THE ALGEBRA OF THE SUBSPACE SEMIGROUP OF $M_2(\mathbb{F}_q)$

BY

JAN OKNIŃSKI (Warszawa)

Abstract. The semigroup $S = S(M_2(\mathbb{F}_q))$ of subspaces of the algebra $M_2(\mathbb{F}_q)$ of $2 \times 2$ matrices over a finite field $\mathbb{F}_q$ is studied. The ideal structure of $S$, the regular $J$-classes of $S$ and the structure of the complex semigroup algebra $\mathbb{C}[S]$ are described.

1. Introduction. Let $M_n(K)$ be the algebra of $n \times n$ matrices over a field $K$. By $S(M_n(K))$ we denote the subspace semigroup of $M_n(K)$, defined as the set of all $K$-subspaces equipped with the operation $V \ast W = \text{lin}_K(VW)$. This semigroup arose in the context of discrete dynamical systems, [3], and was first studied in [6]. It was shown that there exists a finite ideal chain $I_1 \subset \ldots \subset I_t = S(M_n(K))$ such that $I_1$ and every Rees factor $I_k/I_{k-1}$ are either nil or 0-disjoint unions of completely 0-simple ideals.

In this paper we consider the case where $K$ is a finite field. A natural problem is to determine the complex irreducible representations of $S(M_n(K))$ and to study the structure and symmetries of the algebra $\mathbb{C}[S(M_n(K))]$. It is well known that a description of the regular $J$-classes of the semigroup is needed in this context. Our aim is to deal with these problems in the case where $n = 2$. A characterization of non-regular elements of $S(M_2(K))$ is obtained and regular $J$-classes are fully described. Moreover, the ideal structures of $S(M_2(K))$ and of its complex semigroup algebra are determined.


2. Regular $J$-classes. Let $S = S(M_2(K))$ for a finite field $K$. Since the idempotents of $S$ play a crucial role, first we list unitary subalgebras of $M_2(K)$:

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1. \( A_1 = M_2(K) \).

2. \( A_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in K \right\} \).

For simplicity, we write \( A_2 = (a \ b \\ 0 \ c) \), if unambiguous.

3. \( A_3 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \).

4. \( A_4 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \).

5. \( A_5 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \).

6. \( A_6 = \) a field extension \( F \) of dimension 2 over \( K \).

(By the Noether–Skolem theorem any two such subfields are conjugate, as they are isomorphic).

7. \( A_7 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \).

Let \( J(A) \) be the radical of an algebra \( A \subseteq M_2(K) \). Recall that by Wedderburn’s structure theorem for algebras over a perfect field [1] we know that \( A = B + J(A) \) where \( B \) is a subalgebra such that \( B \cong A/J(A) \). Also, every nil subalgebra of \( M_2(K) \) is conjugate to \( \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \). So it is easy to see that, up to conjugation, the above list exhausts all unitary subalgebras of \( M_2(K) \).

From [6] we know that every non-zero regular \( J \)-class of \( S = S(M_2(K)) \) contains a unitary algebra, whence it contains one of the algebras \( A_i \). Recall that this \( J \)-class consists of subspaces \( V \subseteq M_2(K) \) such that \( V \) and \( A_i \) generate the same ideal of \( S(M_2(K)) \). Clearly, \( A_5 \) is the identity of \( S(M_2(K)) \).

Any two of the elements \( A_1, A_2, A_3, A_4, A_5, A_6 \) are in different \( J \)-classes of \( S \). This can be checked directly but it also follows from the fact that \( A JB \) implies that \( A, B \) are Morita equivalent [6]. Clearly \( A_1 \) and \( A_7 \) are in the same \( J \)-class of \( S \).

For any \( n \geq 2 \), let \( A \) be a subalgebra of \( M_n(K) \) which is basic. That is, \( A \) has a unity and \( A/J(A) \) has no non-zero nilpotents. Let \( U = U(A) \) be the unit group of \( A \) and \( N = N(A) \) be the normalizer of \( A \) in \( \text{Gl}_n(K) \). So \( N = \{ g \in \text{Gl}_n(K) \mid gA = Ag \} \). Notice that \( \text{lin}_K U(A) = A \) if \( K \neq \mathbb{F}_2 \), the field of two elements (it is enough to assume that \( A/J(A) \) has at most one copy of \( \mathbb{F}_2 \) as a direct summand). Therefore, in this case \( N = \{ g \in \text{Gl}_n(K) \mid gU = Ug \} \). By \( H_A \) we denote the maximal subgroup of \( S \) containing \( A \), treated as an idempotent of \( S \). In other words, \( H_A \) consists of all subspaces \( V \) of \( M_n(K) \) such that \( V = AV = VA \) and \( VW = WV = A \) for some subspace \( W \). Let \( e \) be the identity of \( A \). Then \( eN = Ne \) is a subgroup of
$U(eM_n(K)e) \cong M_{\text{rank}(e)}(K)$, which we denote by $N_e$. It is easy to see that

