ON INDECOMPOSABLE PROJECTIVE REPRESENTATIONS OF FINITE GROUPS OVER FIELDS OF CHARACTERISTIC \( p > 0 \)

BY

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Abstract. Let \( G \) be a finite group, \( F \) a field of characteristic \( p \) with \( p \mid |G| \), and \( F^\lambda G \) the twisted group algebra of the group \( G \) and the field \( F \) with a 2-cocycle \( \lambda \in Z^2(G, F^*) \). We give necessary and sufficient conditions for \( F^\lambda G \) to be of finite representation type. We also introduce the concept of projective \( F \)-representation type for the group \( G \) (finite, infinite, mixed) and we exhibit finite groups of each type.

Introduction. Let \( F \) be a field of characteristic \( p > 0 \), \( F^* \) the multiplicative group of the field \( F \), \( F^p = \{ a^p : a \in F \} \), \( G \) a finite group of order \( |G| \), where \( p \mid |G| \), and \( G_p \) a Sylow \( p \)-subgroup of \( G \). Let \( G' \) be the commutant of \( G \), \( C_p \) a Sylow \( p \)-subgroup of \( G' \), \( C_p \subset G_p \), \( G_p \) the commutant of \( G_p \), and \( Z^2(G, F^*) \) the group of all \( F^* \)-valued normalized 2-cocycles of the group \( G \), where we assume that \( G \) acts trivially on \( F^* \) (see [26, Chapter 1]). Denote by \( F^\lambda G \) the twisted group algebra of the group \( G \) and the field \( F \) with a cocycle \( \lambda \in Z^2(G, F^*) \) and by \( \text{rad} F^\lambda G \) the radical of \( F^\lambda G \). An \( F \)-basis \( \{ u_g : g \in G \} \) of \( F^\lambda G \) satisfying \( u_a u_b = \lambda_{a,b} u_{ab} \) for all \( a, b \in G \) is called natural. By an \( F^\lambda G \)-module we mean a finitely generated left \( F^\lambda G \)-module. If \( H \) is a subgroup of \( G \), then the restriction of \( \lambda \in Z^2(G, F^*) \) to \( H \times H \) will also be denoted by \( \lambda \). In this case, \( F^\lambda H \) is a subalgebra of \( F^\lambda G \).

Higman [21] proved that a group algebra \( FG \) is of finite representation type if and only if \( G_p \) is a cyclic group. In this case Kasch, Kneser and Kupisch [27] gave a sharper upper bound of the number of indecomposable \( FG \)-modules. They also obtained conditions on \( G \) under which the bound is attained. Later Janusz [22] gave a formula for the exact number of indecomposable \( FG \)-modules for the case when \( F \) is an algebraically closed field. In [23] he determined the structure of indecomposable modules in more detail. Indecomposable \( FG \)-modules with \( G_p \) being cyclic are also investigated in [5], [11], [24], [25], [28], [29] (see as well [16, Chapter VII]). The representation type of group rings \( SG \), where \( S \) is an arbitrary commutative artinian ring or a local artinian ring whose quotient ring \( S/\text{rad} S \) is finitely generated over its center, is determined by Gustafson [20] and Dowbor and Simson [14].

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Generalizations to the case when $S$ is an arbitrary finite-dimensional algebra over a field $F$ and $G$ is a finite group have been found by Meltzer and Skowroński [30], [31] and Skowroński [35], [36]. Representation-infinite group algebras $SG$ of polynomial growth are classified in [36]. Gudivok [18] and Janusz [24], [25] showed that if $F$ is an infinite field and $G$ is an abelian $p$-group which is neither cyclic nor of order 4, then there exist infinitely many non-isomorphic indecomposable $FG$-modules of $F$-dimension $n$ for every natural number $n > 1$. If $G$ is the non-cyclic group of order 4, then the preceding result is valid for even natural numbers $n$.

Higman [21] proved, in fact, that the first Brauer–Thrall conjecture holds for group algebras of finite groups. Results by Gudivok [18] and Janusz [24], [25] give the solution of the second Brauer–Thrall conjecture for group algebras of finite groups. As is well known, the first Brauer–Thrall conjecture for finite-dimensional algebras over an arbitrary field was solved by Roiter [34]. The second Brauer–Thrall conjecture was proved by Nazarov and Roiter [32], Bautista [3], Bongartz [6], Bautista, Gabriel, Roiter and Salmerón [4].

In [7], Conlon developed the theory of twisted group algebras $F^\lambda G$ by exploiting their analogy with group algebras $FG$ assuming that $F$ is large enough. In this case $F^\lambda G_p$ is a group algebra and therefore $F^\lambda G$ is of finite representation type if and only if $G_p$ is cyclic. Moreover, in the same paper Conlon established that if $G_p$ is a cyclic group then a rough upper bound for the number of indecomposable $FG$-modules which was found in [21] also holds for the number of indecomposable $F^\lambda G$-modules. It should be noted that Reynolds [33] computed the number of non-isomorphic simple $K^\mu G$-modules where $K$ is an arbitrary field, $G$ is a finite group and $\mu \in Z^2(G, K^*)$. We also remark that if the characteristic of $K$ does not divide the order of the group $G$, then $K^\mu G$ is a semisimple algebra for any $\mu \in Z^2(G, K^*)$, and hence is of finite representation type. Using Green’s results [17], for the case when $G$ is a finite abelian $p$-group and the radical of $F^\lambda G$ is not cyclic, Sobolewska [37] constructed increasing functions $f_\lambda : \mathbb{N} \to \mathbb{N}$ such that there exist infinitely many isomorphism classes of indecomposable $F^\lambda G$-modules of $F$-dimension $f_\lambda(n)$ for every natural number $n > 1$.

In the present paper we shall characterize twisted group algebras $F^\lambda G$ of finite representation type. We shall also describe finite groups depending on a projective representation type over the field $F$.

Let us briefly present the main results of the paper. In Section 1, we prove that an algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda G_p$ is a uniserial algebra (Theorem 1.1; we use the terminology introduced in [15]). We also establish (Theorem 1.2) that if $p \neq 2$, then $F^\lambda G_p$ is a uniserial algebra if and only if $C_p$ is cyclic and one of the following conditions holds:
(1) the quotient algebra $F^\lambda G_p/F^\lambda G_p\cdot \text{rad} F^\lambda C_p$ is a field;
(2) $C_p = \{e\}$ and there exists a decomposition $G_p = H \times N$ such that $H$ is cyclic and $F^\lambda N$ is a field;
(3) $C_p \neq \{e\}$ and there exists a decomposition $G_p/C_p = \langle a_1C_p \rangle \times \ldots \times \langle a_sC_p \rangle$ such that $C_p \subseteq \langle a_1 \rangle$, $C_p \nsubseteq \langle a_j \rangle$ for every $j = 2, \ldots, s$ and $F^\lambda D/F^\lambda D \cdot \text{rad} F^\lambda C_p$ is a field, where $D$ is the subgroup of $G_p$ generated by $C_p, a_2, \ldots, a_s$.

