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A PINCHING THEOREM ON COMPLETE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTORS

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Abstract. Let M be an n-dimensional complete immersed submanifold with parallel mean curvature vectors in an $(n+p)$-dimensional Riemannian manifold N of constant curvature $c>0$. Denote the square of length and the length of the trace of the second fundamental tensor of M by S and H, respectively. We prove that if

$$S \le \frac{1}{n-1}\,H^2 + 2c, \qquad n \ge 4,$$

or

$$S \le \frac{1}{2}\,H^2 + \min\left(2, \frac{3p-3}{2p-3}\right)c, \qquad n = 3,$$

then M is umbilical. This result generalizes the Okumura–Hasanis pinching theorem to the case of higher codimensions.

1. Introduction. Let M be an n-dimensional complete immersed submanifold with parallel mean curvature vector in an $(n+p)$-dimensional Riemannian manifold N of constant curvature $c>0$, and let h denote the second fundamental tensor of M. We denote the square of the length of h by S and the length of the trace of h by H. It is well known that M is a totally umbilical submanifold of N if and only if $S = H^2/n$.

When $p=1$, i.e., when M is a complete hypersurface of N, a classical pinching theorem has been obtained by Okumura and Hasanis. They proved [O], [H]: If $n \ge 3$ and

$$S \le \frac{1}{n-1}\,H^2 + 2c,$$

then M is umbilical.

The purpose of this paper is to generalize the above result to the case of higher codimensions. We first prove that, when $n \ge 4$, Okumura–Hasanis's pinching theorem also holds in the case of high codimension.

THEOREM 1. *Let M be an n-dimensional complete immersed submanifold with parallel mean curvature vector in an $(n+p)$-dimensional Riemannian manifold N of constant curvature $c>0$. If $n \ge 4$, $p \ge 2$, and*

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$$S \leq \frac{1}{n-1} H^2 + 2c,$$

then M is totally umbilical.

In the case of $n = 3$ and $p \geq 2$, we have the following pinching theorem, which is slightly weaker than Okumura–Hasanis's result in the case of $n = 3$ and $p = 1$.

THEOREM 2. *Let M be an 3-dimensional complete immersed submanifold with parallel mean curvature vector in a $(3+p)$-dimensional Riemannian manifold N of constant curvature $c > 0$. If $p \geq 2$ and*

$$S \leq \frac{1}{2} H^2 + \min\left(2, \frac{3p-3}{2p-3}\right) c,$$

then M is totally umbilical.

We refer the reader to [CN] for other related results in the case of submanifolds in Euclidean spheres. In Section 2 we prepare some fundamental formulas, and in Section 3 we prove two lemmas. The proof of Theorems 1 and 2 is given in Section 4.

2. Fundamental formulas. We shall use the following convention on the ranges of indices:

$$1 \leq A, B, C, \ldots \leq n + p,$$
$$1 \leq i, j, k, \ldots \leq n,$$
$$n + 1 \leq u, v, y, \ldots \leq n + p.$$

Let M be an n-dimensional complete immersed submanifold with parallel mean curvature vector in an $(n+p)$-dimensional Riemannian manifold N. We choose a local orthonormal frame field $e_1, \ldots, e_n, e_{n+1} \ldots, e_{n+p}$ in N such that, when restricted to M, $e_1, \ldots, e_n$ are tangent to M, and consequently, $e_{n+1}, \ldots, e_{n+p}$ will be the normal frame on M. Let $\omega_1, \ldots, \omega_{n+p}$ be the dual frame. Then the structure equations of N are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}$$

where K_{ABCD} is the Riemannian curvature of N. If we restrict these forms to M, then $\omega_u = 0$. Thus,

$$0 = d\omega_u = \sum_j \omega_{uj} \wedge \omega_j.$$

By Cartan's lemma we can write

$$\omega_{ui} = \sum_j h^u_{ij}\omega_j, \qquad h^u_{ij} = h^u_{ji}. \tag{2.3}$$

From these formulas we obtain

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

$$R_{ijkl} = K_{ijkl} + \sum_u (h^u_{ik}h^u_{jl} - h^u_{il}h^u_{jk}), \tag{2.4}$$

$$d\omega_{uv} = \sum_y \omega_{uy} \wedge \omega_{yv} - \frac{1}{2}\sum_{k,l} R_{uvkl}\omega_k \wedge \omega_l, \tag{2.5}$$