$$H_A = \{Ax \mid x \in N\} \quad \text{and} \quad H_A \cong N_e/U.$$

In particular, if $A$ contains the identity matrix, then $U \subseteq N = N_e$ and $H_A = [N : U]$. Moreover

$$[\text{Gl}_n(K) : N] = \text{the number of } \mathcal{H}\text{-classes of } S \text{ of the form } gH_A, \ g \in \text{Gl}_n(K)$$

$$= \text{the number of } \mathcal{H}\text{-classes of } S \text{ of the form } H_Ag, \ g \in \text{Gl}_n(K).$$

We shall consider the case where $K = \mathbb{F}_q$, a finite field of $q$ elements. We count the subspaces of $M_2(\mathbb{F}_q)$ of any given dimension:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of subspaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$(q^4 - 1)/(q - 1) = q^3 + q^2 + q + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$(1 + q + q^2)(1 + q^2)$</td>
</tr>
<tr>
<td>3</td>
<td>$q^3 + q^2 + q + 1$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

It follows that $|S| = q^4 + 3q^3 + 4q^2 + 3q + 5$.

Write $G = \text{Gl}_2(K)$. We have seen above that $\{gAh \mid g, h \in G\}$ yields $[N : U][G : N]^2$ elements in the $\mathcal{J}\text{-class of } A$ in the subspace semigroup $S = S(M_2(K))$. We discuss the seven cases listed above.

1) $A = M_2(K)$. Then $A \mathcal{J} B$ for $B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Every non-zero subspace $V \subseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is a left $B$-module and satisfies $VW = B$ for some right $B$-module $W$. So the $\mathcal{R}\text{-class of } B$ consists of all such subspaces $V$, whence it has $q + 2$ elements. As the same holds for the $\mathcal{L}\text{-class of } B$, it follows that the $\mathcal{J}\text{-class of } B$ has $\geq (q + 2)^2$ elements.

2) $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. It is easy to see that $N = U$ and $[G : N] = q + 1$. So the $\mathcal{J}\text{-class of } A$ has $\geq (q + 1)^2$ elements.

3) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then $N$ consists of invertible matrices of the form $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ or $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$. Hence $[N : U] = 2$ and $[G : N] = (q + 1)q/2$. Therefore the $\mathcal{J}\text{-class of } A$ has $\geq q^2(q + 1)^2/2$ elements.

4) $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Then $N$ consists of invertible matrices of the form $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$. So $|U| = (q - 1)q$ and $|N| = (q - 1)^2q$. Hence $[N : U] = q - 1$ and $[G : N] = q + 1$ and therefore the $\mathcal{J}\text{-class of } A$ has $\geq (q + 1)^2(q - 1)$ elements.

5) $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Then $N = G$ and $U \cong K^*$. So $[N : U] = q(q^2 - 1)$ and $[G : N] = 1$. It follows that the $\mathcal{J}\text{-class of } A$ has $\geq q(q^2 - 1)$ elements.

6) $A$ is a subfield of dimension 2 over $K$. Now $|A| = q^2$, so that $|U| = q^2 - 1$. Let $C$ be the centralizer of $A$ in $M_2(K)$. Then $C$ is a simple algebra, so it is a maximal subfield of $M_2(K)$ containing $A$, [1]. Hence $C = A$. The
Galois group $G(A/K)$ is $\{\text{Id}, \phi\}$, where $\phi(x) = x^q$. So $n \in N$ if and only if $nan^{-1} = a$ or $nan^{-1} = a^q$ for $a \in A$ (as there are no other automorphisms). Hence $n \in C = A$ or $nan^{-1} = a^q$. By the Noether–Skolem theorem there exists an element $n \in N$ of the latter type. Then any other $y \in N$ satisfies either $y^{-1}n \in C$ or $y \in C$. So $n \in N$ if and only if $nan - 1 = a$ or $nan - 1 = a^q$ for $a \in A$ (as there are no other automorphisms).

