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ISOMETRIES OF NORMED SPACES

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Abstract. We improve the Mazur–Ulam theorem by relaxing the surjectivity condition.

By a classical result of S. Mazur and S. Ulam [3], [1, 14.1], a surjective isometry of a real normed space onto a real normed space is affine. We replace the surjectivity assumption by a weaker assumption that every direction in Y can be approximated by a direction between two points in the range of the isometry f and obtain a sharp bound for this approximation which still implies that f is affine.

THEOREM. *Let X, Y be normed spaces and let f be an isometry of X into Y. Assume that for every unit vector $y \in Y$ there exist $a, b \in X$ and a real number s such that*

$$\|y - s(f(a) - f(b))\| < 1/2. \tag{1}$$

Then f is affine and $f(X)$ is a dense subspace of Y. Moreover, if X is complete then $f(X) = Y$.

Proof. Clearly, it is enough to prove the theorem in the special case where $f(0) = 0$. Denote the linear span of $f(X)$ by Y'. By the result of [2] there is a linear map $F : Y' \to X$ such that $\|F\| \le 1$ and $F(f(x)) = x$ for every $x \in X$.

Suppose that F has a nontrivial kernel. Then there is a unit vector $y \in Y'$ with $F(y) = 0$. We can find $a, b \in X$ and a real number s satisfying (1). It follows that

$$1/2 > \|y - s(f(a) - f(b))\| \ge 1 - \|s(f(a) - f(b))\|,$$

and consequently, $\|s(a - b)\| = \|s(f(a) - f(b))\| > 1/2$. On the other hand,

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since $\|F\| \le 1$ we have

$$1/2 > \|y - s(f(a) - f(b))\| \ge \|F(y) - s(F(f(a)) - F(f(b)))\| = \|s(a-b)\|,$$

a contradiction. Thus, F is a bijective linear map of Y' onto X, and hence its inverse f is a linear isometry of X onto Y'. Clearly, our assumption yields that Y' is dense in Y.

Finally, observe that if X is complete, then so is Y', being isometric to X. Consequently, Y' is closed in Y. Since it is dense in Y, we have $Y' = Y$, and hence f is a surjection of X onto Y.

Let us remark that condition (1) is rather sharp. Namely, the theorem becomes false when (1) is replaced by $\|y - s(f(a) - f(b))\| \le 1/2$.

Indeed, let $X = \mathbb{R}$, $Y = l^2_\infty$, and define $f : X \to Y$ by $f(x) = (x, |x|)$. Then f is a nonlinear isometry and yet, if $y \in Y$ is such that $\|y\| = 1$, then $\|y - sf(x)\| = \|y - s(f(x) - f(0))\| \le 1/2$ for some $x \in X$ and scalar s. The verification is an easy exercise, since each of the vectors $(\pm 1/2, \pm 1/2)$ is of the form $(s/2)f(x)$ for some $s, x \in \{-1, 1\}$.

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