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OPERATOR MATRIX OF MOORE–PENROSE INVERSE OPERATORS ON HILBERT C^*-MODULES

BY

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Abstract. We show that the Moore–Penrose inverse of an operator T is idempotent if and only if it is a product of two projections. Furthermore, if P and Q are two projections, we find a relation between the entries of the associated operator matrix of PQ and the entries of associated operator matrix of the Moore–Penrose inverse of PQ in a certain orthogonal decomposition of Hilbert C^*-modules.

1. Introduction and preliminaries. Hilbert C^*-modules are objects like Hilbert spaces, except that the inner product takes its values in a C^*-algebra, instead of being complex-valued. Throughout the paper $\mathcal{A}$ is a C^*-algebra (not necessarily unital). A (right) *pre-Hilbert module* over $\mathcal{A}$ is a complex linear space $\mathcal{X}$ which is an algebraic right $\mathcal{A}$-module such that $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in \mathcal{X}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying

(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$,

for all $x, y, z \in \mathcal{X}$, $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a *Hilbert $\mathcal{A}$-module* if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example, every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to the inner product $\langle x, y\rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$, called the adjoint of T, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a bounded linear operator, which

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is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [LA, p. 8]. We write $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and $\ker(\cdot)$ and $\mathrm{ran}(\cdot)$ for the kernel and range of operators, respectively. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{Y}$ is *orthogonally complemented* if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^{\perp}$, where $\mathcal{Y}^{\perp} := \{y \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for all } x \in \mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$. The reader is referred to [F2, F1, LA, MT] and the references cited therein for more details.

Throughout this paper, $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented; however, Lance proved the following:

Theorem A ([LA, Theorem 3.2]). *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then*

- $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\mathrm{ran}(T^*)$.
- $\mathrm{ran}(T)$ *is orthogonally complemented in $\mathcal{Y}$, with complement* $\ker(T^*)$.
- $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ *has closed range.*

Definition 1.1. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The *Moore–Penrose inverse* $T^\dagger$ of T (if it exists) is an element in $L(\mathcal{Y}, \mathcal{X})$ which satisfies:

(a) $TT^\dagger T = T$,
(b) $T^\dagger T T^\dagger = T^\dagger$,
(c) $(TT^\dagger)^* = TT^\dagger$,
(d) $(T^\dagger T)^* = T^\dagger T$.

The operator $T^\dagger$ (if it exists) is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections, that is, selfadjoint idempotent operators. Clearly, T is Moore–Penrose invertible if and only if T^* is Moore–Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$. The following theorem is known.

Theorem B ([XS, Theorem 2.2]). *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the Moore–Penrose inverse $T^\dagger$ of T exists if and only if T has closed range.*

By Definition 1.1, we have

$$\mathrm{ran}(T) = \mathrm{ran}(TT^\dagger), \quad \mathrm{ran}(T^\dagger) = \mathrm{ran}(T^\dagger T) = \mathrm{ran}(T^*),$$
$$\ker(T) = \ker(T^\dagger T), \quad \ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*),$$

and by Theorem A,

$$\mathcal{X} = \ker(T) \oplus \mathrm{ran}(T^\dagger) = \ker(T^\dagger T) \oplus \mathrm{ran}(T^\dagger T),$$
$$\mathcal{Y} = \ker(T^\dagger) \oplus \mathrm{ran}(T) = \ker(TT^\dagger) \oplus \mathrm{ran}(TT^\dagger).$$

A matrix form of an adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$

are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$, $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as a 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{1.1}$$

where $T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$ and $T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$. Note that $P_{\mathcal{M}}$ denotes the projection corresponding to $\mathcal{M}$.

In fact $T_1 = P_{\mathcal{N}} T P_{\mathcal{M}}$, $T_2 = P_{\mathcal{N}} T(1 - P_{\mathcal{M}})$, $T_3 = (1 - P_{\mathcal{N}})TP_{\mathcal{M}}$ and $T_4 = (1 - P_{\mathcal{N}})T(1 - P_{\mathcal{M}})$.

Recall that if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed renge, then $TT^\dagger = P_{\mathrm{ran}(T)}$ and $T^\dagger T = P_{\mathrm{ran}(T^*)}$.