Hence $n \in C = A$ or $nan - 1 = a$. By the Noether–Skolem theorem there exists an element $n \in N$ of the latter type. Then any other $y \in N$ satisfies either $y - 1 n \in C$ or $y \in C$. So $N \subseteq C \cup Cn$ and consequently $N = U \cup Un$.

We now add the numbers of subspaces produced in cases 1)–6) (note that they are in different $J$-classes of $S$):

1) $(q + 1)^2$ spaces of dimension 1,
   2) $2q + 2$ spaces of dimension 2,
   1 space of dimension 4,
2) $(q + 1)^2$ spaces of dimension 3,
3) $q^2(q + 1)^2/2$ spaces of dimension 2,
4) $(q + 1)^2(q - 1)$ spaces of dimension 2,
5) $q(q^2 - 1)$ spaces of dimension 1,
6) $q^2(q - 1)^2/2$ spaces of dimension 2.

So we have constructed

$$q^4 + q^3 + 2q^2 + q + 1 = (1 + q + q^2)(1 + q^2)$$

subspaces of dimension 2, whence these are all such subspaces. Also, we have got $q^3 + q^2 + q + 1$, hence all, subspaces of dimension 1. Moreover, there are

$$|S| - 1 - |\{\text{elements listed in 1)–6}\}| = q^3 - q$$

remaining non-zero elements of $S$ (all of them of dimension 3). We will show that they are all not regular. So, it will follow that the elements listed in 1)–6) cover all non-zero regular $J$-classes of $S$, and hence they exhaust all non-zero regular elements of $S$. It also follows that the regular $J$-classes of $S$ consist of unit regular elements of $S$.

**Proposition 2.1.** Assume that $K$ is any field and let $n \geq 2$. Let $V \in S = S(M_n(K))$ be a subspace of dimension $n^2 - 1$. Let $V$ be described by a linear equation $\sum_{i,j=1}^n a_{ij} x_{ij} = 0$, $a_{ij} \in K$. If $VwV \subseteq V$ for some non-zero $w \in M_n(K)$, then the rank of the matrix $A = (a_{ij})$ is 1. Moreover, the latter is equivalent to the fact that $V$ is a regular element of $S$.

**Proof.** Assume that $h \in M_n(K)$ is an elementary matrix. So it is a transposition or $h = 1 + \lambda e_{pq}$ for some $p \neq q$ and $\lambda \in K^*$, where $e_{pq}$ denotes a matrix unit. Let $B = (b_{ij})$ be the matrix determined by an equation describing the subspace $hV$. If $h$ is a transposition with non-diagonal entries $h_{pq}, h_{qp}$, then clearly we may take $b_{qj} = a_{pj}, b_{pj} = a_{qj}$ and $b_{ij} = a_{ij}$ if $i \neq p, q$, for $j = 1, \ldots, n$. If $h = 1 + \lambda e_{pq}$, then it is easy to see that we
may take $b_{qj} = a_{qj} - \lambda a_{pj}$ and $b_{ij} = a_{ij}$ for $i \neq q$, and for all $j$. It follows that $\text{rank}(A) = \text{rank}(B)$. Hence every $hV$ is described by an equation with the corresponding matrix having the same rank as $A$. The same holds if $h = 1 + \lambda e_{pp}$ with any $p \in \{1, \ldots, n\}$ and $\lambda \neq -1$, and therefore for every $h \in \text{GL}_n(K)$. The same applies to $Vh$.

Suppose that $VwV \subseteq V$ for some $w \in M_n(K)$. If $g, h \in \text{GL}_n(K)$, then

$$g^{-1}Vhh^{-1}wgg^{-1}Vh \subseteq g^{-1}Vh.$$  

Clearly, $V$ is a regular element of $S$ if and only if so is $g^{-1}Vh$. It follows that, when proving both statements, we may replace $(V, 1)$ matrices, respectively. It is easy to see that $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfies $VWV = V$. Therefore $V$ is a regular element of $S$.

It is clear that if $V$ is a regular element of $S$ then there exists a non-zero $w \in M_n(K)$ such that $VwV \subseteq V$.