The proofs of these theorems are based on the characterization of local rings of finite representation type which was obtained in [12]–[14]. A special case of such rings was investigated in [19]. In Section 1 of this paper, we also obtain indecomposable $F^\lambda G$-modules for the case when $G_p$ is a normal subgroup of $G$ and $F^\lambda G_p$ is a uniserial algebra (Theorems 1.3 and 1.4).

We say that a group $G$ is of finite (resp. infinite) PFR-type (Projective F-Representation type) if the algebra $F^\lambda G$ is of finite (resp. infinite) representation type for every cocycle $\lambda \in Z^2(G, F^*)$. Otherwise, $G$ is said to be of mixed PFR-type.

In Section 2, we classify finite groups depending on their PFR-type (Theorems 2.1 and 2.2, Proposition 2.1). We also state necessary and sufficient conditions for $G$ and $G_p$ to be of the same PFR-type (Propositions 2.2–2.3).

1. Twisted group algebras of finite representation type and their representations

**Lemma 1.1.** Let $\lambda \in Z^2(G, F^*)$. Every $F^\lambda G$-module is isomorphic to an $F^\lambda G$-component of an induced $F^\lambda G$-module $F^\lambda G \otimes_{F^\lambda G_p} V$, where $V$ is some $F^\lambda G_p$-module.

**Lemma 1.2.** Let $H$ be a subgroup of $G$ and $\lambda \in Z^2(G, F^*)$. If $F^\lambda H$ is of infinite representation type, then $F^\lambda G$ is also of infinite representation type.

**Lemma 1.3.** An algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda G_p$ is of finite representation type.

The proofs of Lemmas 1.1–1.3 are similar to those of the corresponding propositions about group algebras (see [8, §63]).

**Lemma 1.4 ([21]).** A group algebra $FG$ is of finite representation type if and only if $G_p$ is a cyclic group.

**Lemma 1.5.** Suppose $p ||G'||$, $C_p \subset G_p$ and $\lambda \in Z^2(G, F^*)$. Then:

(1) Up to cohomology
\begin{equation}
\lambda_{g,h} = \lambda_{h,g} = 1
\end{equation}
for any $g \in G_p$ and any $h \in C_p$. 

(2) Suppose $\lambda$ satisfies condition (1.1), $\overline{G}_p = G_p/C_p$, $\overline{g} = gC_p$ for $g \in G_p$, and $\overline{\lambda}_{a,b} = \lambda_{a,b}$ for any $a, b \in G_p$. Then $\overline{\lambda} \in Z^2(\overline{G}_p, F^*)$ and

$$F^\lambda \overline{G}_p \cong F^\lambda G_p/F^\lambda G_p \cdot \text{rad} F^\lambda C_p.$$ 

Proof. In view of [26, Proposition 5.17, p. 48] the restriction of every cocycle $\lambda \in Z^2(G, F^*)$ to $C_p \times C_p$ is a coboundary. Therefore, statements (1) and (2) follow from the properties of natural homomorphisms of twisted group algebras ([26, pp. 87–93]).

In what follows, we assume that every cocycle $\lambda \in Z^2(G, F^*)$ under consideration satisfies condition (1.1). In particular, $F^\lambda C_p$ will always be the group algebra $FC_p$.

The number $i_F = \sup \{0, m\}$ is important in describing twisted group algebras of abelian $p$-groups which are of finite representation type, where $m$ is a natural number such that for some $\gamma_1, \ldots, \gamma_m \in F^*$ the algebra

$$F[x]/(x^p - \gamma_1) \otimes_F \cdots \otimes_F F[x]/(x^p - \gamma_m)$$

is a field. If $F$ is a perfect field, then $i_F = 0$, otherwise $i_F \neq 0$.

**Proposition 1.1.** Let $K$ be a perfect field of characteristic $p$ and $F = K(x_1, \ldots, x_n)$ the quotient field of the polynomial ring $K[x_1, \ldots, x_n]$. Then $i_F = n$.

Proof. By induction on $i$ we prove that the algebra

$$A_i = F[y]/(y^p - x_1) \otimes_F \cdots \otimes_F F[y]/(y^p - x_i)$$

is a field for every $i \in \{1, \ldots, n\}$. From this it follows that $i_F \geq n$. Suppose that for some $\lambda_1, \ldots, \lambda_m \in F^*$ the algebra

$$B = F[y]/(y^p - \lambda_1) \otimes_F \cdots \otimes_F F[y]/(y^p - \lambda_m)$$

is a field. Let $C = B \otimes_F A_n$. The algebra $A_n$ is isomorphic to the field $K(y_1, \ldots, y_n)$, where $y_j^p = x_j$ ($j = 1, \ldots, n$). Every element of $F$ is the $p$th power of some element of $A_n$. It follows that $C \cong A_n[y]/(y^p - 1) \otimes_{A_n} \cdots \otimes_{A_n} A_n[y]/(y^p - 1)$ ($m$ factors).

Consequently, $C/\text{rad} C \cong A_n$. On the other hand, $C$ can be viewed as a twisted group algebra of an elementary abelian $p$-group of order $p^n$ over the field $B$. Therefore, $C/\text{rad} C$ is isomorphic to a purely inseparable extension of the field $B$ of degree $p^s$, where $s \leq n$. It follows that $p^n = p^s \cdot [B : F]$ or $p^n = p^s \cdot p^m$, whence $m \leq n$. Hence $i_F \leq n$, and the proof is complete.

**Proposition 1.2.** Let $K$ be a field of characteristic $p$, $X = \{x_i : i = 1, 2, \ldots\}$, and $F$ the quotient field of the polynomial ring $K[X]$. Then $i_F = \infty$. 

Theorem 1.1. Let $G$ be a finite group, $p || G$ and $\lambda \in Z^2(G, F^*)$. The algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda G_p$ is a uniserial algebra.

Proof. By Lemma 1.3, we may assume that $G$ is a $p$-group. Let $\{u_g : g \in G\}$ be a natural $F$-basis of the algebra $F^\lambda G$ and $e$ be the identity element of $G$. It is known (see [26, p. 74]) that $F^\lambda G / \text{rad} F^\lambda G \cong K$, where $K$ is a purely inseparable extension of the field $F$. Suppose $F^\lambda G$ is of finite representation type. Then by Lemmas 1.2, 1.4 and 1.5, $G'$ is a cyclic group and $F^\lambda G'$ is a group algebra. Let $G' = \langle c \rangle$, $A = F^\lambda G$, $V = \text{rad} A / (\text{rad} A)^2$, $m = \dim_K V$ and $m' = \dim V_K$. We know (see [12]–[14]) that in the case under consideration we have $m \cdot m' \leq 3$.

Suppose $m = 1$. If $u_c - u_e \notin (\text{rad} A)^2$, then $\{u_c - u_e + (\text{rad} A)^2\}$ is a basis of the left vector space $V$ over the field $K$. It follows that any element of $V$ is of the form
\[ \overline{x}(u_c - u_e + (\text{rad} A)^2) = x(u_c - u_e) + (\text{rad} A)^2, \]
where $x \in A$, $\overline{x} = x + \text{rad} A$. Since for each $x \in A$ there exists $y \in A$ such that $x(u_c - u_e) = (u_c - u_e)y$, we have
\[ \overline{x}(u_c - u_e + (\text{rad} A)^2) = (u_c - u_e + (\text{rad} A)^2)y. \]
Hence, $m' = 1$. Suppose now that $u_c - u_e \in (\text{rad} A)^2$. Since for arbitrary $x, y \in A$ there exists $z \in A$ such that $xy - yx = (u_c - u_e)z$, we obtain
\[ \overline{x}(y + (\text{rad} A)^2) = (y + (\text{rad} A)^2)\overline{x} \]
for any $x, y \in A$. In this case $m' = 1$. By the same arguments we can establish that if $m' = 1$ then $m = 1$.

