POSITIVE SOLUTIONS FOR SUBLINEAR ELLIPTIC EQUATIONS

BY

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Abstract. The existence of a positive radial solution for a sublinear elliptic boundary value problem in an exterior domain is proved, by the use of a cone compression fixed point theorem. The existence of a nonradial, positive solution for the corresponding nonradial problem is obtained by the sub- and supersolution method, under an additional monotonicity assumption.

1. Introduction. In the first part of the paper we consider the problem

\[ -\Delta u = f(\|x\|, u) \quad \text{for } \|x\| > 1, \quad x \in \mathbb{R}^n, \quad n \geq 3, \]
\[ u = 0 \quad \text{for } \|x\| = 1, \]
\[ u \to 0 \quad \text{as } \|x\| \to \infty. \]

Looking for its radial solutions \( u(x) = z(\|x\|) \), where \( z : [1, \infty) \to \mathbb{R} \), one can substitute \( v(t) = z((1 - t)^{1/(2-n)}) \), thus reducing the elliptic BVP (1) to the following BVP for ODE, which is singular at 1:

\[ v''(t) + g(t, v(t)) = 0 \quad \text{for } t \in (0, 1), \]
\[ v(0) = v(1) = 0, \]

where

\[ g(t, v) = \frac{1}{(n-2)^2} (1 - t)^{(2n-2)/(2-n)} f((1 - t)^{1/(2-n)}, v(t)). \]

Using some fixed point theorem in a cone [6] we obtain the existence of at least one positive solution for (2) and therefore a radial positive solution for BVP (1). The nonlinearity \( g \) (or \( f \)) is assumed to be sublinear with respect to the second variable both at 0 and \( \infty \). We relax the sublinearity assumption on \( g \) at 0 used in [8], where some results for BVP (2) were obtained by means of lower and upper solutions. BVP (2) generalizes the Emden–Fowler equation considered in [23]. Related problems were considered in [3], [7], [10], [11], [21]. A similar method (with another cone) has been used in [4]. Problems of the form (1) but with superlinear nonlinearity were considered in [9], [22]. In problem (1) we have used the exterior of the unit ball only.
for the sake of convenient notation (one could replace it with the exterior of a ball with an arbitrary radius).

In the second part using the existence result for the radial case together with the existence theorem of Noussair [13] we obtain the existence of a nonradial solution for the following nonradial BVP:

\[
\begin{align*}
-\Delta u &= f(x, u) \quad \text{for } \|x\| > 1, \ x \in \mathbb{R}^n, \ n \geq 3, \\
u(x) &= 0 \quad \text{for } \|x\| = 1, \\
u(x) &\to 0 \quad \text{as } \|x\| \to \infty.
\end{align*}
\]

(4)

The method of sub- and supersolutions developed in [13] cannot be applied directly, since we do not know any positive subsolution. If we allow \( f(x, \cdot) \equiv 0 \) then we have the trivial solution which cannot be used as a subsolution to produce a new one (the theorem of Noussair does not exclude that a solution is different from a subsolution). The existence of a nonnegative subsolution (which is neither zero nor positive) was used in [14] to obtain a nonnegative solution; in our approach we obtain a positive one. The problem of symmetry for BVP (4) was studied in the autonomous case in [19], while the multiplicity result was obtained in [12], [16]. Related problems were considered in [15], [17].

In our paper we treat the case of \( f \) decaying as \( x \) tends to infinity, therefore the autonomous case cannot be considered in this framework.

2. **Radial case.** First we establish the existence result for the following BVP (possibly singular at 0 and 1):

\[
\begin{align*}
v''(t) + g(t, v(t)) &= 0 \quad \text{for } t \in (0, 1), \\
v(0) &= v(1) = 0,
\end{align*}
\]

(5)

where \( g : (0, 1) \times [0, \infty) \to [0, \infty) \). We shall use the following theorem [6, Theorem 2.3.4]:

**Theorem 2.1.** Let \( E \) be a Banach space, and let \( P \subset E \) be a cone in \( E \). Let \( \Omega_1 \) and \( \Omega_2 \) be two bounded open sets in \( E \) such that \( 0 \in \Omega_1 \) and \( \overline{\Omega}_1 \subset \Omega_2 \). Let \( A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P \) be a completely continuous operator. Suppose that either

\[
\begin{align*}
\|Ax\| &\leq \|x\| \quad \text{for any } x \in P \cap \partial \Omega_1, \\
\|Ax\| &\geq \|x\| \quad \text{for any } x \in P \cap \partial \Omega_2, \\
or
\end{align*}
\]

(6)

\[
\begin{align*}
\|Ax\| &\geq \|x\| \quad \text{for any } x \in P \cap \partial \Omega_1, \\
\|Ax\| &\leq \|x\| \quad \text{for any } x \in P \cap \partial \Omega_2.
\end{align*}
\]

(7)

Then \( A \) has at least one fixed point in \( P \cap (\overline{\Omega}_2 \setminus \Omega_1) \).
Let $E$ be the space $C([0,1])$ of continuous functions with the norm $\|v\|_\infty = \sup_{t \in [0,1]} |v(t)|$. Define

$$H = \left\{ h \in C((0,1)) : h > 0, \int_0^1 t(1-t)h(t) \, dt < \infty \right\}.$$

**Theorem 2.2.** Let $g : (0,1) \times [0,\infty) \to \mathbb{R}$ be a continuous function. Assume that:

(A1) for any $M > 0$ there exists a function $h_M \in H$ such that for any $0 \leq v \leq M$, $t \in (0,1)$ we have

$$0 \leq g(t,v) \leq h_M(t), \quad \limsup_{M \to \infty} \frac{\int_0^1 s(1-s)h_M(s) \, ds}{M} < 1,$$

(A2) there exists a set $A \subset (0,1)$ of positive measure such that

$$\liminf_{v \to 0^+} \frac{g(t,v)}{v} = \infty \quad \text{uniformly w.r.t. } t \in A.$$

Then BVP (5) has at least one positive solution.

**Proof.** The Green function corresponding to the linear homogeneous problem has the form

$$G(t,s) = \begin{cases} 
  s(1-t) & \text{for } 0 \leq s \leq t, \\
  t(1-s) & \text{for } t \leq s \leq 1,
\end{cases}$$

and satisfies the following estimate:

$$|G(t,s)| \leq s(1-s) \quad \text{for all } t,s \in [0,1].$$

Taking any set of positive measure $B \subset A \cap (\delta, 1-\delta)$, for some positive $\delta$, where $A$ is the set from assumption (A2), we can define a cone $P$ in $E$ by

$$P = \{ v \in E : v(t) \geq 0, \ t \in [0,1], \ \inf_{t \in B} v(t) \geq \min\{a,1-b\} \|v\|_\infty \},$$

where $a = \inf B$ and $b = \sup B$. Then BVP (5) can be restated as an equation in $E$:

$$Sv = v,$$

where $S : P \to E$ is defined by

$$Sv(t) := \frac{1}{0} \int G(t,s)g(s,v(s)) \, ds.$$

By (A1) and (9) one can see that $S$ is well defined on the set of all nonnegative, continuous functions and maps it into $E$. Moreover, it maps all nonnegative functions into the cone $P$. Indeed, for $v \geq 0$ we have $g(t,v) \geq 0$ by (A1), therefore

$$\inf_{t \in B} Sv(t) = \inf_{t \in B} \frac{t}{0} \int (1-t)sg(s,v(s)) \, ds + \frac{1}{t} \int t(1-s)g(s,v(s)) \, ds.$$
\[
\geq \min\{1 - b, a\} \inf_{t \in B} \left( \int_0^t sg(s, v(s)) ds + \int_t^1 (1 - s)g(s, v(s)) ds \right)
\]
\[
\geq \min\{1 - b, a\} \frac{1}{0} s(1 - s)g(s, v(s)) ds
\]
\[
= \min\{1 - b, a\} \sup_{t \in [0,1]} \left( \int_0^t s(1 - s)g(s, v(s)) ds + \int_t^1 s(1 - s)g(s, v(s)) ds \right)
\]
\[
\geq \min\{1 - b, a\} \sup_{t \in [0,1]} \left( \int_0^t s(1 - t)g(s, v(s)) ds + \int_1^t (1 - s)g(s, v(s)) ds \right)
\]
\[
= \min\{1 - b, a\} \|Sv\|_{\infty}.
\]

By standard reasoning one can show that assumption (A1) guarantees that \(S\) maps \(E\) into itself and is also continuous. To prove that \(S\) is compact take any closed ball \(B(0, M)\) in \(E\). We shall show that the functions from \(S(B(0, M)) = \{Sv : \|v\|_{\infty} \leq M, \ v \in P\}\) are equicontinuous and equibounded.