Finally, suppose that $VwV \subseteq V$ for a non-zero matrix $w$. Because of the diagonal idempotent form of $A$ we have

$$V = \{x = (x_{ij}) \in M_n(K) \mid x_{11} + \ldots + x_{rr} = 0\}.$$  

If $r = 1$ then $\text{rank}(A) = 1$ and we are done. So suppose that $r \geq 2$. Let $w = (w_{ij})$ and suppose that $w_{kl} \neq 0$ for some $k, t$. If $k, t \neq 1$ then let $v = (v_{ij}), v' = (v'_{ij})$ be such that $v_{1k} = 1 = v'_{tt}$ and all the remaining entries are 0. Then $v, v' \in V$, so that $vvv' \in V$. But $vvv'$ has only one non-zero entry and it is in position $(1,1)$. This contradicts the above description of $V$. It follows that $w_{ij} = 0$ if $i, j \neq 1$. The same argument applied to position $(2,2)$ implies that also $w_{ij} = 0$ if $i, j \neq 2$. So $w_{12}, w_{21}$ can be the only non-zero entries of $w$. Choose a matrix $u = (u_{ij})$ whose only non-zero entry is $u_{21}$ and let $u' = (u'_{ij})$ be such that $u_{11} = -1$ and $u_{22} = 1$ and all other entries are zero. Then $u, u' \in V$ and $uuu' \in V$. The second row of $uuu'$ is equal to $(0, w_{12}, 0, \ldots, 0)$ and all other rows are zero. So the description of $V$ yields $w_{12} = 0$. A similar argument applied to the product $uuu'$ (where $u'$ is the transpose of $u$) yields $w_{21} = 0$. Therefore $w = 0$. This contradiction shows that $r = 1$, completing the proof of the proposition.

We come back to the case $K = \mathbb{F}_q$ and $n = 2$. Notice that there are

$$|\text{GL}_2(K)|/(q - 1) = (q^2 - q)(q^2 - 1)/(q - 1) = q^3 - q$$  

subspaces of dimension 3 defined by an equation $\alpha x_{11} + \beta x_{12} + \gamma x_{21} + \delta x_{22} = 0$ such that $\det \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \neq 0$.  


Let \( V = \{(a b c) \mid a, b, c \in K\} \). So, \( V \) is defined by the equation \( x_{12} - x_{21} = 0 \), and is of the desired type. We determine the stabilizer \( C \) of \( V \) under the action of \( \text{Gl}_2(K) \) on \( S \) by left multiplication. So, let \( g = (g_{ij}) \in \text{Gl}_2(K) \) satisfy

\[
\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} g_1a + g_2b & g_1b + g_2c \\ g_3a + g_4b & g_3b + g_4c \end{pmatrix} \in V
\]

for all \( a, b, c \in K \). Then \( g_3a + g_4b = g_1b + g_2c \), whence \( g_3 = 0 = g_2 \) and \( g_4 = g_1 \). So \( C \) consists of scalar matrices and

\[ |\{gV \mid g \in \text{Gl}_2(K)\}| = |\text{Gl}_2(K)| \cdot |K^*|^{-1} = q^3 - q. \]

Clearly, every element of the form \( gV, g \in \text{Gl}_2(K) \), satisfies \( gV \Lambda V \) in \( S \), whence it is not regular by Proposition 2.1. It then follows that we have constructed \( q^3 - q \) non-regular elements of the form \( gV \). Therefore, comparing the cardinality of \( S \) and the number of regular elements constructed before, we see that the elements listed in cases 1)–6) exhaust all non-zero regular \( J \)-classes of \( S \) and the elements \( gV, g \in \text{Gl}_2(K) \), exhaust all non-regular elements of \( S \).

**Corollary 2.2.** Let \( V = \{x = (x_{ij}) \in M_2(K) \mid x_{12} = x_{21}\} \). Then the \( J \)-class of \( V \) in \( S \) is equal to \( \{gV \mid g \in \text{Gl}_2(K)\} = \{Vg \mid g \in \text{Gl}_2(K)\} \) and it coincides with the \( \mathcal{H} \)-class of \( V \). Moreover \( S \) has exactly eight \( J \)-classes, namely the classes of \( A_1, \ldots, A_6, V, \{0\} \).

**Proof.** We have seen that \( \{gV \mid g \in \text{Gl}_2(K)\} \) exhaust all non-regular elements in \( S \). A symmetric argument shows that \( \{Vg \mid g \in \text{Gl}_2(K)\} \) also is the set of all non-regular elements of \( S \) and hence \( \{gV \mid g \in \text{Gl}_2(K)\} = \{Vg \mid g \in \text{Gl}_2(K)\} \). Therefore non-regular elements of \( S \) form a single \( \mathcal{H} \)-class of \( S \) and the assertion follows.