Corollary 1.2. *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions $\mathcal{X} = \mathrm{ran}(T^*) \oplus \ker(T)$ and $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix}$$

where T_1 is invertible. Moreover

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix}.$$

Proof. According to the above discussion, it is enough to set $\mathcal{M} = \mathrm{ran}(T^*)$ and $\mathcal{N} = \mathrm{ran}(T)$. Then

$$\begin{aligned} T_2 &= P_{\mathrm{ran}(T)} T(1 - P_{\mathrm{ran}(T^*)}) = P_{\mathrm{ran}(T)} T(P_{\ker T}) = 0, \\ T_3 &= (1 - P_{\mathrm{ran}(T)}) T P_{\mathrm{ran}(T^*)} = T P_{\mathrm{ran}(T^*)} - P_{\mathrm{ran}(T)} T P_{\mathrm{ran}(T^*)} = 0, \\ T_4 &= (1 - P_{\mathrm{ran}(T)}) T(1 - P_{\mathrm{ran}(T^*)}) \\ &= T(1 - P_{\mathrm{ran}(T^*)}) - P_{\mathrm{ran}(T)} T(1 - P_{\mathrm{ran}(T^*)}) = 0. \end{aligned}$$

Now we show that T_1 is invertible. Let $x \in \ker(T_1)$, so $0 = T_1 x = P_{\mathrm{ran}(T)} T P_{\mathrm{ran}(T^*)} x = Tx$, which means that $x \in \ker(T)$. On the other hand $T_1 \in \mathcal{L}(\mathrm{ran}(\mathcal{T}^*), \mathrm{ran}(\mathcal{T}))$, so $x \in \ker(T_1) \subseteq \mathrm{ran}(T^*)$. Hence $x \in \ker(T) \cap \mathrm{ran}(T^*) = \{0\}$. Therefore $x = 0$. By Definition 1.1, we conclude that $T^\dagger = \left[\begin{smallmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. ∎

Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $P \in L(\mathcal{X})$ be an orthogonal projection with $\mathrm{ran}(P) = \mathcal{K}$. Since $\mathcal{X} = \mathrm{ran}(P) \oplus \mathrm{ran}(P)^\perp = \mathcal{K} \oplus \mathcal{K}^\perp$, we have the following representations of the projections $P, 1 - P \in \mathcal{L}(\mathcal{X})$ with respect to the decomposition $\mathcal{X} = \mathcal{K} \oplus \mathcal{K}^\perp$:

$$P = \begin{bmatrix} 1_\mathcal{K} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{bmatrix}, \tag{1.2}$$

$$1 - P = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{K}^\perp} \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{bmatrix}. \tag{1.3}$$

If $Q \in L(\mathcal{X})$ is an orthogonal projection and

$$Q = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{bmatrix}, \tag{1.4}$$

then $A \in \mathcal{L}(\mathcal{K})$ and $D \in \mathcal{L}(\mathcal{K}^\perp)$ are selfadjoint, and since $Q = Q^2$, we have

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = \begin{bmatrix} A^2 + BB^* & AB + BD \\ B^*A + DB^* & B^*B + D^2 \end{bmatrix} = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix},$$

which implies that

$$\begin{aligned} A &= A^2 + BB^*, \\ B &= AB + BD, \\ D &= D^2 + B^*B. \end{aligned} \tag{1.5}$$

In the next section we shall use the following result.

THEOREM C ([MS, Corollary 2.4]). *Suppose that* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *has closed range. Then* $(TT^*)^\dagger = (T^*)^\dagger T^\dagger$.

Closedness of the range of operators and the structure of Moore–Penrose inverses are important topics in operator theory. Xu and Sheng [XS] showed that a bounded adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore–Penrose inverse if and only if the operator has closed range. In this paper we state conditions equivalent to the Moore–Penrose inverse being idempotent, and we find a relation between the entries of the operator matrix associated to PQ and the entries of the operator matrix associated to $(PQ)^\dagger$ for a certain orthogonal decomposition of Hilbert C^*-modules, where P and Q are two projections in Hilbert C^*-modules.

2. Operator matrix of the Moore–Penrose inverse of an operator. We begin this section with the following useful facts about products of module maps with closed range.