To this end take \(\varepsilon > 0\) and notice that by the integrability of the function \(s \mapsto s(1 - s)h_M(s)\), there exists \(\delta > 0\) such that
\[
\int_t^{t'} s(1 - s)h_M(s) ds < \varepsilon \quad \text{if} \ |t' - t| < \delta.
\]

Thus, for such \(t\) and \(t'\), we have
\[
|Sv(t) - Sv(t')| \leq \int_\min\{t,t'\}^{\max\{t,t'\}} s(1 - s)h_M(s) ds < \varepsilon
\]
where we have used (9). This proves that the \(Sv\) are equicontinuous. Similarly, one can show they are equibounded. Hence the functions from the set \(\{Sv : \|v\|_{\infty} \leq M\}\) satisfy the assumptions of the Ascoli–Arzelà Theorem and in consequence this set must be compact in \(E\). Since \(M > 0\) was arbitrary we get compactness of the operator \(S : P \rightarrow E\).

Now we shall show that assumptions (7) from Theorem 2.1 are satisfied. By (A1) we can choose \(M > 0\) large enough so that \(\int_0^1 t(1 - t)h_M(t) dt \leq M\), whence for any \(0 \leq v \leq M\) and \(t \in (0, 1)\),
\[
0 \leq g(t, v) \leq h_M(t).
\]

Then for any \(v \in P\) such that \(\|v\|_{\infty} = M\), by (9) one obtains
\[
Sv(t) = \int_0^1 G(t, s)g(s, v(s)) ds \leq \int_0^1 s(1 - s)g(s, v(s)) ds
\]
\[ \leq \int_0^1 s(1-s)h_M(s) \, ds \leq M = \|v\|_\infty \]

for any \( t \in [0,1] \). Therefore \( \|Sv\|_\infty \leq \|v\|_\infty \) for any \( v \in \partial \Omega_2 \cap P \), where
\[
\Omega_2 := \{ v \in E : \|v\|_\infty < M \}.
\]

Finally, choose \( \mu > 0 \) such that \( \mu \min\{a,1-b\} \int_B G((a+b)/2, s) \, ds \geq 1 \) \((B \subset A \cap (\delta, 1-\delta))\) for some positive constant \( \delta \) as in the definition of the cone \( P \). By (A2), there exists \( R < M \) such that for \( 0 \leq v \leq R \),
\[
\inf_{t \in B} g(t,v) \geq \mu v.
\]

Let \( \Omega_1 = \{ v \in E : \|v\|_\infty < R \} \). If \( v \in P \cap \partial \Omega_1 \), then \( g(t,v(t)) \geq \mu v(t) \) for all \( t \in B \). Then
\[
Sv \left( \frac{a+b}{2} \right) = \int_0^1 G \left( \frac{a+b}{2}, s \right) g(s,v(s)) \, ds \geq \int_B G \left( \frac{a+b}{2}, s \right) g(s,v(s)) \, ds \geq \mu \int_B G \left( \frac{a+b}{2}, s \right) v(s) \, ds \geq \|v\|_\infty \mu \min\{a,1-b\} \int_B G \left( \frac{a+b}{2}, s \right) ds \geq \|v\|_\infty.
\]

Therefore \( \|Sv\|_\infty \geq \|v\|_\infty \) for \( v \in P \cap \partial \Omega_1 \). Applying Theorem 2.1 to \( S \) one obtains a fixed point \( v_0 \) in \( P \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

Moreover, since \( v_0 \) is nonnegative and \( \|v_0\|_\infty \geq R \) we have \( v_0(t_0) > 0 \) at some point \( t_0 \). Since \( v_0 \) satisfies integral equation (11), the function \( g(\cdot,v_0(\cdot)) \) cannot vanish on the whole interval \((0,1)\) (otherwise \( v_0 \equiv 0 \) by (11)) so once again by (11), \( v_0 \) must be positive in \((0,1)\).