Therefore, if $F^\lambda G$ is of finite representation type, then $F^\lambda G$ is a uniserial algebra. Conversely, every uniserial algebra is of finite representation type ([15, p. 171]).

Proposition 1.3. Let $F$ be a field of characteristic $p$, $G$ a finite abelian $p$-group and $\lambda \in Z^2(G, F^*)$. The algebra $F^\lambda G$ is of finite representation type if and only if $G = H \times N$, where $H$ is a cyclic group and $F^\lambda N$ is a field.

Proof. Let $G = H \times N$, where $H$ is cyclic and $F^\lambda N$ is a field. Then $F^\lambda G$ is a uniserial algebra, and hence it is of finite representation type. Now we suppose that there is no decomposition $G = H \times N$ such that $H$ is a cyclic group and $F^\lambda N$ is a field. Let $\overline{G}$ be the socle of $G$. Then $F^\lambda \overline{G} \cong F^\mu B$, where $B$ is an elementary abelian $p$-group of order $|\overline{G}|$ and the following conditions are satisfied: $B = L \times M$, $L$ is a non-cyclic group of order $p^2$ and $F^\mu L$ is the group algebra of the group $L$ over the field $F$. By Lemmas 1.2 and 1.4, the algebra $F^\mu B$ is of infinite representation type. Applying again Lemma 1.2 to $F^\lambda \overline{G}$ and $F^\lambda G$, we conclude that the algebra $F^\lambda G$ is of infinite representation type.
Corollary 1. Let $G$ be a finite abelian $p$-group and $\lambda \in Z^2(G, F^*)$. Assume that $G = H \times N$, where $H$ is a cyclic group and $F^\lambda H$ is not a field. The algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda N$ is a field.

Corollary 2. Let $G$ be a finite abelian $p$-group, $\bar{G}$ the socle of $G$, and $\lambda \in Z^2(G, F^*)$. The algebra $F^\lambda G$ is of infinite representation type if and only if $\bar{G}$ is a non-cyclic group of order $p^2$ and $F^\mu H$ is the group algebra.

Corollary 3. Let $G = \langle a_1 \rangle \times \ldots \times \langle a_s \rangle$ be an abelian $p$-group. If $s \geq i_F + 2$ then $F^\lambda G$ is of infinite representation type for every $\lambda \in Z^2(G, F^*)$. If $s \leq i_F + 1$ then there exists an algebra $F^\lambda G$ which is of finite representation type. If $s = 1$ then $F^\lambda G$ is of finite representation type for every $\lambda \in Z^2(G, F^*)$.

Lemma 1.6. Let $p \neq 2$, $G$ be a non-abelian $p$-group with $G' = \langle c \rangle$ of order $p$, and $\{u_g : g \in G\}$ be a natural $F$-basis of $F^\lambda G$. Then:

1. $(u_a u_b)^p = u_a^p u_b^p$ for any $a, b \in G$.
2. If $y \in F^\lambda G$, $g \in G$, then
   
   $\begin{align*}(u_g y)^p &= y u_g + (u_c - u_e) y' u_g, \\
   (y u_g)^p &= y^p u_g^p + (u_c - u_e)^2 z \end{align*}$

for some $y', z \in F^\lambda G$.

3. If
   
   $x = \sum_{g \in G} \alpha_g u_g$

is an element of $F^\lambda G$, then
   
   $x^p = \sum_{g \in G} \alpha_g^p u_g^p + (u_c - u_e)^2 z$, \quad z \in F^\lambda G.$

Proof. We remark that $u_c$ belongs to the center of $F^\lambda G$ and if $ab = c^j ba$, then $u_a u_b = u_c^j u_b u_a$. From this we obtain (1) and formula (1.2). Then

$\begin{align*}(y u_g)^p &= y[y + (u_c - u_e) y'] [y + 2(u_c - u_e)y'] \ldots \\
   & \ldots [y + (p - 1)(u_c - u_e)y'] u_g^p + (u_c - u_e)^2 z' \\
   &= y^p u_g^p + (u_c - u_e)^2 z', \quad z \in F^\lambda G.$

Hence, formula (1.3) holds.

It remains to prove (3). Suppose $\alpha_b \neq 0$. Applying (1.3) and induction on the number of non-zero summands of $x$, we obtain
\[ x^p = \left\{ \alpha_b u_e + \sum_{g \neq b} \alpha_g (u_g u^{-1}_b) \right\}^p u_b \]
\[ = \left[ \alpha_b u_e + \sum_{g \neq b} \alpha_g (u_g u^{-1}_b) \right]^p u_b^p + (u_c - u_e)^2 z' \]
\[ = \left[ \alpha_b^p u_e + \sum_{g \neq b} \alpha_g^p (u_g u^{-1}_b)^p + (u_c - u_e)^2 z'' \right] u_b^p + (u_c - u_e)^2 z' \]
\[ = \sum_{g \in G} \alpha_g^p u_g^p + (u_c - u_e)^2 z. \]

**Lemma 1.7.** Suppose \( p \neq 2, i_F \neq 0, p | |G^\prime|, \) and \( \lambda \in \mathbb{Z}^2(G, F^*) \). Assume that \( C_p \) is cyclic, \( G_p/C_p = \langle a_1 C_p \rangle \times \cdots \times \langle a_m C_p \rangle \) and \( C_p \not\subset \langle a_i \rangle \) for all \( i \in \{1, \ldots, m\} \). The algebra \( F^\lambda G \) is of finite representation type if and only if \( F^\lambda G_p/F^\lambda G_p \cdot \text{rad} \cdot F C_p \) is a field.

**Proof. Necessity.** I. First we examine the case when \( G_p \) is a group of exponent \( p \). Taking into consideration Corollary 1 to Proposition 1.3 we may assume that \( G_p \) is non-abelian. Let \( C_p = \langle \xi \rangle \) and suppose \( F^\lambda G_p \) is of finite representation type. We prove that \( V = F^\lambda G_p (u_e - u_c) \) is the radical of the algebra \( F^\lambda G_p \).

Any element \( g \in G_p \) can be uniquely represented in the form
\[ g = a_1^{i_1} \cdots a_m^{i_m} c^j, \]
where \( 0 \leq i_r, j < p \). Up to cocycle cohomology we may suppose
\[ u_g = u_1^{i_1} \cdots u_m^{i_m} u_c^j, \]
where
\[ u_{a_r}^p = \gamma_r u_e, \quad u_c^p = u_c \quad (\gamma_r \in F^*, 1 \leq r \leq m). \]
Let \( F^\lambda G_p = F^\lambda G_p/V \) and \( \bar{x} = x + V \) for every \( x \in F^\lambda G_p \). The algebra \( F^\lambda G_p \) is the commutative twisted group algebra \( F^\lambda G_p \) of the group \( \bar{G}_p = G_p/C_p \) and the field \( F \) with the cocycle \( \bar{\lambda} \), where \( \bar{\lambda}_{g_1, g_2} = \lambda_{g_1, g_2} \) for any \( g_1, g_2 \in G_p \).