THEOREM 2.1. *Suppose that* $Q \in \mathcal{L}(\mathcal{X})$ *and* $P \in \mathcal{L}(\mathcal{Y})$ *are orthogonal projections and* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. *If* PTQ *has closed range, then* $T(PTQ)^\dagger$ *and* $(PTQ)^\dagger T$ *are idempotent closed range operators.*

Proof. Since $\operatorname{ran}(PTQ)$ is closed, the operator $U = (PTQ)^\dagger$ exists and $\operatorname{ran}(U) = \operatorname{ran}((PTQ)^\dagger) = (\operatorname{ran}(PTQ)^*) = \operatorname{ran}(QT^*P)$, so $\operatorname{ran}(U) \subseteq \operatorname{ran}(Q)$. Also

$$\operatorname{ran}(U^*) = \operatorname{ran}((PTQ)^\dagger)^* = \operatorname{ran}(((PTQ)^*)^\dagger) = \operatorname{ran}((PTQ)^*)^* = \operatorname{ran}(PTQ) \subseteq \operatorname{ran}(P).$$

Hence

$$QU = U, \quad PU^* = U^*, \quad UP = U. \tag{2.1}$$

Therefore

$$UTU = UPTQU = U(PTQ)U = UU^\dagger U = U. \tag{2.2}$$

By multiplying (2.2) on the left by T, we get $TU = TUTU = (TU)(TU) = (TU)^2$. Again by multiplying (2.2) on the right by T, we obtain $UT = UTUT = (UT)(UT) = (UT)^2$. Hence $T(PTQ)^\dagger$ and $(PTQ)^\dagger T$ are idempotent. Then [LA, Corollary 3.3] implies that $T(PTQ)^\dagger$ and $(PTQ)^\dagger T$ have closed range. ■

Corollary 2.2. *Suppose that $P, Q \in \mathcal{L}(\mathcal{X})$ are orthogonal projections. If PQ has closed range, then $U = (PQ)^\dagger$ is idempotent and $U = QUP$.*

Proof. Set $T = 1_{\mathcal{X}}$ in Theorem 2.1. Then $U = (PQ)^\dagger$ is idempotent. By using (2.1), we obtain $QU^2P = U^2$. Since U is idempotent, the desired result follows. ■

The following theorem states some equivalent conditions under which the Moore–Penrose inverse of an operator is idempotent.

Theorem 2.3. *Suppose that $T \in L(\mathcal{X})$ has closed range. Then the following assertions are equivalent:*

(i) *$T = PQ$ for some projections P and Q,*
(ii) *$T^2 = TT^*T$,*
(iii) *$T^* = T^\dagger T^2 T^\dagger$,*
(iv) *$T = T(T^\dagger)^2 T$,*
(v) *$(T^\dagger)^2 = T^\dagger$,*
(vi) *$|Tx|^2 = \langle Tx, x\rangle$ for all $x \in (\ker(T))^\perp$,*
(vii) *$T^\dagger T^* = T^*$.*

Proof. (i)⇒(ii): Obvious.

(ii)⇒(iii): Multiplying $T^2 = TT^*T$ on the left by $T^\dagger$ yields $T^\dagger TT = T^\dagger TT^*T = (T(T^\dagger T))^*T = T^*T$. Now, multiplying by $T^\dagger$ on the right, we get the desired result.

(iii)⇒(iv): If $T^* = T^\dagger T^2 T^\dagger$, then $(T^*)^* = ((T^\dagger T)(TT^\dagger))^* = (TT^\dagger)(T^\dagger T)$. Hence $T = T(T^\dagger)^2 T$.

(iv)⇒(v): If $T = T(T^\dagger)^2T$, then multiplying by $T^\dagger$ both on the left and on the right, we get $T^\dagger TT^\dagger = T^\dagger TT^\dagger T^\dagger TT^\dagger$, which implies that $(T^\dagger)^2 = T^\dagger$.

(v)⇒(i): By multiplying $(T^\dagger)^2 = T^\dagger$ on both sides by T, we get $TT^\dagger T^\dagger T = TT^\dagger T$, so $P_{\mathrm{ran}(T)} P_{\mathrm{ran}(T^*)} = T$.

