$

where

$$n_1 = \prod_{\substack{p^a \| \phi(n)\\ p\le\log_2(n)}} p^a, \tag{53}$$

$$n_2 = \prod_{\substack{p^a \| \phi(n)\\ p>\log_2(n)}} p^a. \tag{54}$$

Clearly,

$$\frac{\sigma(\phi(n))}{\phi(n)} = \frac{\sigma(n_1)}{n_1}\cdot\frac{\sigma(n_2)}{n_2} < \frac{n_1}{\phi(n_1)}\cdot\frac{n_2}{\phi(n_2)}. \tag{55}$$

Notice that

$$\frac{n_1}{\phi(n_1)} \le \prod_{p<\log_2(x)}\left(1-\frac{1}{p}\right)^{-1} < e^{\gamma}\log_3(x) + c_{13} \tag{56}$$

for all $n \le x$, and

$$\frac{n_2}{\phi(n_2)} \le \exp(c_{14}h_3(n)) \tag{57}$$

for all $n > \sqrt{x}$, where c_{14} can be taken to be any constant larger than 1 provided that x is large enough. By combining (55)–(57), it follows that if $n \in B(x)$, then

$$\begin{aligned} \frac{\sigma(\phi(n))}{\phi(n)} &< (e^\gamma \log_3(x) + c_{13}) \exp\left(\frac{c_{14}}{\sqrt{\log_3(x)}}\right) \\ &< e^\gamma \log_3(x) + c_{15}\sqrt{\log_3(x)}. \end{aligned} \tag{58}$$

Hence,

$$\begin{aligned} S_B &< (e^\gamma \log_3(x) + c_{15}\sqrt{\log_3(x)}) \cdot \frac{1}{x} \sum_{1 \le n \le x} \frac{\phi(n)}{n} \\ &< \frac{6e^\gamma}{\pi^2} \log_3(x) + c_{16}\sqrt{\log_3(x)}, \end{aligned} \tag{59}$$

where the last inequality follows from (12).

Finally, when $n \in C(x)$, let $t(x) = \lfloor (\log_2(x))^3 \rfloor$. For a given $n \in C(x)$, let $p_1 < \ldots < p_{t(x)}$ be the first $t(x)$ primes with p_1 larger than $\log_2(n)$. Then obviously

$$h_3(n) \le \sum_{j=1}^{t(x)} \frac{1}{p_j} < \sum_{c_{17}\log_2(x) < p < c_{18}(\log_2(x))^3 \log_3(x)} \frac{1}{p} < c_{19}. \tag{60}$$

Hence, by (57) and (60), we have

$$\frac{n_2}{\phi(n_2)} < c_{20} \tag{61}$$

when $n \in C(x)$. Combining (55) with (56) and (61), we get

$$\frac{\sigma(\phi(n))}{\phi(n)} < c_{21} \log_3(x) \tag{62}$$

whenever $n \in C(x)$. Thus,

$$S_C < c_{21} \frac{|C(x)|}{x} \log_3(x) < c_{22}\sqrt{\log_3(x)}, \tag{63}$$

where the last inequality in (63) follows from the fact that $|C(x)| = O(x/\sqrt{\log_3(x)})$. The assertion (iii) follows now by combining inequalities (50), (51), (59) and (63).

This concludes the proof of Theorem 1.

Proof of Theorem 2. By inequality (2), we know that $0 < \alpha \le 1/2 + 1/(2^{34} - 1)$. We shall make use of the following theorem due to Chen (see [5], or Chapter 11 of [14], or Lemma 1.2 in [13]):

Chen's Theorem. *For each even natural number m and $x \ge x_0(m)$ there exists a prime number $s \in (x/2, x]$ with $s \equiv 1 \pmod{m}$ such that $(s-1)/m$ has at most two prime factors each of which exceeds $x^{1/10}$.*

We shall distinguish two cases:

CASE 1: *There exists a sequence of integers* $(m_n)_{n\geq 1}$ *such that* $\lim_n S(m_n) = \alpha$ *and* $\lim_n \mathrm{ord}_2(\phi(m_n)) = \infty$. Fix any real number $\beta \geq 1$. Let ε be any small positive number, and let m be a positive integer such that

$$|S(m) - \alpha| < \varepsilon \quad \text{and} \quad \mathrm{ord}_2(\phi(m)) > 1/\varepsilon. \tag{64}$$