**3. Structure of the algebra.** In this section we describe the radical of \( \mathbb{C}[S] \) and we show that, for every regular principal factor \( T \) of \( S \), the contracted semigroup algebra \( \mathbb{C}_0[T] \) is semisimple. Hence \( \mathbb{C}[S]/J(\mathbb{C}[S]) \) is a direct product of all \( \mathbb{C}_0[T] \) (see [4]). As \( \mathbb{C}[S] = B + J(\mathbb{C}[S]) \), a direct sum of subspaces, for a subalgebra \( B \cong \mathbb{C}[S]/J(\mathbb{C}[S]) \), this yields a description of the structure of the algebra.

**Lemma 3.1.** Let \( A \subseteq M_n(K) \) be a subalgebra with \( 1 \in A \). Then \( A = \{gAh \mid g, h \in G\} \) with zero adjoined is a completely 0-simple inverse subsemigroup of the principal factor \( J_A \) of \( A \) in \( S(M_n(K)) \). Moreover, \( A \) is a union of \( \mathcal{H} \)-classes of \( J_A \).

**Proof.** We know that \( H_A = \{Ax \mid x \in N\} \), where \( N \) is the normalizer of \( A \) in \( \text{Gl}_n(K) \). It follows that \( A \) is a union of \( \mathcal{H} \)-classes of \( J_A \). Moreover every non-empty intersection \( R \) of \( A \) with an \( \mathcal{R} \)-class of \( S \) contains an idempotent. Namely, if \( uAv \in R \) for some \( u, v \in G \), then \( uAu^{-1} \in R \).
Suppose that $B \in A$ is an idempotent from the $\mathcal{R}$-class of $A$ in $S$. Since $B \cap G \neq \emptyset$ and $B$ is a subalgebra of $M_n(K)$, we must have $1 \in B$. But $AB = B$ and $BA = A$. It follows that $A = B$. Now, if $gAh$ is an idempotent, where $g, h \in G$, then $Ahg$ is also an idempotent and $A \Delta Ahg$. So $Ahg = A$ by the preceding part of the proof. Then $gAh = gAg^{-1}$. Now, suppose that two idempotents $gAg^{-1}, fAf^{-1}$ ($g, f \in G$) are in the same $\mathcal{R}$-class of $S$. Then $A \Delta fAf^{-1}g$ and again we get $A = g^{-1}fAf^{-1}g$. Hence $gAg^{-1} = fAf^{-1}$. Similarly one proves that every non-empty intersection of $A$ with an $\mathcal{L}$-class of $S$ contains exactly one idempotent. The assertion follows.

We have seen that the regular $\mathcal{J}$-classes of $S$ described in cases 2)–6) are of the form $A$, where $A$ is a subalgebra containing 1. So, the lemma above applies to these $\mathcal{J}$-classes.

**Proposition 3.2.** Let $J$ be a completely $0$-simple principal factor of the semigroup $S = S(M_2(\mathbb{F}_q))$. Then $\mathbb{C}_0[J]$ is a semisimple algebra.

**Proof.** Let $J$ be one of the regular $\mathcal{J}$-classes of $S$ described in 2)–6), with zero adjoined. Then by Lemma 3.1, $\mathbb{C}_0[J] \cong M_k(\mathbb{C}[H])$ for the maximal subgroup $H$ of $J$ and some $k$ (see [4], Corollary 5.26). It remains to consider the $\mathcal{J}$-class $J$ containing $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. The maximal subgroup of $J$ is trivial. So, to consider the Rees presentation of $J$ (see [2]) in the coordinate system corresponding to the maximal subgroup $\{A\}$ of $J$, we list the elements of the $\mathcal{R}$-class of $A$ (in the leading column) and of the $\mathcal{L}$-class of $A$ (in the leading row). This yields the following form of the sandwich matrix $P$ of $J$:

$$
\begin{pmatrix}
0 & 0 \\
\begin{pmatrix} b \\ 0 \end{pmatrix} & \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} b \\ 0 \end{pmatrix} & \begin{pmatrix} -a^{-1}b \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} b \\ 0 \end{pmatrix} \\
\begin{pmatrix} d \\ 0 \end{pmatrix} & \begin{pmatrix} b \\ 0 \end{pmatrix}
\end{pmatrix}
$$