$$R_{uvkl} = K_{uvkl} + \sum_i (h^u_{ik}h^v_{il} - h^u_{il}h^v_{ik}), \tag{2.6}$$

where R_{ijkl} is the Riemannian curvature of M. The symmetric 2-form

$$h = \sum_{i,j,u} h^u_{ij}\omega_i\omega_j e_u$$

and the vector

$$q = \sum_{i,u} h^u_{ii} e_u \tag{2.7}$$

are the second fundamental form and the mean curvature vector of M, respectively. If q is parallel in the normal bundle of M, then M is called a *submanifold with parallel mean curvature vector*. The *length* of q is defined by

$$H = \|\mathrm{tr}\, h\| = \Big(\sum_u \Big(\sum_i h^u_{ii}\Big)^2\Big)^{1/2}.$$

Define the covariant derivative Dh of h (with components h^u_{ijk}) by

$$\sum_k h^u_{ijk}\omega_k = dh^u_{ij} + \sum_m h^u_{im}\omega_{mj} + \sum_m h^u_{mj}\omega_{mi} + \sum_v h^v_{ij}\omega_{vu}. \tag{2.8}$$

Taking the exterior derivative of (2.3) and using the structure equations in (2.1) to (2.6), one can show [Y1]

$$h^u_{ijk} - h^u_{ikj} = K_{uijk}. \tag{2.9}$$

Next, we take the exterior derivative of (2.8) and define h^u_{ijkl} by

$$\sum_k h^u_{ijkl}\omega_l = dh^u_{ijk} + \sum_l h^u_{ljk}\omega_{li} + \sum_l h^u_{ilk}\omega_{lj} + \sum_l h^u_{ijl}\omega_{lk} + \sum_v h^v_{ijk}\omega_{vu}.$$

Then we can show [Y1]

$$h^u_{ijkl} - h^u_{ijlk} = \sum_m h^u_{im} R_{mjkl} + \sum_m h^u_{mj} R_{mikl} + \sum_v h^v_{ij} R_{vukl}. \tag{2.10}$$

From now on, we assume that N is a Riemannian manifold of constant curvature c. Then we have

$$K_{ABCD} = (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})c. \tag{2.11}$$

In this case we deduce from (2.9) that

$$h^u_{ijk} = h^u_{ikj}. \tag{2.12}$$

We further assume that M is a submanifold with parallel mean curvature vector. A direct computation shows that $H = \text{const}$ under this assumption.

We choose $e_{n+1} = q/\|q\|$ with q defined in (2.7) in our local orthonormal frame, and denote the $n \times n$ matrix (h^u_{ij}) by H_u. Then clearly $\operatorname{tr} H_v = 0$ if $v \neq n+1$. Therefore,

$$H = \sum_i h^{n+1}_{ii}.$$

Since e_{n+1} is parallel in the normal bundle of M, we have

$$\omega_{n+1,v} = 0. \tag{2.13}$$

Taking the exterior derivative of (2.13) and using (2.2) yields

$$\sum_i \omega_{n+1,i} \wedge \omega_{vi} = 0,$$

which, together with (2.5), implies

$$R_{n+1,vkl} = 0. \tag{2.14}$$

From (2.6) and (2.14) we have

$$H_{n+1}H_u = H_u H_{n+1}. \tag{2.15}$$

Define

$$D = \operatorname{tr} H^2_{n+1} = \sum_{i,j} (h^{n+1}_{ij})^2,$$

$$Q = S - D = \sum_{v\neq n+1} \operatorname{tr} H^2_v = \sum_{i,j,v\neq n+1} (h^v_{ij})^2.$$

We now compute the Laplacian of D and Q:

$$\Delta D = \sum_i D_{ii}, \qquad \Delta Q = \sum_i Q_{ii}.$$

From (2.10)–(2.12) and (2.15), one can show [Y1]

$$\Delta h^u_{ij} = \sum_k h^u_{ijkk} = \sum_{k,m} h^u_{mk} R_{mijk} + \sum_{k,m} h^u_{im} R_{mkjk} + \sum_{k,v} h^v_{ik} R_{vujk}. \tag{2.16}$$