After fixing such an m, let t be an odd number such that

$$t > 1/\varepsilon, \qquad |\sigma(t)/t - \beta| < \varepsilon, \qquad \gcd(t, \phi(m)) = 1. \tag{65}$$

It is easy to see that one can find such a number t. We now use Chen's theorem to find a prime number $s > m$ such that

$$s - 1 = 2tl \tag{66}$$

where l has at most two prime factors larger than $\max(\phi(m), t, 1/\varepsilon)$ each. Write

$$\phi(m) = 2^\gamma m_1, \qquad m_1 \equiv 1 \pmod 2. \tag{67}$$

Then

$$S(m) = \frac{\sigma(\phi(m))}{m} = \frac{\sigma(2^\gamma m_1)}{m} = \frac{(2^{\gamma+1} - 1)\sigma(m_1)}{m}.$$

Notice that m and s are coprime, t and l are also coprime (because the smallest prime factor of l is larger than t), and tl is coprime to $\phi(m)$; therefore

$$S(ms) = \frac{\sigma(\phi(ms))}{ms} = \frac{\sigma(2^{\gamma+1} m_1 tl)}{m(2tl+1)} = \frac{(2^{\gamma+2} - 1)\sigma(m_1)\sigma(t)\sigma(l)}{m(2tl+1)}. \tag{68}$$

Thus,

$$\frac{S(ms)}{S(m)} = \frac{2^{\gamma+2} - 1}{2(2^{\gamma+1} - 1)} \cdot \frac{\sigma(t)}{t} \cdot \frac{\sigma(l)}{l} \cdot \frac{1}{1 + (2tl)^{-1}}. \tag{69}$$

Formula (69) together with the previous assumptions on the parameters γ, t and l implies that

$$\left| \frac{S(ms)}{S(m)} - \beta \right| < c_{23}\varepsilon, \tag{70}$$

where c_{23} depends only on β. Inequalities (64) and (70) imply that

$$|S(ms) - \alpha\beta| < c_{24}\varepsilon, \tag{71}$$

where c_{24} depends only on β (because α is absolute). Since ε can be taken to be arbitrarily small after β was fixed, it follows that $\alpha\beta$ is a cluster point of $\{S(n) \mid n \geq 1\}$, which settles this case.

Case 2: *There exist two constants* K_1 *and* K_2 *such that* $|S(n)-\alpha| < K_1$ *implies* $\mathrm{ord}_2(\phi(n)) < K_2$. Notice first of all that since

$$S(p^2m) = \frac{\sigma(\phi(p^2m))}{p^2m} = \frac{\sigma(p\phi(pm))}{p^2m} > \frac{\sigma(\phi(pm))}{pm} = S(pm), \tag{72}$$

it follows that there exists a sequence $(m_n)_{n\geq 1}$ of squarefree integers such that $S(m_n)$ converges to α. Moreover, in this case we know that $\mathrm{ord}_2(\phi(m_n)) < K_2$ when n is large enough. In particular, $\omega(m_n) < K_2+1$ when n is large enough.

Define k to be the smallest positive integer such that there exists a sequence $(m_n)_{n\geq 1}$ of squarefree numbers for which $S(m_n)$ converges to α and such that $\omega(m_n) = k$ for all $n \geq 1$. For each n let

$$m_n = p_1(n)\dots p_k(n), \tag{73}$$

where $p_1(n) < \dots < p_k(n)$. Write $m_n = p_k(n)m'_n$. Notice first of all that $k > 1$. Indeed, if $k = 1$, it follows that α is a cluster point of $\{S(p) \mid p \text{ prime}\}$. However, for $p \geq 7$,

$$S(p) = \frac{\sigma(p-1)}{p} \geq \frac{1}{p}\left(1+2+\frac{p-1}{2}+(p-1)\right) = \frac{3(p+1)}{2p} \geq \frac{3}{2},$$

which would imply that $\alpha \geq 3/2$, contradicting the result of Mąkowski and Schinzel (see (2)). Thus, $k > 1$. Since $p_k(n) = P(m_n)$ tends to infinity with n, we get

$$\begin{aligned}\alpha = \lim_n S(m_n) &= \lim_n \frac{\sigma(\phi(m'_n)(p_k(n)-1))}{m'_n p_k(n)} \\ &\geq \liminf_n \frac{p_k(n)-1}{p_k(n)} S(m'_n) = \liminf_n S(m'_n).\end{aligned} \tag{74}$$