Here \( \bar{g} = g C_p \) for every \( g \in G_p \). A natural \( F \)-basis of \( F^\lambda G_p \) is formed by elements \( \bar{u}_g \) \((g \in G_p)\) which by (1.4) can be uniquely represented in the form
\[ \bar{u}_g = \bar{u}_1^{i_1} \cdots \bar{u}_m^{i_m}, \]
where \( \bar{u}_{a_r} = \gamma_r \bar{u}_e, 1 \leq r \leq m \).

Suppose that \( V \) is not the radical of the algebra \( F^\lambda G_p \). From Proposition 1.3 we conclude that up to reindexing \( a_1, \ldots, a_m \) the algebra \( F[\bar{u}_{a_1}, \ldots, \bar{u}_{a_{m-1}}] \) is a field and \( F[\bar{u}_{a_1}, \ldots, \bar{u}_{a_{m-1}}, \bar{u}_a] \) is not. In this case
\[ \gamma_m^{-1} \bar{u}_e = \bar{x}^p \]
for some 
\[ x = \sum_{i_1, \ldots, i_{m-1}} \alpha_{i_1, \ldots, i_{m-1}} u_{a_{i_1}}^{i_1} \cdots u_{a_{i_{m-1}}}^{i_{m-1}}, \]
where \( \alpha_{i_1, \ldots, i_{m-1}} \in F, 0 \leq i_j < p \) for \( j = 1, \ldots, m-1 \). In view of Lemma 1.6,
\[ x^p = \gamma_m^{-1} u_e + (u_c - u_e)^2 z', \quad z' \in F^\lambda G_p, \]
and consequently
\[ (x u_m)^p = x^p u_m^p + (u_c - u_e)^2 z'' = u_e + (u_c - u_e)^2 z, \]
where \( z'' \in F^\lambda G_p, z = \gamma_m z' + z''. \) Let \( w = x u_m - u_e \). Then \( w^p = (u_c - u_e)^2 z. \) We also have \( \text{rad } F^\lambda G_p = \text{rad } F^\lambda G_p \cdot \bar{w}. \)

By Theorem 1.1 the algebra \( F^\lambda G_p \) is uniserial. Applying the Morita Theorem (see [10, p. 507]) and [10, Corollary 62.31, p. 510] we conclude that \( \text{rad } F^\lambda G_p = F^\lambda G_p \cdot \theta = \theta \cdot F^\lambda G_p \), where \( \theta^p = 0 \) and \( \theta^l \neq 0 \) for every \( l < p^2 \). We also obtain \( \text{rad } F^\lambda G_p = F^\lambda G_p \cdot \bar{\theta} \). It follows that \( \bar{w} = \bar{\theta} \cdot y' \), where \( y' \) is an invertible element of \( F^\lambda G_p \). The equality \( u_c - u_e = \theta^p y'' \), \( y'' \in F^\lambda G_p \), now shows that \( w = \theta y = z \theta \), where \( y \) and \( z \) are invertible in \( F^\lambda G_p \). This makes it possible to take \( \theta = w \). However,
\[ w^{p(p+1)/2} = (u_c - u_e)^{p+1} \bar{z} = 0 \quad \text{and} \quad p + 1 < 2. \]

This contradiction shows that \( V \) is the radical of \( F^\lambda G_p \).

II. Now we examine the general case. Let \( C_p = \langle c \rangle, \tilde{G}_p = G_p / \langle c^p \rangle, \tilde{C}_p = C_p / \langle c^p \rangle, \tilde{g} = g \langle c^p \rangle \) for every \( g \in G_p \), and \( \lambda_{a, b} = \lambda_{a, b} \) for any \( a, b \in G_p \). Then \( \tilde{\lambda} \in Z^2(\tilde{G}_p, F^*) \), \( F^\lambda \tilde{C}_p \) is the group algebra, \( F^\lambda \tilde{G}_p \) is a quotient algebra of \( F^\lambda G_p \) and \( F^\lambda \tilde{G}_p / F^\lambda \tilde{C}_p \cdot \text{rad } F^\lambda C_p \cong F^\lambda G_p / F^\lambda G_p \cdot \text{rad } F^\lambda C_p \). Suppose that \( F^\lambda G_p \) is of finite representation type. Then so is \( F^\lambda \tilde{G}_p \). We have \( \tilde{G}_p \subset \tilde{C}_p \) and \( \tilde{c} \) is a central element of order \( p \). Let
\[ \tilde{b}_i = \tilde{a}_i^{p^{r_i} - 1}, \]
where \( p^{r_i} \) is the order of \( a_i C_p \), \( 1 \leq i \leq m \). Denote by \( \tilde{T} \) the subgroup of \( \tilde{G}_p \) generated by \( \tilde{c}, \tilde{b}_1, \ldots, \tilde{b}_m \). The exponent of \( \tilde{T} \) is \( p \). From Lemma 1.2 and the result of case I, we conclude that \( F^\lambda \tilde{T} / F^\lambda \tilde{T} \cdot \text{rad } F^\lambda \tilde{C}_p \) is a field. Then so is \( F^\lambda \tilde{G}_p / F^\lambda \tilde{G}_p \cdot \text{rad } F^\lambda \tilde{C}_p \), and hence also \( F^\lambda G_p / F^\lambda G_p \cdot \text{rad } F^\lambda C_p \).

Sufficiency. If \( F^\lambda G_p / F^\lambda G_p \cdot \text{rad } F^\lambda C_p \) is a field, then \( F^\lambda G_p \) is uniserial, and hence by Theorem 1.1 the algebra \( F^\lambda G \) is of finite representation type. ■

Remark 1.1. If \( p = 2 \), then the necessity in Lemma 1.7 does not hold. Indeed, let \( F \) be a field of characteristic 2 with \( i_F \neq 0 \), and \( G_2 = \langle a, b \rangle \) the dihedral group of order 8. Assume that \( F^\lambda G_2 \) is given by the defining
relations
\[ u_a^4 = u_c, \quad u_b^2 = \gamma u_e, \quad u_b^{-1} u_a u_b = u_a^3, \]
where \( \gamma \in F^* \) and \( \gamma \notin F^2 \). In this case, \( \text{rad} \, F^G_2 = F^G_2 (u_a - u_e) \). The algebra \( F^G_2 \) is uniserial, and hence of finite representation type. At the same time, \( C_1 = G_2' = \langle a^2 \rangle \), \( G_2/C_2 = \langle abC_2 \rangle \times \langle bcC_2 \rangle \), \( C_2 \subset \langle ab \rangle \) and \( F^G_2/F^G_2 \cdot \text{rad} \, FC_2 \) is not a field.