Choosing $u = n+1$ in (2.16) and using (2.14) yields

$$\Delta h_{ij}^{n+1} = \sum_{k,m} h_{mk}^{n+1} R_{mijk} + \sum_{k,m} h_{im}^{n+1} R_{mkjk}.$$

Therefore,

$$\begin{aligned} \frac{1}{2}\,\Delta D &= \sum_{i,j} h_{ij}^{n+1} \Delta h_{ij}^{n+1} + \sum_{i,j,k} (h_{ijk}^{n+1})^2 \\ &\geq \sum_{i,j,k,m} h_{ij}^{n+1} h_{mk}^{n+1} R_{mijk} + \sum_{i,j,k,m} h_{ij}^{n+1} h_{im}^{n+1} R_{mkjk}. \end{aligned} \tag{2.17}$$

For a given point $p \in M$, we choose the frame field $e_1, \ldots, e_n$ so that the matrix (h_{ij}^{n+1}) is diagonal at p. Thus we may assume that at p,

$$h_{ij}^{n+1} = L_i \delta_{ij}.$$

In this frame field, the inequality in (2.17) can be simplified at p:

$$\Delta D \geq \sum_{i,j} (L_i - L_j)^2 R_{ijij}. \tag{2.18}$$

Let

$$f^2 = D - \frac{1}{n}\,H^2.$$

Substituting (2.4) and (2.11) into (2.17) and using Okumura's computation [O], we can get the following estimate:

$$\begin{aligned} \frac{1}{2}\,\Delta D \geq f^2 &\left(cn + \frac{1}{n}\,H^2 - \frac{n-2}{\sqrt{n(n-1)}}\,|H| f - f^2 \right) \\ &- \sum_{v \neq n+1} [\operatorname{tr}(H_{n+1} H_v)]^2 \end{aligned} \tag{2.19}$$

Choosing $u = v \neq n+1$ in (2.16), one can show

$$\begin{aligned} \sum_{i,j,v \neq n+1} h_{ij}^v \Delta h_{ij}^v = &\sum_{u,v \neq n+1} \operatorname{tr}(H_u H_v - H_v H_u)^2 - \sum_{u,v \neq n+1} [\operatorname{tr}(H_u H_v)]^2 \\ &+ ncQ + \sum_{v \neq n+1} H \operatorname{tr}(H_v^2 H_{n+1}) - \sum_{v \neq n+1} [\operatorname{tr}(H_v H_{n+1})]^2. \end{aligned}$$

Using the techniques of [CDK], one can prove

$$\sum_{u,v \neq n+1} \operatorname{tr}(H_u H_v - H_v H_u)^2 - \sum_{u,v \neq n+1} [\operatorname{tr}(H_u H_v)]^2 \geq -\left(2 - \frac{1}{p-1}\right) Q^2,$$

and, when combined with (2.16), this estimate implies

$$\begin{aligned} \frac{1}{2}\,\Delta Q &= \sum_{i,j,v \neq n+1} h_{ij}^v \Delta h_{ij}^v + \sum_{i,j,k,v \neq n+1} (h_{ijk}^v)^2 \\ &\geq -\left(2 - \frac{1}{p-1}\right) Q^2 + ncQ + \sum_{v \neq n+1} H \operatorname{tr}(H_v^2 H_{n+1}) - \sum_{v \neq n+1} [\operatorname{tr}(H_v H_{n+1})]^2. \end{aligned} \tag{2.20}$$

3. Lemmas

LEMMA 1. *Under the assumptions of Theorems* 1 *and* 2, *if*, *in addition*, M *is umbilical with respect to* e_{n+1}, *then* M *must be totally umbilical.*

Proof. Since M is umbilical with respect to e_{n+1}, we have

$$h_{ij}^{n+1} = L\delta_{ij} \tag{3.1}$$

for some constant L, and $\operatorname{tr} H_v = 0$ for $v \neq n+1$. A direct computation shows that in this case,

$$\sum_{v\neq n+1} H \operatorname{tr}(H_v^2 H_{n+1}) - \sum_{v\neq n+1} [\operatorname{tr}(H_v H_{n+1})]^2 = \frac{H^2}{n}\, Q. \tag{3.2}$$

So, by substituting (3.1) into (2.20) and using (3.2) we get

$$\frac{1}{2}\,\Delta Q \geq Q\left[-\left(2 - \frac{1}{p-1}\right)Q + nc + \frac{H^2}{n}\right]. \tag{3.3}$$

This is our main estimate. We shall come back to it later.