From the definition of k, it follows that $(m'_n)_{n\geq 1}$ cannot have infinitely many distinct terms and is, therefore, bounded. These arguments show that one may assume that $m_n = ap_k(n)$ where $a > 1$ is a fixed integer. Notice that $a > 2$. Indeed, if $a = 2$, then, for $p \geq 7$, one has

$$S(2p) = \frac{\sigma(p-1)}{2p} \geq \frac{1}{2p}\left(1+2+\frac{p-1}{2}+p-1\right) = \frac{3(p+1)}{4p} > \frac{3}{4},$$

therefore $\alpha \geq 3/4$, contradicting again the result of Mąkowski and Schinzel. Thus, $a > 2$, therefore $\phi(a) > 1$. We next show that

$$\alpha = \min\left(\frac{\sigma(\phi(a)d)}{ad} \;\middle|\; d \text{ an even squarefree divisor of } \phi(a)\right). \tag{75}$$

To see why α is at least what is claimed by formula (75), let p be a large prime and write

$$p-1 = d_p n_p,$$

where $\gcd(d_p, n_p) = 1$ and d_p is a number whose prime factors are exactly the prime factors of $\gcd(p-1, \phi(a))$. Notice that d_p is even. Then

$$S(ap) = \frac{\sigma(\phi(a)d_pn_p)}{a(d_pn_p+1)} = \frac{\sigma(\phi(a)d_p)}{ad_p} \cdot \frac{\sigma(n_p)}{n_p} \cdot \frac{1}{1+(p-1)^{-1}}. \tag{76}$$

Since $\sigma(n_p) \geq n_p$, the above argument shows that

$$\alpha \geq \liminf_m{}' \frac{\sigma(\phi(a)m)}{am}, \tag{77}$$

where the $\liminf'$ means that we are allowing m to run only over those even positive integers whose prime divisors are among the prime divisors of $\phi(a)$. If d is the largest squarefree divisor of m, then

$$\frac{\sigma(\phi(a)m)}{am} \geq \frac{\sigma(\phi(a)d)}{ad},$$

so α is at least as large as the number appearing on the right hand side of (75). To see that α is at most that number, choose d to be any even squarefree divisor of $\phi(a)$ and use Chen's theorem to construct large primes s such that $l = (s-1)/d$ is an integer composed of at most two primes, each of them large. Now

$$S(as) = \frac{\sigma(\phi(a)dl)}{a(dl+1)} = \frac{\sigma(\phi(a)d)}{ad} \cdot \frac{\sigma(l)}{l} \cdot \frac{1}{1+(s-1)^{-1}}, \tag{78}$$

and notice that the right hand side of (78) tends to $\sigma(\phi(a)d)/(ad)$ when s tends to infinity through such numbers. This proves formula (75).

Finally, having (75) at hand one can again use Chen's theorem to prove our Theorem 2. Indeed, assume that d is an even squarefree divisor of $\phi(a)$ realizing the minimum of the expression appearing on the right hand side of (75). Let β be an arbitrary real number ≥ 1. Fix $\varepsilon > 0$ arbitrarily small and choose a number t coprime to $\phi(a)$ (in particular, to $\phi(a)d$) such that both

$$t > 1/\varepsilon \quad \text{and} \quad |\sigma(t)/t - \beta| < \varepsilon. \tag{79}$$

Now use Chen's theorem to construct a prime number s such that $l = (s-1)/(td)$ is an integer composed of at most two primes larger than $\max(t\phi(a), 1/\varepsilon)$ each. Now t and l are coprime and tl is coprime to $\phi(a)d$, therefore

$$S(sa) = \frac{\sigma(\phi(a)tdl)}{a(tdl+1)} = \alpha \cdot \frac{\sigma(t)}{t} \cdot \frac{\sigma(l)}{l} \cdot \frac{1}{1+(s-1)^{-1}}. \tag{80}$$

It is easily seen that formulae (79) and (80) together with our assumptions on s and l imply that

$$|S(sa) - \alpha\beta| < c_{25}\varepsilon, \tag{81}$$

where c_{25} depends only on β. Since β was first fixed and then ε was chosen arbitrarily small, it follows that $\alpha\beta$ is a cluster point of $\{S(n) \mid n \geq 1\}$. Finally, since $\beta \geq 1$ was arbitrary, it follows that $\{S(n) \mid n \geq 1\}$ is dense in $[\alpha, \infty]$.

