STABLE FAMILIES OF ANALYTIC SETS

BY

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Abstract. We give a different proof of the well-known fact that any uncountable family of analytic subsets of a Polish space with the point-finite intersection property must contain a subfamily whose union is not analytic. Our approach is based on the Kunen–Martin theorem.

1. Introduction and notations. It is well known that any uncountable family of analytic subsets of a Polish space with the point-finite intersection property must contain a subfamily whose union is not analytic (see [1], [2], [3] and [5]). In [1], this (and in fact a much stronger) result is proved but the proof heavily depends on the Axiom of Choice. In [2], [3] and [5], the proofs are effective but the arguments are more complicated. In this note we give a short proof by using the Kunen–Martin theorem.

Notations. In what follows $X$ and $Y$ will be uncountable Polish spaces. By $\mathcal{N}$ we denote the Baire space. If $A \subseteq X \times Y$ and $U \subseteq Y$ is an arbitrary open set, we put

$$A(U) = \text{proj}_X \{A \cap (X \times U)\}.$$ 

All the other notations we use are standard (for more information we refer to [4]).

2. Stable families of analytic sets. Departing from standard terminology, we make the following definition.

Definition 1. A family $\mathcal{F} = (A_i)_{i \in I}$ of analytic subsets of $X$ will be called stable if for every $J \subseteq I$ the set $\bigcup_{i \in J} A_i$ is an analytic subset of $X$.

Clearly any subfamily of a stable family is stable. Furthermore any countable family of analytic sets is stable. There exist however uncountable stable families of analytic sets.

Example 1. Let $A \subseteq X$ be an analytic non-Borel set. By a classical result of Sierpinski (see [4, p. 201]), there exists a transfinite sequence $(B_\xi)_{\xi < \omega_1}$ of Borel sets such that $A = \bigcup_{\xi < \omega_1} B_\xi$. Clearly we may assume that

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the sequence \((B_\xi)_{\xi \in \omega_1}\) is increasing. As \(A\) is not Borel, there exists \(\Lambda \subseteq \omega_1\) uncountable such that \(B_\xi \subseteq B_\zeta\) for every \(\xi, \zeta \in \Lambda\) with \(\xi < \zeta\). Then the family \(\mathcal{F} = (B_\xi)_{\xi \in \Lambda}\) is an uncountable stable family of mutually different analytic sets (note that the members of \(\mathcal{F}\) are actually Borel sets).

DEFINITION 2. A family \(\mathcal{F} = (A_i)_{i \in I}\) of subsets of \(X\) is said to have the point-finite intersection property (abbreviated as p.f.i.p.) if for every \(x \in X\), the set \(I_x = \{i \in I : x \in A_i\}\) is finite.

As before, any subfamily of a family with the point-finite intersection property has the point-finite intersection property. We will show that stable families of analytic sets with the p.f.i.p. are necessarily countable. First a couple of lemmas. The one that follows is elementary.

**Lemma 3.** Let \(X\) and \(Y\) be Polish spaces. If \(A \in \Pi_1^1(X)\) and \(U \subseteq Y\) is open, then \(A \times U \in \Pi_1^1(X \times Y)\).

**Lemma 4.** Let \(X\) and \(Y\) be Polish spaces. Assume that \(A \subseteq X \times Y\) has closed sections (i.e. for every \(x \in X\), the set \(A_x = \{y \in Y : (x, y) \in A\}\) is closed) and moreover for every \(U \subseteq Y\) open the set \(A(U)\) is analytic. Then \(A\) is also analytic.

**Proof.** Let \(B = (V_n)_n\) be a countable base for \(Y\). Observe that \((x, y) \notin A\) if and only if there exists a basic open subset \(V_n\) of \(Y\) such that \(x \notin A(V_n)\) and \(y \in V_n\). It follows that

\[
(X \times Y) \setminus A = \bigcup_n (X \setminus A(V_n)) \times V_n
\]

and so, by Lemma 3, \(A\) is analytic. \(\blacksquare\)

We have the following stability result.

**Lemma 5.** Let \(\mathcal{F} = (A_i)_{i \in I}\) be a stable family of analytic subsets of \(X\) with the point-finite intersection property. Then for every Polish space \(Y\) and every family \((B_i)_{i \in I}\) of analytic subsets of \(Y\), the set

\[
A = \bigcup_{i \in I} (A_i \times B_i)
\]

is an analytic subset of \(X \times Y\).

**Proof.** Let \(\mathcal{F} = (A_i)_{i \in I}\) and \((B_i)_{i \in I}\) as above. As every \(B_i\) is analytic, there exists \(C_i \subseteq Y \times \mathcal{N}\) closed such that \(B_i = \text{proj}_Y C_i\). Define \(\tilde{A} \subseteq X \times Y \times \mathcal{N}\) by

\[
\tilde{A} = \bigcup_{i \in I} (A_i \times C_i).
\]

Clearly \(A = \text{proj}_{X \times Y} \tilde{A}\). Note that for every \(x \in X\) the section

\[
\tilde{A}_x = \{(y, z) \in Y \times \mathcal{N} : (x, y, z) \in \tilde{A}\}
\]
is exactly the set $\bigcup_{i \in I_x} C_i$. As the family $\mathcal{F}$ has the point-finite intersection property, for every $x \in X$ the section $\tilde{A}_x$ of $\tilde{A}$ is closed. Observe that for every $U \subseteq Y \times \mathcal{N}$ open, we have

$$
\tilde{A}(U) = \text{proj}_X \{\tilde{A} \cap (X \times U)\} = \{x \in X : \exists i \in I_x \text{ such that } C_i \cap U \neq \emptyset\} = \bigcup \{A_i : C_i \cap U \neq \emptyset\}.
$$

It follows directly from the stability of the family that $\tilde{A}(U)$ is analytic. By Lemma 4, $\tilde{A}$ is an analytic subset of $X \times Y \times \mathcal{N}$. Hence so is $A$. □

Let $\prec$ be a strict well-founded binary relation on $X$. By recursion, we define the rank function $\varrho_\prec : X \to \text{Ord}$ as follows. We set $\varrho_\prec(x) = 0$ if $x$ is minimal, otherwise we set $\varrho_\prec(x) = \sup\{\varrho_\prec(y) + 1 : y \prec x\}$. Finally we define the rank of $\prec$ to be $\varrho(\prec) = \sup\{\varrho_\prec(x) + 1 : x \in X\}$. We will need the following boundedness principle of analytic well-founded relations due to Kunen and Martin (see [4] or [6]).