By assumption, we have

$$S \leq \frac{1}{n-1}\, H^2 + 2c, \qquad n \geq 4, \tag{3.4}$$

and

$$S \leq \frac{1}{2}\, H^2 + \min\left(2, \frac{3p-3}{2p-3}\right)c, \qquad n = 3. \tag{3.5}$$

Also, from (3.1) we have

$$D = \frac{1}{n}\, H^2. \tag{3.6}$$

(3.4)–(3.6) imply

$$Q \leq \frac{1}{n(n-1)}\, H^2 + 2c, \qquad n \geq 4, \tag{3.7}$$

and

$$Q \leq \frac{1}{6}\, H^2 + \min\left(2, \frac{3p-3}{2p-3}\right)c, \qquad n = 3. \tag{3.8}$$

Since

$$\frac{p-1}{2p-3} > \frac{1}{n-1} \qquad \text{when } n \geq 3 \text{ and } p \geq 2, \tag{3.9}$$

and

$$\frac{n(p-1)}{2p-3} > 2 \qquad \text{when } n \geq 3 \text{ and } p \geq 2, \tag{3.10}$$

(3.7)–(3.10) imply that for $n \geq 3$,

$$Q < \frac{p-1}{n(2p-3)}\, H^2 + \frac{n(p-1)}{2p-3}\, c = \frac{H^2}{n(2-\frac{1}{p-1})} + \frac{n}{2-\frac{1}{p-1}}\, c,$$

i.e.

$$A = -\Big(2 - \frac{1}{p-1}\Big) Q + nc + \frac{H^2}{n} > 0. \tag{3.11}$$

We now come back to (3.3), and deduce from (3.11) that

$$\frac{1}{2}\, \Delta Q \geq AQ, \qquad A > 0. \tag{3.12}$$

Now since S is bounded, so are Q and the Ricci curvature of M. We apply Yau's generalized maximal principle [Y2] to conclude that there exists a sequence $\{p_s\}$ of points of M such that

$$\lim_{s\to\infty} Q(p_s) = \sup_M Q, \tag{3.13}$$

$$\lim_{s\to\infty} \Delta Q(p_s) \leq 0. \tag{3.14}$$

From (3.12)–(3.14), we have

$$0 \geq A \sup_M Q.$$

This implies that $\sup_M Q = 0$, i.e., $Q \equiv 0$. Hence M is umbilical with respect to e_u, and consequently, M is totally umbilical. ■

LEMMA 2. *Let* $a_1, \ldots, a_n$, b *be* $n+1$ $(n > 1)$ *real numbers satisfying*

$$\Big(\sum_{i=1}^n a_i\Big)^2 \geq (n-1) \sum_{i=1}^n a_i^2 + b.$$

Then for $1 \leq i \neq j \leq n$,

$$2a_i a_j \geq \frac{b}{n-1}.$$

Proof. See [C, p. 55]. ■

4. Proof of the theorems. According to Lemma 1, we need only prove that M is umbilical with respect to e_{n+1}. In other words, we need to show $D = H^2/n$. Suppose $D \neq H^2/n$. Then

$$\sup_M D > \frac{1}{n} H^2.$$

According to Yau's generalized maximal principle [Y2], there exists a sequence $\{p_s\} \subset M$ such that

$$\lim_{s\to\infty} D(p_s) = \sup_M D,$$

$$\lim_{s\to\infty} \Delta D(p_s) \leq 0.$$

For each s, we choose a local orthonormal frame $e_1^{p_s}, \dots, e_n^{p_n}$ in a neighborhood of p_s. We denote the components of h in this frame by h^v_{ij,p_s}, and $h^{n+1}_{ij,\,p_s} = L_{i,\,p_s}\delta_{ij}$. Moreover, we denote the Riemannian curvature tensor by R_{ijkl,p_s}, the n-matrix (h^v_{ij,p_s}) by H_{v,p_s} with $v \neq n+1$, and the n-matrix $(L_{i,p_s}\delta_{ij})$ by H_{n+1,p_s}. Since S is bounded, so are the sequences $\{(h^v_{ij,p_s})\}_{s\in\mathbb{Z}^+}$ and $\{(L_{i,p_s}\delta_{ij})\}_{s\in\mathbb{Z}^+}$. Therefore, by choosing subsequences if necessary, we can assume they are convergent, and we can write