**Remark 1.** One can see from the proof of the theorem that the assumption (A2) could be replaced by

(A2') \quad \text{there exists a set } B \subset (0,1) \text{ of positive measure and } \varepsilon > 0 \text{ such that}
\[
g(t,v) \geq \mu_B v \quad \text{for } t \in B, \ 0 < v < \varepsilon,
\]

where
\[
\mu_B := \inf_{C \subset B} \left( \min\{\inf C, 1-\sup C\} \sup_{t \in (0,1)} \int_C G(t,s) \, ds \right)^{-1}.
\]

Obviously, if \( C \subset B \) then the inequality holds on \( C \) as well, but the constant in round brackets can be less than the one for \( B \).

**Remark 2.** For example, if \( B = (0,1) \), then \( \mu_B = 24\sqrt{3} \), which is obtained for \( C = (1/(2\sqrt{3}), 1-1/(2\sqrt{3})) \). In this case assumption (A2') can be compared with the assumption
(A) there exists $k > 1$ and for any compact set $K \subset (0,1)$, there is $\varepsilon > 0$ such that
\[
g(t,v) \geq k^2 v \quad \text{for all } t \in K, \ u \in (0,\varepsilon]
\]
from [8]. Obviously, our constant is worse than this one, but our theorem also works for nonlinearities that satisfy the inequality on smaller sets than $t \in (0,1)$.

Define $K := \{ p \in C((1,\infty)) : \int_1^\infty s(1 - s^{2-n})p(s) \, ds < \infty \}$. Now we are ready to formulate the main result for elliptic BVP (1) in the radial case, which is an immediate consequence of Theorem 2.2:

**Theorem 2.3.** Let $f : (1,\infty) \times [0,\infty) \to [0,\infty)$ be a continuous function satisfying

\[\text{(B1)} \quad \text{for any } M > 0 \text{ there exists a function } p_M \in K \text{ such that, for any } 0 \leq u \leq M, s > 1, \quad 0 \leq f(s,u) \leq p_M(s), \quad \limsup_{M \to \infty} \frac{\int_1^\infty s(1 - s^{2-n})p_M(s) \, ds}{M} < 1,\]

\[\text{(B2)} \quad \text{there exists a set } B \text{ of positive Lebesgue measure such that} \]
\[
\lim_{u \to 0^+} \frac{f(s,u)}{u} = \infty \quad \text{uniformly w.r.t. } s \in B.
\]

Then BVP (1) has at least one positive solution.

**Remark 3.** It is worth noticing that even if $f(s,0) \equiv 0$ then by the above theorem we obtain additionally a positive solution.

**Remark 4.** One can see from the proof of the theorem that the assumption (B2) could be replaced by

\[\text{(B2')} \quad \text{there exists a set } B \subset (1,\infty) \text{ of positive measure and a sufficiently large constant } L_B \text{ such that} \]
\[
f(s,u) \geq L_B u \quad \text{for } s \in B, u \geq R.
\]

**Remark 5.** The assumption (B1) excludes the case of the nonlinearity $f$ depending only on $u$. In fact $f$ not only has to depend on $s$ but also to decay sufficiently fast as $s \to \infty$. The case of slower decay was considered in [2].

**Corollary 1.** If $f(s,u) = p(s)h(u)$, where $p : (1,\infty) \to (0,\infty)$ and $h : [0,\infty) \to [0,\infty)$ are continuous functions, then the assumptions of Theorem 2.3 reduce to:

\[\text{(C1)} \quad p_0 := \int_1^\infty s(1 - s^{2-n})p(s) \, ds < \infty,
\]
(C2) \( \limsup_{u \to \infty} \frac{h(u)}{u} < \frac{1}{p_0} \),

(C3) \( \liminf_{u \to 0^+} \frac{h(u)}{u} = \infty \).