\textbf{Theorem 1.2.} Let \( G \) be a finite group, \( p \neq 2 \), \( \overline{G}_p = G_p/C_p \), \( \overline{g} = gC_p \) for every \( g \in G_p \), \( \lambda \in Z^2 (G, F^*) \) and \( \overline{\lambda}_{a,b} = \lambda_{a,b} \) for any \( a,b \in G_p \). The algebra \( F^G \) is of finite representation type if and only if \( C_p \) is cyclic and one of the following conditions is satisfied:

(1) \( F^G_0 \) is a field;
(2) there is a decomposition \( \overline{G}_p = \langle \overline{a}_1 \rangle \times \overline{D} \) with \( \overline{D} = \langle \overline{a}_2 \rangle \times \cdots \times \langle \overline{a}_s \rangle \) such that \( F^\lambda \overline{D} \) is a field, and if \( C_p \neq \{ e \} \) then \( C_p \subset \langle a_1 \rangle \) and \( C_p \nsubseteq \langle a_j \rangle \) for all \( j \in \{ 2, \ldots, s \} \).

\textbf{Proof.} Suppose \( F^G_0 \) is of finite representation type. From Lemmas 1.2, 1.4 and 1.5 we deduce that \( C_p \) is a cyclic group. Let \( C_p = \langle c \rangle \). Assume that \( G_p \) is not cyclic. In view of Proposition 1.3 we also suppose \( c \neq e \). Suppose \( \overline{G}_p = \langle \overline{a}_1 \rangle \times \cdots \times \langle \overline{a}_s \rangle \) is a group of type \( (p^{m_1}, \ldots, p^{m_s}) \). If
\[
a_i^{p^{m_i}} = c^{pt},
\]
for all \( i \in \{ 1, \ldots, s \} \), then by Lemma 1.7, \( F^G_0 \) is a field. Suppose
\[
a_1^{p^{m_1}} = c^{k_1}, \quad a_2^{p^{m_2}} = c^{k_2},
\]
where \( (k_1, p) = 1 \), \( (k_2, p) = 1 \) and \( m_1 \geq m_2 \). There exists an integer \( l \) such that \( lk_1 + k_2 \equiv 0 \) (mod \( p \)). Let \( \overline{G}_p = G_p/\langle c^p \rangle \) and \( \overline{g} = g/\langle c^p \rangle \) for any \( g \in G_p \). From the equality
\[
\left( a_1^{l p^{m_1-m_2}} \cdot \overline{a}_2 \right)^{p^{m_2}} = \overline{a}_1^{l p^{m_1}} \cdot \overline{a}_2^{p^{m_2}} = c^{lk_1+k_2} = \overline{c}.
\]
it follows that
\[
(a_1^{l p^{m_1-m_2}} \cdot a_2)^{p^{m_2}} = c^{pt},
\]
so we may assume that
\[
(1.5) \quad C_p = \langle a_1^{p^{m_1}} \rangle \quad \text{and} \quad a_j^{p^{m_j}} = c^{pt_j}
\]
for all \( j \in \{ 2, \ldots, s \} \). Let \( \overline{D} = \langle \overline{a}_2 \rangle \times \cdots \times \langle \overline{a}_s \rangle \) and \( D \) be the subgroup of \( G_p \) generated by \( c, a_2, \ldots, a_s \). By Lemma 1.2 the algebra \( F^G D \) is of finite representation type. In view of Lemma 1.7, \( F^G \overline{D} \) is a field. This proves the necessity.

Let us prove the sufficiency. Keep the notation used in the proof of the necessity, and suppose that conditions (1.5) are satisfied. Assume also that
$F^\lambda D$ is a field and $F^\lambda G_p$ is not. Let $\{u_g : g \in G_p\}$ be a natural $F$-basis of $F^\lambda G_p$ and
\begin{equation}
 (1.6) \quad u_{a_1}^{m_1} = \gamma_1 u_e, \quad u_{a_j}^{m_j} = \gamma_j u_{p_j}, \quad 2 \leq j \leq s,
 \end{equation}
where $\gamma_i \in F^*$, $1 \leq i \leq s$. Let $c \neq e$, $U = F^\lambda G_p(\bar{u}_c - u_e)$, and $V = F^\lambda G_p(\bar{u}_c - u_e)$. We have
\begin{equation}
 (1.7) \quad u_c u_g \equiv u_g u_c \pmod{V}, \quad u_{a_j}^p u_g \equiv u_g u_{a_j}^p \pmod{V}
 \end{equation}
for all $a, g \in G_p$. We suppose that $F^\lambda G_p = F^\lambda G_p/U$ and a natural $F$-basis of $F^\lambda G_p$ is formed by elements $u_g$, where $u_g := u_g + U$. Let $K$ be the $F$-subalgebra of $F^\lambda G_p/U$ generated by $u_{a_j}^p + U$, $2 \leq j \leq s$, and $L$ the $F$-subalgebra of $F^\lambda G_p/V$ generated by $u_{a_j} + V$, $2 \leq j \leq s$. By (1.7), $L$ is commutative. In view of (1.6) the correspondence
\[ u_{a_j}^p + U \mapsto u_{a_j}^p + V, \quad 2 \leq j \leq s, \]
extends to an $F$-homomorphism $f$ of the field $K$ onto $L$. Hence $f$ is an isomorphism and $L$ is a field.

Let $p^d$ be the nilpotency index of the radical of the algebra $F^\lambda G_p/U$. Evidently $d \leq m_1$. There exists an element
\[ x = \sum_{i_2, \ldots, i_s} \alpha_{i_2, \ldots, i_s} u_{a_2}^{i_2} \cdots u_{a_s}^{i_s}, \]
where $\alpha_{i_2, \ldots, i_s} \in F$, $0 \leq i_j < p^{m_j}$, such that
\[ x^{p^d} = g_1^{-1} u_e \pmod{U}. \]
Applying the isomorphism $f$, we obtain
\begin{equation}
 (1.8) \quad \sum_{i_2, \ldots, i_s} \alpha_{i_2, \ldots, i_s} u_{a_2}^{i_2 p^d} \cdots u_{a_s}^{i_s p^d} \equiv g_1^{-1} u_e \pmod{V}. \]
Let
\[ w = x u_{a_1}^{p^{m_1-d}} - u_e. \]
Then $(F^\lambda G_p w + U)/U$ is the radical of the algebra $F^\lambda G_p/U$. By Lemma 1.6,
\begin{equation}
 (1.9) \quad w^p \equiv x^p u_{a_1}^{p^{m_1-d+1}} - u_e + (u_c - u_e)^2 z' \pmod{V},
 \quad x^p \equiv \sum_{i_2, \ldots, i_s} \alpha_{i_2, \ldots, i_s} u_{a_2}^{p i_2} \cdots u_{a_s}^{p i_s} + (u_c - u_e)^2 z'' \pmod{V},
 \end{equation}
where $z', z'' \in F^\lambda G_p$. It follows from (1.6), (1.8) and (1.9) that
\[ w^{p^d} \equiv u_c - u_e + (u_c - u_e)^2 p^{d-1} z \pmod{V}, \quad z \in F^\lambda G_p, \]
and hence
\[ w^{p^d} = (u_c - u_e)^d. \]
where $y$ is an invertible element of $F^\lambda G_p$. We proved that $F^\lambda G_p w$ is the radical of the algebra $F^\lambda G_p$. Therefore, $F^\lambda G_p$ is uniserial. By Theorem 1.1 the algebra $F^\lambda G$ is of finite representation type. ■

**Corollary.** Let $G$ be a finite group. If the algebra $F^\lambda G$ is of finite representation type for some $\lambda \in Z^2(G, F^*)$, then $C_p$ is a cyclic group and the number of invariants of the group $G_p / C_p$ does not exceed $i_F + 1$.