REMARK. Instead of using Chen's theorem, we could have used a weaker fact, namely that for every even m there exist infinitely many primes p such that $(p-1)/m$ is an integer whose smallest prime factor is at least $\log p$. This statement follows from simpler sieve methods such as Brun's or Selberg's. Of course, when p goes to infinity through such primes, $S(p)$ approaches $\sigma(m)/m$, which is enough for the arguments employed in the proof of our Theorem 2 to go through.

Proof of Theorem 3. We use the notations from the proofs of Lemmas 2 and 3. Let A be the set of all integers n which are not primes and for which $M(n) \mid \phi(n)$ and inequalities (18) and (19) hold. By Lemmas 2 and 3, A has asymptotic density 1. We now show that inequality (9) holds for all values of $n \in A$ which are large enough. We start by finding suitable upper and lower bounds on $\phi(n - \phi(n))$. Notice that

$$\phi(n-\phi(n)) = \sum_{\substack{m<n-\phi(n)\\(m,n-\phi(n))=1}} 1 \leq \sum_{\substack{m<n-\phi(n)\\(m,n)=1}} 1 + \sum_{\substack{m<n-\phi(n)\\(m,n-\phi(n))=1,\,(m,n)>1}} 1 = \sum\nolimits_1 + \sum\nolimits_2. \tag{82}$$

Clearly,

$$\sum\nolimits_1 = (n-\phi(n)) - \sum_{p|n}\left\lfloor\frac{n-\phi(n)}{p}\right\rfloor + \sum_{pq|n,\,p<q}\left\lfloor\frac{n-\phi(n)}{pq}\right\rfloor - \dots < (n-\phi(n))\frac{\phi(n)}{n} + \tau(n). \tag{83}$$

Here, $\tau(n)$ is the number-of-divisors function of n. For the second sum, we use the fact that $M(n) \mid \phi(n)$. Hence, if m is a number such that $(m, n) > 1$, but $(m, n-\phi(n)) = 1$, it follows that m has to be a multiple of a prime $p \mid n$ with $p > g(n)$. Of course, for such n, one has

$$\sum\nolimits_2 < \sum_{\substack{p|n\\p>g(n)}}\left\lfloor\frac{n-\phi(n)}{p}\right\rfloor < (n-\phi(n))\sum_{\substack{p|n\\p>g(n)}}\frac{1}{p} = (n-\phi(n))h_1(n) < \frac{(n-\phi(n))\log_3(n)}{\log_2(n)}. \tag{84}$$

Putting together inequalities (82)–(84), we get

$$\frac{\phi(n-\phi(n))}{n-\phi(n)} - \frac{\phi(n)}{n} < \frac{\tau(n)}{n-\phi(n)} + \frac{\log_3(n)}{\log_2(n)} < \frac{2\log_3(n)}{\log_2(n)}. \tag{85}$$

For the rightmost inequality in (85) we used the fact that $n - \phi(n) \geq n^{1/2}$, which holds because n is not prime, and the fact that $\tau(n) < n^{1/4}$, which holds for all n large enough, thus

$$\frac{\tau(n)}{n-\phi(n)} < \frac{1}{n^{1/4}} < \frac{\log_3(n)}{\log_2(n)}$$

for $n \in A$ large enough. Inequality (85) proves half of (9). For the other half, we write

$$\begin{aligned} \phi(n-\phi(n)) &= \sum_{\substack{m<n-\phi(n)\\ (m,n-\phi(n))=1}} 1 \\ &\geq \sum_{\substack{m<n-\phi(n)\\ (m,n)=1}} 1 - \sum_{\substack{m<n-\phi(n)\\ (m,n-\phi(n))>1,\,(m,n)=1.}} 1 \\ &= \sum\nolimits_1 - \sum\nolimits_3. \end{aligned} \tag{86}$$

From elementary arguments similar to the ones employed above, we get

$$\sum\nolimits_1 > (n-\phi(n)) \cdot \frac{\phi(n)}{n} - \tau(n) \tag{87}$$

and

$$\begin{aligned} \sum\nolimits_3 &< (n-\phi(n)) \sum_{\substack{p>g(n)\\ p\mid(n-\phi(n)),\,p\nmid n}} \frac{1}{p} \\ &= (n-\phi(n))h_2(n) < \frac{(n-\phi(n))\log_3(n)}{\log_2(n)}. \end{aligned} \tag{88}$$

Putting together inequalities (86)–(88), we get

$$\frac{\phi(n-\phi(n))}{n-\phi(n)} - \frac{\phi(n)}{n} > -\left(\frac{\tau(n)}{n-\phi(n)} + \frac{\log_3(n)}{\log_2(n)}\right) > -\frac{2\log_3(n)}{\log_2(n)}$$

for $n \in A$ large enough. With (85) we thus have (9).