This includes the following

**Example 1.** The equation
\[
-\Delta u = \frac{u^\alpha}{\|x\|^\beta} \quad \text{for } \|x\| \geq 1,
\]
where \( \alpha < 1, \beta > 2 \), has a positive solution satisfying the boundary conditions as in (1).

Theorem 2.3 also applies to the case where one cannot separate the variables of the function \( f \), as in the following:

**Example 2.** Consider the equation
\[
-\Delta u = \frac{u^\alpha (\|x\|)}{\|x\|^\beta} \quad \text{for } \|x\| \geq 1,
\]
where \( \beta > 2 \) and \( \alpha : [1, \infty) \to \mathbb{R} \) satisfies \( \sup_{s \in [1, \infty)} \alpha(s) < 1 \). Then (15) with the boundary conditions as in (1) has a positive solution.

**Remark 6.** Considering \( f : (1, \infty) \times \mathbb{R} \to \mathbb{R} \) and assuming only (B1), by the Schauder Theorem, one can obtain the existence of at least one solution to BVP (1). If \( f(\cdot, 0) \neq 0 \) then the solution is not trivial, otherwise \( f(\cdot, 0) \equiv 0 \) we do not know whether a nonzero solution exists. Therefore the situation is quite different from the one considered above.

### 3. Nonradial case.
Consider the following BVP in an exterior domain \( \Omega \subset \mathbb{R}^n \) \( (n \geq 3) \):
\[
-\Delta u = f(x, u, \nabla u) \quad \text{for } x \in \Omega,
\]
\[u(x) = 0 \quad \text{for } x \in \partial \Omega.
\]

We look for classical solutions \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \). We recall a well known notion of sub- and supersolution for BVP (16). We shall call \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) a **subsolution** for (16) if it satisfies
\[
-\Delta u \leq f(x, u, \nabla u) \quad \text{for } x \in \Omega,
\]
\[u(x) \leq 0 \quad \text{for } x \in \partial \Omega,
\]
In a similar way we define a **supersolution** \( \bar{u} \) for (16)—it suffices to reverse the inequalities in (17). A good survey of results obtained by the sub- and supersolution method is the book of Pao [18].

Now we recall the existence result for a nonradial elliptic BVP in an exterior domain due to Noussair [13]:

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\( \ldots \)
**Theorem 3.1.** Let $\Omega$ be an exterior domain. Assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions:

(D1) for each bounded domain $M \subset \Omega$ there exists a continuous function $\varphi_M : \mathbb{R} \to \mathbb{R}$ such that for all $x \in M$, $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$,

$$|f(x,u,p)| \leq \varphi_M(u)(1 + |p|^2),$$

(D2) $f$ is Hölder continuous ($C^{0,r}$) with respect to $(x,u,p)$ and $C^1$ with respect to $u,p$,

(D3) there exist sub- and supersolutions for (16), $u$ and $\overline{u}$ respectively, such that $u(x) \leq \overline{u}(x)$ for any $x \in \Omega$.

Then there exists at least one solution $u$ of (16) such that $u(x) \leq u(x) \leq \overline{u}(x)$ for any $x \in \Omega$.

Let $\Omega$ be the exterior of the closed ball $B(0,1)$ of radius 1 in $\mathbb{R}^n$ and let $n \geq 3$. We consider the following BVP in $\Omega$ with a not necessarily radial nonlinearity $f$:

$$-\Delta u = f(x,u) \quad \text{in } \Omega,$$

$$u(x) = 0 \quad \text{on } \partial \Omega,$$

$$u(x) \to 0 \quad \text{as } \|x\| \to \infty.$$

**Theorem 3.2.** Assume that a function $f : \Omega \times [0,\infty) \to [0,\infty)$ satisfies:

(E1) $f$ is Hölder continuous ($C^{0,r}$) with respect to $(x,u)$ and continuously differentiable and nonincreasing with respect to the second variable,

(E2) for any $M > 0$ there exists a function $p_M : (1,\infty) \to (0,\infty)$ such that $\int_1^\infty s(1 - s^{2-n})p_M(s)\,ds < \infty$ and

$$0 \leq f(x,u) \leq p_M(\|x\|) \quad \text{for any } 0 \leq u \leq M, \ 1 < \|x\|,$$

(E3) there exists a set $B \subset (1,\infty)$ of positive measure such that

$$\liminf_{u \to 0^+} \frac{f(x,u)}{u} = \infty \quad \text{uniformly w.r.t. } \|x\| \in B.$$

Then there exists at least one positive solution to BVP (18).