$$\lim_{s\to\infty} h^v_{ij,p_s} = \bar{h}^v_{ij}, \tag{4.1}$$

$$\lim_{s\to\infty} L_{i,p_s} = \bar{L}_i, \tag{4.2}$$

$$\lim_{s\to\infty} R_{ijkl,p_s} = \bar{R}_{ijkl}. \tag{4.3}$$

Since $H = \text{const}$, we have

$$\sum_i \bar{L}_i = H.$$

If we define

$$\bar{Q} = \sum_{i,j,v\neq n+1} (\bar{h}^v_{ij})^2, \qquad \bar{D} = \sum_i (\bar{L}_i)^2,$$

$$\bar{S} = \bar{Q} + \bar{D}, \qquad \bar{f} = \bar{D} - \frac{1}{n}H^2,$$

$$\bar{H}_v = (\bar{h}^v_{ij}), \qquad \bar{H}_{n+1} = (\bar{L}_i\delta_{ij}),$$

$$\bar{R}_{ijkl} = \bar{L}_i\bar{L}_j + \sum_{v\neq n+1} (\bar{h}^v_{ii}\bar{h}^v_{jj} - (\bar{h}^v_{ij})^2) + c,$$

then, in addition to (4.1)–(4.3), we have

$$\lim_{s\to\infty} Q(p_s) = \bar{Q}, \qquad \lim_{s\to\infty} D(p_s) = \bar{D},$$

$$\lim_{s\to\infty} S(p_s) = \bar{S}, \qquad \lim_{s\to\infty} f(p_s) = \bar{f},$$

$$\lim_{s\to\infty} \operatorname{tr}(H_{v,p_s}H_{n+1,p_s}) = \operatorname{tr}(\bar{H}_v\bar{H}_{n+1}).$$

From (2.18) and (2.19), we have

$$\Delta D(p_s) \geq \sum_{i,j} (L_{i,p_s} - L_{j,p_s})^2 R_{ijij,p_s}.$$

Therefore

$$\frac{1}{2}\,\Delta D(p_s) \geq f^2(p_s)\left(cn + \frac{1}{n}\,H^2 - \frac{n-2}{\sqrt{n(n-1)}}\,|H|f(p_s) - f^2(p_s)\right) - \sum_{v\neq n+1} [\operatorname{tr}(H_{n+1,p_s}H_{v,p_s})]^2.$$

Letting $s \to \infty$, we have

$$\sum_{i,j} (\bar{L}_i - \bar{L}_j)^2 \bar{R}_{ijij} \le 0, \tag{4.4}$$

$$\bar{f}^2 \left(cn + \frac{1}{n} H^2 - \frac{n-2}{\sqrt{n(n-1)}} |H| \bar{f} - \bar{f}^2 \right) - \sum_{v \ne n+1} [\mathrm{tr}(\overline{H}_{n+1} \overline{H}_v)]^2 \le 0. \tag{4.5}$$

We now divide the rest of the proof into two cases.

CASE 1: $Q > 0$. Since

$$S(p_s) \le \frac{1}{n-1} H^2 + 2c,$$

we have

$$\bar{S} \le \frac{1}{n-1} H^2 + 2c,$$

and consequently,

$$H^2 \ge (n-1)\bar{S} - 2(n-1)c,$$

i.e.,

$$\left(\sum_i \bar{L}_i \right)^2 \ge (n-1) \sum_i \bar{L}_i^2 + (n-1) \sum_{i,j} (\bar{h}^v_{ij})^2 - 2(n-1)c. \tag{4.6}$$

Applying Lemma 2 to (4.6) yields

$$2\bar{L}_i \bar{L}_j \ge \frac{(n-1) \sum_{k,l,v \ne n+1} (\bar{h}^v_{kl})^2 - 2(n-1)c}{n-1},$$

or

$$\bar{L}_i \bar{L}_j + c \ge \frac{1}{2} \sum_{k,l,v \ne n+1} (\bar{h}^v_{kl})^2. \tag{4.7}$$