**Theorem 6.** Let $\prec$ be a strict well-founded relation and assume that $\prec$ is analytic (as a subset of $X \times X$). Then $\varrho(\prec)$ is countable.

**Lemma 7.** Let $\mathcal{F} = (A_i)_{i \in I}$ be a stable family of mutually disjoint analytic subsets of $X$. Then $\mathcal{F}$ is countable.

**Proof.** Assume that $\mathcal{F}$ is not countable. Pick an uncountable subfamily $\mathcal{F}'$ of $\mathcal{F}$ with $|\mathcal{F}'| = \aleph_1$ and let $\mathcal{F}' = (A_\xi)_{\xi < \omega_1}$ be a well-ordering of $\mathcal{F}'$. Clearly $\mathcal{F}'$ remains stable. As the sets $A_\xi$ are pairwise disjoint let $\phi : \bigcup_{\xi < \omega_1} A_\xi \to \text{Ord}$, where $\phi(x)$ is the unique $\xi$ such that $x \in A_\xi$. Define the binary relation $\prec$ by

$$
x \prec y \iff \phi(x) < \phi(y).
$$

Clearly $\prec$ is well-founded and strict. Moreover note that $\prec$, as a subset of $X \times X$, is the set

$$
\bigcup_{\xi < \omega_1} (A_\xi \times B_\xi),
$$

where $B_\xi = \bigcup_{\zeta > \xi} A_\zeta$. From the stability of $\mathcal{F}'$, we see that the sets $B_\xi$ are analytic subsets of $X$ for every $\xi < \omega_1$. As $\mathcal{F}'$ is stable and has the p.f.i.p., by Lemma 5 we deduce that $\prec$ is a $\Sigma_1^1$ relation. By Theorem 6, $\varrho(\prec)$ must be countable and we derive a contradiction. □

Finally we have the following.

**Theorem 8.** Let $\mathcal{F}$ be a stable family of analytic sets with the point-finite intersection property. Then $\mathcal{F}$ is countable.
Proof. Assume not. Let $\mathcal{F}'$ be as in Lemma 7. Let $Y$ be an arbitrary uncountable Polish space and let $(y_\xi)_{\xi<\omega_1}$ be a transfinite sequence of distinct members of $Y$. For every $\xi < \omega_1$, set $L_\xi = A_\xi \times \{y_\xi\}$. Clearly every $L_\xi$ is an analytic subset of $X \times Y$ and moreover $L_\xi \cap L_\zeta = \emptyset$ if $\xi \neq \zeta$. As the family (and every subfamily of) $\mathcal{F}'$ is stable and has the p.f.i.p., by Lemma 5, for every $G \subseteq \omega_1$ the set
\[
\bigcup_{\xi \in G} (A_\xi \times \{y_\xi\}) = \bigcup_{\xi \in G} L_\xi
\]
is an analytic subset of $X \times Y$. It follows that the family $\mathcal{L} = (L_\xi)_{\xi<\omega_1}$ is a stable family of mutually disjoint analytic subsets of $X \times Y$. By Lemma 7, the family $\mathcal{L}$ must be countable and again we derive a contradiction.

A corollary of Theorem 8 is the following (see [7]).

**Corollary 9.** Let $X$ be a Polish space, $Y$ a metrizable space and $A \in \Sigma_1^1(X)$. Let also $f : A \to Y$ be a Borel measurable function. Then $f(A)$ is separable.

**Proof.** Assume not. Let $C \subseteq f(A)$ be an uncountable closed discrete set with $|C| > \aleph_0$. For every $y \in C$, put $A_y = f^{-1}\{\{y\}\}$. Then $\mathcal{F} = (A_y)_{y \in C}$ is a stable family of mutually disjoint analytic subsets of $X$. By Theorem 8, $\mathcal{F}$ must be countable and we derive a contradiction.

**Remark 1.** Say that a family $\mathcal{F} = (A_i)_{i \in I}$ has the point-countable intersection property if for every $x \in X$ the set $I_x = \{i \in I : x \in A_i\}$ is countable. One can easily see that Theorem 8 is not valid for stable families with the point-countable intersection property. For instance let $(A_\xi)_{\xi<\omega_1}$ be a strictly decreasing transfinite sequence of analytic sets with $\bigcap_{\xi<\omega_1} A_\xi = \emptyset$. As the sequence is decreasing, the family $\mathcal{F} = (A_\xi)_{\xi<\omega_1}$ is stable. Moreover note that for every $x \in X$ there exists $\xi < \omega_1$ such that $x \notin A_\xi$ for every $\zeta > \xi$ (for if not there would exist an $x \in X$ such that $x \in A_\xi$ for every $\xi < \omega_1$, that is, $x \in \bigcap_{\xi<\omega_1} A_\xi$). Hence the family $\mathcal{F}$ is an uncountable stable family of analytic sets with the point-countable intersection property.

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