**Proof.** Define $f_1 : (1,\infty) \times [0,\infty) \to [0,\infty)$ and $f_2 : (1,\infty) \times [0,\infty) \to [0,\infty)$ by

$$f_1(r,u) = \inf_{\|x\|=r} f(x,u), \quad f_2(r,u) = \sup_{\|x\|=r} f(x,u).$$

Since $f_1$ and $f_2$ satisfy the assumptions of Theorem 2.3 we obtain the existence of two radial solutions $u_1$ and $u_2$ to BVP (1) with $f = f_1$ and $f = f_2$ respectively. But then

$$-\Delta u_1 = f_1(\|x\|,u_1) \leq f(x,u_1)$$
so $u_1$ is a subsolution to BVP (18). By a similar reasoning $u_2$ is a supersolution to BVP (18). Moreover if $u_1(x_0) > u_2(x_0)$ for some $x_0 \in \Omega$ then taking the connected component $U$ of the set \{ $x \in \Omega : u_1(x) > u_2(x)$ \} such that $x_0 \in U$ and considering the function $h(x) = u_2(x) - u_1(x)$ we arrive at

$$-\Delta h(x) = f_2(\|x\|, u_2(x)) - f_1(\|x\|, u_1(x))$$

$$\geq f_1(\|x\|, u_2(x)) - f_1(\|x\|, u_1(x)),$$

which together with the monotonicity (by (E1) the function $f_1(r, \cdot)$ is non-increasing) implies $-\Delta h(x) \geq f_1(\|x\|, u_1(x)) - f_1(\|x\|, u_1(x)) = 0$ for any $x \in U$ (since $u_1(x) > u_2(x)$ for those $x$). Consequently,

$$-\Delta h(x) \geq 0 \quad \text{for } x \in U,$$

$$h(x) = 0 \quad \text{for } x \in \partial U,$$

$$h(x) \to 0 \quad \text{as } \|x\| \to \infty, \; x \in U,$$

so if $h$ is negative at some point it has to attain a negative minimum at some $x_0$. Then take $r_0 > 0$ such that $\|x_0\| < r_0$ and $h(x) \geq \frac{1}{2} h(x_0)$ for $\|x\| \geq r_0$. Consequently, by the maximum principle (see [5] or [20]) applied in the set $U_1 = \{ x \in U : 1 < \|x\| < r_0 \}$, the function $h$ attains its minimum on the boundary of $U_1$ so we get $h(x_0) \geq \inf_{\partial U_1} h(x) \geq \frac{1}{2} h(x_0)$, which contradicts the fact that $x_0 \in U$.

Therefore by Theorem 3.1 we obtain the existence of a solution $u_0$ for (18) such that $u_1 \leq u_0 \leq u_2$. Thus if $f$ is not radial with respect to $x$ then obviously neither is $u_0$. ■

**Remark 7.** The above theorem provides a nonradial solution only if the nonlinearity $f$ is nonradial with respect to $x$.

**Example 3.** Consider the following BVP:

$$-\Delta u = g(x)u^\sigma \quad \text{in } \Omega,$$

$$u(x) = 0 \quad \text{on } \partial \Omega,$$

$$u(x) \to 0 \quad \text{as } \|x\| \to \infty,$$

where $\sigma < 1$ and $g : \Omega \to (0, \infty)$ is a continuous, nonnegative function, strictly positive on some annulus $\Omega_1 = \{ x \in \Omega : a \leq \|x\| \leq b \} \subset \Omega$ ($1 < a < b$ some positive constants) satisfying

$$\int_1^\infty r(1 - r^{2-n}) \sup_{\|x\|=r} g(x) \, dr < \infty.$$

Then BVP (19) has a positive solution (nonradial if $g$ is such).
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