Hence, for $i \ne j$,

$$\begin{aligned} \bar{R}_{ijij} &= \bar{L}_i \bar{L}_j + c + \sum_{i,j,v \ne n+1} (\bar{h}^v_{ii} \bar{h}^v_{jj} - (\bar{h}^v_{ij})^2) \\ &\ge \frac{1}{2} \sum_{k,l,v \ne n+1} (\bar{h}^v_{kl})^2 + \frac{1}{2} \left(-2 \sum_{v \ne n+1} (\bar{h}^v_{ij})^2 + 2 \sum_{v \ne n+1} \bar{h}^v_{ii} \bar{h}^v_{jj} \right) \\ &\ge \frac{1}{2} \sum_{v \ne n+1;\, k \ne i,j \text{ or } l \ne i,j} (\bar{h}^v_{kl})^2 + \frac{1}{2} (\bar{h}^v_{ii} + \bar{h}^v_{jj})^2 \ge 0. \end{aligned} \tag{4.8}$$

From the above inequality and (4.4) we obtain

$$\sum_{i,j} (\bar{L}_i - \bar{L}_j)^2 \bar{R}_{ijij} = 0. \tag{4.9}$$

We claim that for $i < j$ there is at most one $\bar{R}_{ijij}$ equal to zero, and the others are positive. If not, we may assume $\bar{R}_{ijij} = 0$ and $\bar{R}_{pqpq} = 0$ for two pairs (i,j) and (p,q) with $i \neq p$, $i<j$ and $p<q$. Then from (4.7) we have

$$\bar{h}^v_{ii} + \bar{h}^v_{jj} = 0, \quad \bar{h}^v_{kl} = 0, \quad k \neq i,j \text{ or } l \neq i,j, \quad v \neq n+1, \tag{4.10}$$

and

$$\bar{h}^v_{pp} + \bar{h}^v_{qq} = 0, \quad \bar{h}^v_{pq} = 0, \quad k \neq p,q \text{ or } l \neq p,q, \quad v \neq n+1.$$

From (4.9) and (4.10) we can deduce that

$$\bar{h}^v_{kl} = 0, \quad 1 \le k,l \le n, \quad v \neq n+1,$$

and consequently, $\bar{Q} = 0$, contrary to the assumption of Case 1. Hence the claim is proven. We now assume without loss of generality that only $\bar{R}_{ijij} = 0$. Then from (4.8) we have

$$\bar{L}_1 = \bar{L}_3 = \ldots = \bar{L}_n, \quad \bar{L}_2 = \bar{L}_3 = \ldots = \bar{L}_n.$$

Hence $\bar{L}_2 = \bar{L}_2 = \ldots = \bar{L}_n$, i.e., $D = H^2/n$. However, this contradicts the assumption $\sup_M D > \frac{1}{n}H^2$. This completes the proof in Case 1.

CASE 2: $\bar{Q} = 0$. From $\bar{Q} = 0$ it is easy to see that

$$\sum_{v \neq n+1} [\mathrm{tr}(\bar{H}_{n+1}\bar{H}_v)]^2 = 0.$$

Hence from (4.5) we get

$$\bar{f}^2\left(cn + \frac{1}{n}H^2 - \frac{n-2}{\sqrt{n(n-1)}}|H|\bar{f} - \bar{f}^2\right) \le 0. \tag{4.11}$$

But $\bar{f} = \bar{D} - \frac{1}{n}H^2 > 0$, so

$$cn + \frac{1}{n}H^2 - \frac{n-2}{\sqrt{n(n-1)}}|H|\bar{f} - \bar{f}^2 \le 0. \tag{4.12}$$

Solving the equality (4.11) we get

$$\bar{f} \ge \frac{2-n}{2\sqrt{n(n-1)}}|H| + \frac{1}{2}\sqrt{\frac{n}{n-1}H^2 + 4nc}. \tag{4.13}$$

On the other hand, from the fact $\bar{D} \le \frac{1}{n-1}H^2 + 2c$ we get

$$\bar{f} \le \sqrt{\frac{1}{n(n-1)}H^2 + 2c}.$$

Combining (4.12) and (4.13) gives

$$\sqrt{\frac{1}{n(n-1)}H^2 + 2c} \ge \frac{2-n}{2\sqrt{n(n-1)}}|H| + \frac{1}{2}\sqrt{\frac{n}{n-1}H^2 + 4nc}.$$

The above inequality implies

$$(n-2)^2c^2 \leq 0.$$

Therefore $n = 2$. However, this contradicts our hypothesis, and this completes the proof of Case 2.

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