**Remark 1.2.** Theorem 1.2 is true for $p = 2$ as well if we suppose that $G'_2 \neq C_2$ in the case when $G'_2$ is not the identity subgroup and $C_2$ is a cyclic group.

**Theorem 1.3.** Suppose $G = G_p \times B$, $\lambda \in Z^2(G, F^*)$, and $F^\lambda G_p$ is a uniserial algebra. Then every indecomposable $F^\lambda G$-module can be uniquely represented, up to isomorphism, in the form $V \# W$, where $V$ is an indecomposable $F^\lambda G_p$-module and $W$ is a simple $F^\lambda B$-module. Moreover, the outer tensor product of any indecomposable $F^\lambda G_p$-module and any simple $F^\lambda B$-module is an indecomposable $F^\lambda G$-module.

The proof of Theorem 1.3 is analogous to the one of Theorem 3.1 in [1], where the case of $G_p$ abelian is investigated.

**Lemma 1.8.** Suppose $p \neq 2$, $p | |G'|$ and $C_p$ is cyclic. Assume that $G$ contains $G_p \times B$, where $[G_p, B] \neq \{e\}$. Then $G_p = C_p \times H$, where $H$ is an abelian subgroup and $[B, H] = \{e\}$.

**Proof.** By hypothesis, $C_p = \langle c \rangle$, $|c| = p^n$ and $n \geq 1$. Let $T = G_p \times B$. The subgroup $C_p$ is normal in $T$. Let $b \in B$ and $\varphi_b$ be the automorphism of $C_p$ such that $\varphi_b(c) = bcb^{-1}$. The mapping $\varphi : b \mapsto \varphi_b$ is a homomorphism of the group $B$ into $\text{Aut} C_p$. Since $\text{Aut} C_p$ is a cyclic group it follows that $\varphi(B)$ is cyclic. Let $K$ be the kernel of $\varphi$. If $B/K = \langle gK \rangle$, then

$$(g^i k) c (g^i k)^{-1} = g^i c g^{-i}, \quad k x k^{-1} = x$$

for all $k \in K$ and $x \in G_p$.

Let $gcg^{-1} = c^i$. Then $i \neq 1 \pmod{p}$. Let $h \in G_p$ and $ghg^{-1} = hc^i$. Then $g (hc^s)g^{-1} = hc^{l+si}$. We choose $s$ in such a way that $l + si \equiv s \pmod{p^n}$. If $gc^j g^{-1} = c^j$, then $j \equiv 0 \pmod{p^n}$. From this and the equality $h = hc^s c^{-s}$ it follows that $G_p = C_p \times H$, where $H = \{h \in G_p : ghg^{-1} = h\}$. ■

**Remark 1.3.** Suppose $p = 2$, $G = G_2 \times B$ and $[G, G_2]$ is a cyclic group. Then $G = G_2 \times B$.

**Theorem 1.4.** Suppose $p \neq 2$, $G = G_p \times B$, $[G, G_p] = \langle c \rangle$, $|c| = p^n$ ($n > 0$) and $[B, G_p] \neq \{e\}$. Then:

1. $G_p = \langle c \rangle \times H$, where $H$ is abelian and $[B, H] = \{e\}$.
2. Let $\lambda \in Z^2(G, F^*)$. The algebra $F^\lambda G$ is of finite representation type if and only if $F^\lambda H$ is a field.
Suppose that $F^\lambda H$ is a field. Let $e_1, \ldots, e_d$ be a complete system of primitive pairwise orthogonal idempotents of the semisimple algebra $F^\lambda B$, and $V_{ij} = F^\lambda G(u_c - u_e)^i e_j$, where $i \in \{0, 1, \ldots, p^n - 1\}$, $j \in \{1, \ldots, d\}$. Then every left ideal $V_{ij}$ of the algebra $F^\lambda G$ is indecomposable as a left $F^\lambda G$-module and any indecomposable $F^\lambda G$-module is isomorphic to one of these ideals. The ideals $V_{i_1j_1}$ and $V_{i_2j_2}$ are isomorphic if and only if $i_1 = i_2$ and the ideals $F^\lambda Be_{j_1}$, $F^\lambda Be_{j_2}$ of the algebra $F^\lambda B$ are isomorphic as $F^\lambda B$-modules.

Proof. The first statement is a particular case of Lemma 1.8. The second statement follows from Lemma 1.7.

Suppose $F^\lambda H$ is a field. Then $\text{rad } F^\lambda G = F^\lambda G(u_c - u_e)$. From the Morita Theorem (see [10, p. 507]) we conclude that $F^\lambda G$ is a serial algebra. In view of [2, Theorem 2], $e_1, \ldots, e_d$ is a complete system of primitive pairwise orthogonal idempotents of the semisimple algebra $A = F^\lambda H \otimes_F F^\lambda B$. By the Deuring–Noether Theorem ([8, p. 200]), we also have

$$Ae_r \cong Ae_s \Leftrightarrow F^\lambda Be_r \cong F^\lambda Be_s.$$ 

In view of [9, Theorem 6.8, p. 124], $e_1, \ldots, e_d$ is a complete system of primitive pairwise orthogonal idempotents of $F^\lambda G$. Furthermore, for $1 \leq r, s \leq d$ we have

$$F^\lambda Ge_r \cong F^\lambda Ge_s \Leftrightarrow Ae_r \cong Ae_s.$$ 

Applying Lemma 1.1 and [10, Lemma 62.28, p. 508], we finish the proof.

Corollary. Keep the notation of Theorem 1.4 and suppose that $F^\lambda H$ is a field. Then every simple $F^\lambda G$-module is isomorphic to one of the ideals $V_{p^n-1,j}$; moreover, any ideal $V_{p^n-1,j}$, $1 \leq j \leq d$, is minimal.

2. Projective representation types of finite groups. A group $G$ is said to be of finite (resp. infinite) PFR-type if the number of indecomposable projective $F$-representations of the group $G$ with a cocycle $\lambda$ is finite (resp. infinite) for any $\lambda \in Z^2(G, F^*)$. Other groups are said to be of mixed PFR-type.

Let $\Gamma$ and $\Gamma'$ be equivalent projective matrix $F$-representations of $G$ with a cocycle $\lambda$. Then there exists an invertible matrix $C$ over $F$ and a mapping $\alpha : G \to F^*$ such that $C^{-1} \Gamma(g)C = \alpha_g \Gamma'(g)$ for all $g \in G$. In this case,

$$\lambda_{a,b} = \frac{\alpha_a \alpha_b}{\alpha_{ab}} \lambda_{a,b}$$ 

for all $a, b \in G$. Hence, $\alpha$ is a linear $F$-character of the group $G$. But the number of linear $F$-characters of $G$ is finite. Therefore, the number of pairwise inequivalent indecomposable projective $F$-representations of $G$ with a cocycle $\lambda$ is finite if and only if the algebra $F^\lambda G$ is of finite representation.
type. This allows one to define the type of projective $F$-representations of $G$ as in the Introduction.