Here the second row (column, respectively) represents $q - 1$ different rows (columns) of $P$ corresponding to different $q - 1$ elements $\alpha$ of $\mathbb{F}_q^*$. Performing elementary operations on rows and columns of $P$, one brings $P$ to the identity matrix. So, $P$ is invertible as a matrix over $\mathbb{C}$ and consequently $\mathbb{C}_0[J] \cong M_{q+2}(\mathbb{C})$, again by Corollary 5.26 of [4]. The assertion follows.
It is easy to verify that the inverse of the above sandwich matrix is

$$P^{-1} = \begin{pmatrix} -1 & 0 & \ldots & 0 & 1 \\ 0 & -1 & \ldots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 1 \\ 1 & 1 & \ldots & 1 & -q \end{pmatrix}.$$  

Finally, we describe the radical of the algebra $\mathbb{C}_0[S]$. Let $J$ be the $\mathcal{J}$-class containing $M_n(K)$ and $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, together with the zero subspace. Notice that $J$ is an ideal of $S$. Since $\mathbb{C}_0[J]$ is semisimple, it has an identity $E$, which can be effectively determined. Namely, in the Munn algebra notation for $\mathbb{C}_0[J]$ (see [4]), $E$ can be identified with $P^{-1}$. Therefore $E$ can be expressed as a linear combination of elements of $J$ with coefficients 1, $-1$ and $q$ as follows:

$$E = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b & d \end{pmatrix} + \sum_{\alpha \in K} \left( \begin{pmatrix} a & \alpha a \\ c & \alpha c \end{pmatrix} + \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix} \right) - \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} - \sum_{\alpha \in K^*} \begin{pmatrix} a & -\alpha^{-1}a \\ \alpha a & -a \end{pmatrix} - qM_2(K).$$

**Proposition 3.3.** Let $V = \{ x \in M_2(K) \mid x_{12} = x_{21} \}$. Then

$$J(\mathbb{C}_0[S]) = \text{lin}_C \{ gV - EgV \mid g \in \text{Gl}_2(K) \}$$

and $J(\mathbb{C}_0[S])^2 = 0$.

**Proof.** Denote the right hand side by $I$. Let $J$ be the $\mathcal{J}$-class of $S$ containing $M_n(K)$ with zero adjoined. Then $\mathbb{C}_0[J]$ is an ideal of $\mathbb{C}_0[S]$ since $J$ is an ideal of $S$. We have $\mathbb{C}_0[J]I = I\mathbb{C}_0[J] = 0$ because $E$ is a central idempotent in $\mathbb{C}_0[S]$. Moreover $I^2 = 0$. Indeed, if $g \in \text{Gl}_2(K)$ then $gV = Vh$ for some $h \in \text{Gl}_2(K)$ by Corollary 2.2. Therefore

$$(V - EV)(gV - EgV) = (V - EV)(Vh - VhE).$$

Since $V^2 = M_2(K) \in J$, we get $(V - EV)(Vh - VhE) = 0$, so $I^2 = 0$, as desired.

We know that the set $H_V = \{ gV \mid g \in \text{Gl}_2(K) \}$ has cardinality $q^3 - q$, so the dimension of $I$ is at most $q^3 - q$. Since the image of $I$ modulo $\mathbb{C}_0[J]$ is spanned by $H_V$, it has dimension $q^3 - q$. Hence, this is the dimension of $I$ as well.

We claim that $I$ is an ideal of $\mathbb{C}_0[S]$. By symmetry of $H_V$ and since $E$ is central, it is enough to show that $I$ is a left ideal. Let $X \in S$, $X \neq 0$. If $X$ is not regular in $S$, then $X = gV$ for some $g \in \text{Gl}_2(K)$ and $XV = M_2(K) \in J$. If $X$ is in one of the regular $\mathcal{J}$-classes listed in cases 2)–6), then $X$ contains an invertible matrix $u$. Thus, $XV$ is either of the form $uV$ or it is equal to $M_2(K)$. So $X(V - EV) \in I$ in the former case and $X(V - EV) = 0$ in the
latter. Finally, if $X \in J$, then we also get $X(V - EV) = 0$ because $E$ is the identity of $C_0[J]$. So $I$ is a left ideal, as claimed.

It follows that $I \subseteq J(C_0[S])$. By Proposition 3.2, the dimension of $C_0[S]$ modulo its radical is $q^3 - q$. Comparing dimensions we get $J(C_0[S]) = I$. ■

Notice that we have in fact shown that the $\mathcal{H}$-class $H_V$ of $V$ in $S$, with zero adjoined, is a minimal non-zero ideal of the Rees factor $S/J$.

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Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: okninski@mimuw.edu.pl

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