To see (8), say $\varepsilon(n)$ tends to 0 arbitrarily slowly as $n \to \infty$, and let

$$\delta(n) = (\varepsilon(n) + 2\log_3(n)/\log_2(n))^{1/2}.$$

Thus, $\delta(n) \to 0$ as $n \to \infty$. By Lemma 1, we may assume that $\phi(n) > \delta(n)n$, and by (9), which we have just proved, we may assume that

$$\phi(n-\phi(n)) < \frac{\phi(n)}{n}(n-\phi(n)) + \frac{2\log_3(n)}{\log_2(n)}(n-\phi(n)). \tag{89}$$

But for $\phi(n) > \delta(n)n$, the right side of (89) is less than $\phi(n) - \varepsilon(n)n$, which gives (8).

This completes the proof of Theorem 3.

4. Comments and problems. Let $u(n) = n - \phi(n)$. In many respects, the function $u(n)$ resembles the sum of aliquot divisors of n, namely, the function $s(n) = \sigma(n) - n$. It seems interesting to iterate the function u at a starting value of n until 0 is reached. Let u_k be the kth iterate. Let $k(n)$ be the first positive integer k for which $u_k(n) = 0$. It is easy to see that

$$k(n) < c_{26} \log(n) \log_2(n) \tag{90}$$

for all n large enough, where c_{26} can be taken to be any constant strictly larger than e^γ, but we suspect that $k(n) = o(\log(n) \log_2(n))$. It would be interesting to investigate both the average and the normal behavior of the function $k(n)$.

Note also that $u_2(n)$ is defined for all $n > 1$ and that $u_2(n) = 0$ if and only if n is prime, so that $u_2(n)$ is defined and $u_2(n) > 0$ for a set of asymptotic density 1. Note, too, that (ii) of Theorem 3 says that $n/u(n)$ is asymptotically equal to $u(n)/u_2(n)$ on a set of asymptotic density 1. Our Theorem 3 is analogous to some results from [7] and [9], where it was shown that $s(n)/n$ is asymptotically equal to $s_2(n)/s(n)$ on a set of n of asymptotic density 1. In analogy with Conjecture 3 in [9] which deals with higher iterates of the function $s(n)$, we conjecture that for each fixed number k, $u_k(n)$ is defined, $u_k(n) > 0$, and $n/u(n)$ is asymptotically equal to $u_{k-1}(n)/u_k(n)$ on a set of asymptotic density 1. That at least $u_k(n)$ is defined, $u_k(n) > 0$, and

$$\frac{u_{k-1}(n)}{u_k(n)} < (1+\varepsilon)\frac{n}{u(n)} \tag{91}$$

on a set of asymptotic density 1, for any fixed $\varepsilon > 0$, follows by the same methods as in [7].

While as we said above, we know by (ii) of Theorem 3 that $n/u(n)$ and $u_1(n)/u_2(n)$ are asymptotically equal on a set of asymptotic density 1, what is still in doubt is which one is larger, and the same question can also be formulated for the pair of asymptotically equal functions $n/s(n)$ and $s_1(n)/s_2(n)$. Computations revealed that 550177 numbers n smaller than 10^6 satisfy

$$nu_2(n) < u_1(n)^2 \tag{92}$$

and 608799 numbers n smaller than 10^6 satisfy

$$ns_2(n) < s_1(n)^2. \tag{93}$$

Of these, 371154 numbers satisfy both (92) and (93). Notice that when

n is prime, both (92) and (93) hold. Of course, the set of all primes is of asymptotic density 0 but this is less noticeable computationally in small ranges. In particular, (92) holds for 471679 of the 921500 composite numbers n smaller than 10^6, or about 51%, while (93) holds for 530301 composites, or about 58%. Both inequalities hold for 292656 composites, or about 32%. Based on our computations, it may be reasonable to conjecture that both inequalities (92) and (93) hold on a set of asymptotic density 1/2 and that they are independent, that is, that they both hold on a set of asymptotic density 1/4.