(v)⇒(vi): We have shown that if $T^\dagger$ is idempotent, then T can be written as

$$T = P_{(\ker(T^*))^\perp} P_{(\ker(T))^\perp} = P_{\mathrm{ran}(T)} P_{(\ker(T))^\perp}.$$

For all $x \in (\ker(T))^\perp$, we know that $P_{(\ker(T))^\perp} x = x$, so

$$\begin{aligned} |Tx|^2 = \langle T^*Tx, x\rangle &= \langle P_{(\ker(T))^\perp} P_{\mathrm{ran}(T)} P_{(\ker(T))^\perp} x, x\rangle \\ &= \langle P_{\mathrm{ran}(T)} P_{(\ker(T))^\perp} x, P_{(\ker(T))^\perp} x\rangle = \langle Tx, x\rangle. \end{aligned}$$

(vi)⇒(ii): Since $|Tx|^2 = \langle Tx, x\rangle$ for all $x \in (\ker(T))^\perp$, and $P_{(\ker(T))^\perp} y \in (\ker(T))^\perp$, we have $Ty = TP_{(\ker(T))^\perp} y$ and

$$\langle T^*Ty, y\rangle = \langle Ty, Ty\rangle = \langle Ty, P_{(\ker(T))^\perp} y\rangle = \langle P_{(\ker(T))^\perp} Ty, y\rangle$$

for all $y \in \mathcal{X}$. Hence, $T^*T = P_{(\ker(T))^\perp} T$. So $T^*T = T^\dagger TT = T^\dagger T^2$. Multiplying by T on the left, we get $TT^*T = TT^\dagger T^2$ or $TT^*T = T^2$.

(iii)⇒(vii): We have shown above that if (iii) holds then $T^\dagger$ is idempotent. This yields the desired implication.

(vii)⇒(iv): Multiplying $T^\dagger T^* = T^*$ by $(T^*)^\dagger$ on the right and by T on the left, we obtain

$$\begin{aligned} TT^\dagger T^*(T^*)^\dagger &= TT^*(T^*)^\dagger, \\ TT^\dagger T^*(T^\dagger)^* &= TT^*(T^\dagger)^*, \\ TT^\dagger(T^\dagger T)^* &= T(T^\dagger T)^* \quad \text{(by Theorem C)}. \end{aligned}$$

Hence $T(T^\dagger)^2 T = TT^\dagger T = T$. ∎

Corollary 2.4. *Suppose that $T \in L(\mathcal{X})$ has closed range and $T^\dagger$ is idempotent. Then*

$$T^\dagger = T^* - P_{\mathrm{ran}(T^*)}[(1 - P_{\mathrm{ran}(T^*)})(1 - P_{\mathrm{ran}(T)})]^\dagger P_{\mathrm{ran}(T)}.$$

Proof. The proof of Theorem 2.3 implies that $T = P_{\mathrm{ran}(T)} P_{\mathrm{ran}(T^*)}$. Now [LI, Theorem 10] yields the desired formula. ∎

Remark 2.5. A valuable consequence of this theorem is that a closed range operator T is a product of two projections if and only if its Moore–Penrose inverse is idempotent, and we also see that the Moore–Penrose inverse of an idempotent operator is a product of two projections.

Recall that an operator $U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is said to be *unitary* if $U^*U = 1_{\mathcal{X}}$ and $UU^* = 1_{\mathcal{Y}}$. If there exists a unitary element in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, then we say that $\mathcal{X}$ and $\mathcal{Y}$ are *unitarily equivalent* Hilbert $\mathcal{A}$-modules, and we write $\mathcal{X} \approx \mathcal{Y}$. Moreover, obviously if U is unitary, then $U^* = U^\dagger$.

THEOREM 2.6. *Suppose that $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ are Hilbert $\mathcal{A}$-modules, and $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $U \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$ are unitary operators. Then for any $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ with closed range, $(UTV)^\dagger = V^*T^\dagger U^*$.*

Proof. Since $U^*U = 1_{\mathcal{Z}}$, $UU^* = 1_{\mathcal{W}}$ and $V^*V = 1_{\mathcal{X}}$, $VV^* = 1_{\mathcal{Y}}$, by Definition 1.1 we have

(a) $\quad (UTV)V^*T^\dagger U^*(UTV) = (UT(VV^*)T^\dagger(U^*U)TV = UTV,$

(b) $\; V^*T^\dagger U^*(UTV)V^*T^\dagger U^* = V^*T^\dagger(U^*U)T(VV^*)T^\dagger U^* = V^*T^\dagger U^*,$

(c)
$$\begin{aligned}((UTV)V^*T^\dagger U^*)^* &= ((UT1_{\mathcal{Y}}T^\dagger U^*)^* = ((UTT^\dagger U^*)^* = UTT^\dagger U^*\\ &= (UTV)V^*T^\dagger U^*,\end{aligned}$$

(d)
$$\begin{aligned}(V^*T^\dagger U^*(UTV))^* &= (V^*T^\dagger 1_{\mathcal{Z}}TV))^* = (V^*T^\dagger TV))^*\\ &= V^*T^\dagger U^*(UTV).\end{aligned}$$

Hence $(UTV)^\dagger = V^*T^\dagger U^*$. ∎

In the next theorem we find a relation between the entries of the associated operator matrix of operators.