Applying Lemma 1.3 we may establish some connection between PFR-type of a group $G$ and PFR-type of a Sylow $p$-subgroup $G_p$ of $G$. If $G_p$ is of finite (resp. infinite) PFR-type, then so is $G$. Suppose $G_p$ is of mixed PFR-type. In view of Corollary 3 to Proposition 1.3, $G_p$ is not cyclic. By Lemma 1.4 the group algebra $FG$ is of infinite representation type. It follows that $G$ is not of finite PFR-type. If $G$ is of finite PFR-type, then by Lemma 1.4, $G_p$ is cyclic, and hence, in view of Corollary 3 to Proposition 1.3, $G_p$ is of finite PFR-type. If $G$ is of infinite PFR-type, then $G_p$ is not of finite PFR-type. If $G$ is of mixed PFR-type, then $G_p$ is also of mixed PFR-type.

Let $G$ be a finite group and $p | |G^*|$. The group $G/G^*$ can be written as a direct product of its Sylow $q$-subgroups $G_q G'/G'$, where $G_q$ is a Sylow $q$-subgroup of $G$ and $q$ is a prime divisor of $|G : G'|$. Denote by $C_p$ a Sylow $p$-subgroup of $G'$. We shall assume that $C_p \subseteq G_p$ and $C_p \neq G_p$. Then $G'_p \subseteq C_p$, and hence $C_p \lhd G_p$. The group $G_p/C_p$ is isomorphic to the Sylow $p$-subgroup $G_p G'/G'$ of $G/G'$. Let $\varphi : G \to G/G'$ be the canonical homomorphism, $\psi : G/G' \to G_p G'/G'$ a projector and $\chi : G_p G'/G' \to G_p/C_p$ the isomorphism defined by $\chi(aG') = aC_p$ for any $a \in G_p$. Then

$$ (2.1) \quad f = \chi \psi \varphi $$

is a homomorphism of $G$ onto $G_p/C_p$. The restriction of $f$ to $G_p$ is the canonical homomorphism of $G_p$ onto $G_p/C_p$.

**Lemma 2.1.** Let $H = G_p/C_p$, $f : G \to H$ be the epimorphism (2.1), $\mu \in Z^2(H, F^*)$ and $\lambda_{a,b} = \mu_{f(a),f(b)}$ for any $a,b \in G$. Then $\lambda \in Z^2(G, F^*)$ and $\lambda_{x,y} = \lambda_{y,x} = 1$ for all $x \in G_p$, $y \in C_p$. If $V = F^\lambda G_p \cdot \text{rad} F C_p$, then $V$ is an ideal of the algebra $F^\lambda G_p$ and $F^\lambda G_p/V \cong F^\mu H$.

**Proof.** Direct calculation. ■

**Theorem 2.1.** Suppose $i_F \neq 0$, $G$ is a finite group, $p | |G'|$ and $G_p/C_p$ is a direct product of $s$ cyclic $p$-subgroups for $C_p \neq G_p$. Then:

1. If $C_p$ is not cyclic or $s \geq i_F + 2$, then $G$ is of infinite PFR-type.
2. If $G_p$ is cyclic, then $G$ is of finite PFR-type.
3. If $C_p$ is a cyclic group and $G_p$ is not a cyclic group and $1 \leq s \leq i_F$, then $G$ is of mixed PFR-type.
4. Suppose $C_p = \langle c \rangle$, $G_p/C_p = \langle a_1 C_p \rangle \times \ldots \times \langle a_s C_p \rangle$ and $s = i_F + 1$. If $c \in \langle a_r \rangle$ for some $r \in \{1, \ldots, s\}$, then $G$ is of mixed PFR-type. If $c \not\in \langle a_j \rangle$ for every $j \in \{1, \ldots, s\}$ and $C_2 \neq C_2'$ for $p = 2$ then $G$ is of infinite PFR-type.

**Proof.** The assertion for $p \neq 2$ follows from Theorem 1.2 and Lemmas 1.5, 2.1. Now we turn to the case when $p$ is an arbitrary prime.
ments (1)–(3) follow from Lemmas 1.2–1.5, 2.1 and Corollary 3 to Proposition 1.3.

We prove (4). Let
\[ c = a_1^{m_1}, \quad H = G_p/C_p, \quad \overline{H} = G_p/\langle a_1 \rangle. \]
Then
\[ \overline{H} \cong H/(\langle a_1 \rangle/C_p) \cong \langle a_2 C_p \rangle \times \ldots \times \langle a_s C_p \rangle. \]
There is a cocycle \( \mu \in Z^2(H, F^*) \) such that \( F^\mu \overline{H} \) is a field. Let \( \varphi : G_p \to \overline{H} \) be the canonical homomorphism. Put \( \mu_{x,y} = \overline{\varphi(x), \varphi(y)} \) for any \( x, y \in G_p \). Then \( \mu \in Z^2(G_p, F^*) \). Let \( \{ u_x : x \in G_p \} \) be a natural \( F \)-basis of the algebra \( F^\mu G_p \). We have
\[ u_{a_1} = u_c, \quad u_{c^j} = u_e, \]
\[ \text{rad} F^\mu G_p = F^\mu G_p(u_{a_1} - u_c) \quad \text{and} \quad F^\mu G_p/\text{rad} F^\mu G_p \cong F^\mu \overline{H}. \]
Let \( \pi : G_p \to G_p/C_p \) be the canonical homomorphism. If \( \pi(x) = \pi(x') \) then \( \varphi(x) = \varphi(x') \).
It follows that the formula \( \nu_{\pi(x), \pi(y)} = \overline{\varphi(x), \varphi(y)} \), where \( x, y \in G_p \), gives a cocycle \( \nu \in Z^2(H, F^*) \). In view of Lemma 2.1 there is a cocycle \( \lambda \in Z^2(G, F^*) \) such that \( \lambda_{a,b} = \nu_{f(a), f(b)} \) for all \( a, b \in G \), where \( f \) is the epimorphism (2.1).
If \( a, b \in G_p \) then \( \lambda_{a,b} = \nu_{a,b} = \mu_{a,b} \). It follows that \( F^\lambda G_p \cong F^\mu G_p \), and hence \( F^\lambda G_p \) is a uniserial algebra. Applying Theorem 1.1 we conclude that \( F^\lambda G \) is of finite representation type. But \( G_p \) is not cyclic. Therefore, by Lemma 1.4 the group algebra \( FG \) is of infinite representation type. Thus, the group \( G \) is of mixed PFR-type.