Recall that an integer n is called a *cototient* if it is in the range of the function u, that is, $n = m - \phi(m)$ for some integer m. It is not known if the set of cototients has an asymptotic density nor if the upper density of the set is < 1. In fact, until a few years ago it was not even known that there are infinitely many non-cototients, until an infinite family of such was pointed out by Browkin and Schinzel in [3] (see also [12] for more examples of such infinite families of non-cototients). In analogy with the notion of a cototient, let us call a positive integer n a *strong cototient* if the equation $u_k(x) = n$ has a positive solution x for every $k \geq 1$. Does the set of strong cototients have a density and if so, what is it? Clearly, since $u_k(p^{m+k}) = p^m$ holds for all $m \geq 0$, $k \geq 1$ and $p \geq 2$ prime, it follows that all prime powers are strong cototients. Moreover, by looking at the values of $u(pq)$ with p and q odd primes, Goldbach's conjecture would imply that all odd integers are cototients, therefore strong cototients.

REFERENCES

[1] L. Alaoglu and P. Erdős, *A conjecture in elementary number theory*, Bull. Amer. Math. Soc. 50 (1944), 881–882.

[2] U. Balakrishnan, *Some remarks on $\sigma(\phi(n))$*, Fibonacci Quart. 32 (1994), 293–296.

[3] J. Browkin and A. Schinzel, *On integers not of the form $n - \phi(n)$*, Colloq. Math. 68 (1995), 55–58.

[4] N. de Bruijn, *On the number of positive integers $\leq x$ and free of prime factors $> y$*, Nederl. Akad. Wetensch. Proc. Ser. A 54 (1951), 50–60.

[5] J. R. Chen, *On the representation of a large even integer as the sum of a prime and a product of at most two primes*, Sci. Sinica 16 (1973), 157–176.

[6] G. L. Cohen, *On a conjecture of Mąkowski and Schinzel*, Colloq. Math. 74 (1997), 1–8.

[7] P. Erdős, *On asymptotic properties of aliquot sequences*, Math. Comp. 30 (1976), 641–645.

[8] —, *Proposed problem P. 294*, Canad. Math. Bull. 23 (1980), 505.

[9] P. Erdős, A. Granville, C. Pomerance and C. Spiro, *On the normal behavior of the iterates of some arithmetic functions*, in: Analytic Number Theory, Proc. Conf. in Honor of P. T. Bateman, Birkhäuser, Boston, 1990, 165–204.

[10] P. Erdős and C. Pomerance, *On the normal number of prime factors of $\phi(n)$*, Rocky Mountain J. Math. 15 (1985), 343–352.

[11] M. Filaseta, S. W. Graham and C. Nicol, *On the composition of $\sigma(n)$ and $\phi(n)$*, Abstracts Amer. Math. Soc. 13 (1992), no. 4, 137.

[12] A. Flammenkamp and F. Luca, *Infinite families of non-cototients*, Colloq. Math. 86 (2000), 37–41.

[13] K. Ford, *The number of solutions of $\phi(x) = m$*, Ann. of Math. 150 (1999), 283–311.

[14] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[15] A. Grytczuk, F. Luca and M. Wójtowicz, *On a conjecture of Mąkowski and Schinzel concerning the composition of the arithmetical functions σ and ϕ*, Colloq. Math. 86 (2000), 31–36.

[16] —, —, —, *A conjecture of Erdős concerning inequalities for the Euler totient function*, Publ. Math. (Debrecen) 59 (2001), 9–16.

[17] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Oxford Univ. Press, London, 1968.

[18] A. Mąkowski and A. Schinzel, *On the functions $\phi(n)$ and $\sigma(n)$*, Colloq. Math. 13 (1964–1965), 95–99.

[19] H. L. Montgomery and R. C. Vaughan, *The large sieve*, Mathematika 20 (1973), 119–134.

[20] C. Pomerance, *On the composition of the arithmetic functions σ and ϕ*, Colloq. Math. 58 (1989), 11–15.

[21] I. J. Schoenberg, *Über die asymptotische Verteilung reeller Zahlen mod* 1, Math. Z. 28 (1928), 171–199.

[22] R. Warlimont, *On iterates of Euler's function*, Arch. Math. (Basel) 76 (2001), 345–349.

Instituto de Matemáticas de la UNAM
Campus Morelia
Ap. Postal 61-3 (Xangari)
Morelia, Michoacán, Mexico
E-mail: fluca@matmor.unam.mx

Lucent Technologies Bell Laboratories
600 Mountain Avenue
Murray Hill, NJ 07974, U.S.A.
E-mail: carlp@research.bell-labs.com

Received 19 April 2001;
revised 1 September 2001 (4054)