Let \( |a_j C_p| = p^{m_j} \) and
\[ a_j^{p^{m_j}} = c^{p^{m_j}} \]
for every \( j \in \{1, \ldots, s\} \). If \( p \neq 2 \) then by Lemma 1.7, \( G \) is of infinite PFR-type. Suppose \( p = 2 \), \( G_2' \neq C_2 \), \( H = \langle c^2 \rangle \) and \( \lambda \in Z^2(G, F^*) \). Then \( G_2' \subset H \) and \( G_2/H = \langle c \rangle \times \langle a_1 H \rangle \times \ldots \times \langle a_s H \rangle \). In view of Lemma 1.5, \( F^\lambda H \) is a group algebra and the set \( V = F^\lambda G_2 \cdot \text{rad} F^\lambda H \) is a two-sided ideal of the algebra \( F^\lambda G_2 \). The quotient algebra \( F^\lambda G_2/V \) is a commutative twisted group algebra of the group \( G_2/H \) and the field \( F \). From Corollary 3 to Proposition 1.3 we conclude that \( F^\lambda G/V \) is of infinite representation type. From this and Lemma 1.3 it follows that \( G \) is of infinite PFR-type.

**Corollary 1.** Suppose \( i_F = \infty \). If \( C_p \) is a non-cyclic group then \( G \) is of infinite PFR-type. If \( C_p \) is cyclic and \( G_p \) is not cyclic then \( G \) is of mixed PFR-type. If \( G_p \) is a cyclic group then \( G \) is of finite PFR-type.

**Corollary 2.** Suppose \( i_F \neq 0, p \neq 2, G = G_p \times B, [G,G_p] = \langle c \rangle \) and \( [B,G_p] \neq \{c\} \). Suppose \( G_p/\langle c \rangle \) is a direct product of \( s \) cyclic subgroups for \( G_p \neq \langle c \rangle \). If \( 1 \leq s < i_F \) then \( G \) is of mixed PFR-type. If \( s \geq i_F + 1 \) then \( G \) is of infinite PFR-type. For \( G_p = \langle c \rangle \) the group \( G \) is of finite PFR-type.
Proof. Apply Theorems 1.4 and 2.1.

**Theorem 2.2.** Suppose \( i_F \neq 0 \), \( G \) is a finite group and \( p \mid |G'| \). Assume that \( G_p \) is abelian and \( C_p \) is cyclic. Let \( s \) be the number of invariants of \( G_p \). If \( s = 1 \) then \( G \) is of finite PFR-type. If \( 1 < s \leq i_F + 1 \) then \( G \) is of mixed PFR-type. If \( s \geq i_F + 2 \) then \( G \) is of infinite PFR-type.

Proof. From Lemma 1.3 and Corollary 3 to Proposition 1.3 we conclude that if \( s = 1 \) then \( G \) is of finite PFR-type, and if \( s \geq i_F + 2 \) then \( G \) is of infinite PFR-type. Let \( 1 < s \leq i_F + 1 \) and \( C_p = \langle c \rangle \). We have \( G_p/C_p = \langle a_1C_p \rangle \times \ldots \times \langle a_tC_p \rangle \), \( t \leq s \). If \( t \leq i_F \) then by Lemmas 1.3 and 2.1, \( G \) is of mixed PFR-type. Suppose that \( t = i_F + 1 \). If \( c \notin \langle a_i \rangle \) for all \( i \in \{1, \ldots, t\} \) then \( G_p/H = \langle cH \rangle \times \langle a_1H \rangle \times \ldots \times \langle a_tH \rangle \), where \( H = \langle c^p \rangle \). This contradiction shows that \( c \in \langle a_r \rangle \) for some \( r \in \{1, \ldots, t\} \). In this case, \( G \) is also of mixed PFR-type, by Lemmas 1.3 and 2.1, Corollary 3 to Proposition 1.3 and Theorem 2.1.

**Proposition 2.1.** Suppose \( i_F = 0 \). If \( G_p \) is not cyclic then \( G \) is of infinite PFR-type. If \( G_p \) is cyclic then \( G \) is of finite PFR-type.

Proof. The algebra \( F^\lambda G_p \) is the group algebra \( FG_p \) for every \( \lambda \in Z^2(G, F^*) \) (see [26, p. 43]). It remains to apply Lemmas 1.3 and 1.4.

We remark that Proposition 2.1 was, in fact, formulated in [7].

Two groups are said to be PFR-isotypic if they are of the same PFR-type. From the above results, we will derive necessary and sufficient conditions for \( G \) and \( G_p \) to be PFR-isotypic. In view of Lemmas 1.3, 1.5 and 2.1 groups \( G \) and \( G_p \) are PFR-isotypic if \( C_p = G'_p \).

**Proposition 2.2.** Let \( G \) be a finite group with \( p \mid |G'| \) and \( G_p \) an abelian group, and \( s \) the number of invariants of \( G_p \). If \( C_p \) is cyclic then \( G \) and \( G_p \) are PFR-isotypic. If \( C_p \) is not cyclic then \( G \) and \( G_p \) are PFR-isotypic if and only if \( s \geq i_F + 2 \).

Proof. If \( C_p \) is cyclic we apply Theorem 2.2. If \( C_p \) is not cyclic we apply the Corollary of Theorem 1.2 and Theorem 2.2.

**Proposition 2.3.** Suppose \( i_F \neq 0 \), \( G \) is a finite group, \( p \mid |G'| \), and \( s \) is the number of invariants of \( G_p/G'_p \). Assume that \( G_p \) is non-abelian and if \( G'_p \) is cyclic then \( s \neq i_F + 1 \) for \( p = 2 \). The groups \( G \) and \( G_p \) are PFR-isotypic if and only if one of the following conditions holds:

1. \( s \geq i_F + 2 \) or \( G'_p \) is non-cyclic;
2. \( s \leq i_F + 1 \) and \( C_p \) is cyclic;
3. \( s = i_F + 1 \), \( G'_p \) is cyclic, \( C_p \) is non-cyclic and \( G_p/G'_p = \langle b_1G'_p \rangle \times \ldots \times \langle b_sG'_p \rangle \), where \( G_p \not\subset \langle b_j \rangle \) for every \( j \in \{1, \ldots, s\} \).
Proof. Apply Theorem 2.1. If condition (1) holds, then $G_p$ is of infinite PFR-type. If condition (2) holds and $G'_p \neq C_p$, then by the same arguments as in the proof of Theorem 2.2 we can establish that $G$ is of mixed PFR-type. Suppose that conditions (1) and (2) do not hold. Then $s \leq i_F + 1$, $G'_p$ is cyclic and $C_p$ is non-cyclic. In this case, $G$ is of infinite PFR-type. The subgroup $G_p$ is of infinite PFR-type if and only if $s = i_F + 1$ and $G_p / G'_p = \langle b_1 G'_p \rangle \times \ldots \times \langle b_s G'_p \rangle$, where $G'_p \not\subset \langle b_j \rangle$ for every $j \in \{1, \ldots, s\}$. ■

Corollary. Suppose $i_F = \infty$, $G$ is a finite group and $p \mid |G'|$. The groups $G$ and $G_p$ are PFR-isotypic if and only if $C_p$ is cyclic or $G'_p$ is not cyclic.

REFERENCES


[37] K. Sobolewska, On the number of indecomposable representations with a given degree of a twisted group algebra over a field of characteristic $p$, Slupskie Prace Mat.-Fiz. 2 (2002), 81–89.

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