THEOREM 2.7. *Suppose that orthogonal projections $P, Q \in \mathcal{L}(\mathcal{X})$ are represented as in* (1.2) *and* (1.4), *and PQ has closed range. Then*

(i) $(PQ)^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{bmatrix}$ *and* $\operatorname{ran}(PQ) = \operatorname{ran}(A)$.

(ii) $B^*A^\dagger A = B^*$, *equivalently* $AA^\dagger B = B$.

(iii) $A^2 + ABB^*A^\dagger = A$ *and* $B^*A^2 + B^*BB^*A^\dagger = B^*$.

(iv) BB^* *commutes with* A, *and* BB^* *commutes with* $A^\dagger$.

(v) $A + BB^*A = AA^\dagger$ *and* $B^*AA^\dagger + DB^*A^\dagger = B^*A^\dagger$.

Proof. (i) Since PQ has closed range, [MS, Corollary 2.4] implies that $(PQ)^\dagger = (PQ)^*(PQ(PQ)^*)^\dagger$. By using this fact and (1.5), we obtain

$$\begin{aligned}(PQ)^\dagger &= \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix}\begin{bmatrix} A^2+BB^* & 0 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}^\dagger\\ &= \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix}\begin{bmatrix} A^\dagger & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}.\end{aligned}$$

From $(PQ)(PQ)^\dagger = P_{\operatorname{ran}(PQ)}$, we deduce

$$(PQ)(PQ)^\dagger = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix} = \begin{bmatrix} AA^\dagger & 0 \\ 0 & 0 \end{bmatrix}.$$

This immediately implies that $\operatorname{ran}(PQ) = \operatorname{ran}(AA^\dagger) = \operatorname{ran}(A)$.

(ii) By (i), $(PQ)^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}$ and $\mathrm{ran}(PQ)$ is closed. Corollary 2.2 implies that $(PQ)^\dagger$ is idempotent. Applying Theorem 2.3(vii), we get $(PQ)^\dagger(PQ)^* = (PQ)^*$. Hence

$$\begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix}.$$

Therefore, $B^*A^\dagger A = B^*$. As A is selfadjoint, by Theorem C we have $AA^\dagger = (AA^\dagger)^* = A^\dagger A$ and $AA^\dagger B = B$.

(iii) Applying Theorem 2.3(iii) for PQ, we get

$$\begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix} = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}\begin{bmatrix} A^2 & AB \\ 0 & 0 \end{bmatrix}\begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}.$$

A straightforward computation shows that

$$AA^\dagger A^2AA^\dagger + AA^\dagger ABB^*A^\dagger = A \quad \text{and} \quad B^*A^\dagger A^2AA^\dagger + B^*A^\dagger ABB^*A^\dagger = B^*.$$

By (ii), we observe that

$$A^2 + ABB^*A^\dagger = A \quad \text{and} \quad B^*A^2 + B^*BB^*A^\dagger = B^*.$$

(iv) Part (iii) and (1.5) show that $ABB^*A^\dagger = BB^*$. Multiplying by A on the right, we get $ABB^*A^\dagger A = BB^*A$. It follows from (ii) that $ABB^* = BB^*A$, i.e. BB^* commutes with A. For the second part,

$$\begin{aligned} ABB^* &= BB^*A, \\ ABB^*A^\dagger &= BB^*AA^\dagger && \text{(multiplication by } A^\dagger \text{ on the right)}, \\ ABB^*A^\dagger &= BB^* && \text{(by (ii), } B^*AA^\dagger = B^*), \\ A^\dagger ABB^*A^\dagger &= A^\dagger BB^* && \text{(multiplication by } A^\dagger \text{ on the left)}, \\ BB^*A^\dagger &= A^\dagger BB^* && \text{(by (ii), } AA^\dagger B = B), \end{aligned}$$

which means that BB^* commutes with $A^\dagger$.

(v) By (i), we have $(PQ)^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}$. Applying Corollary 2.2 we get $(PQ)^\dagger = Q(PQ)^\dagger P$, which implies that

$$\begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}\begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}\begin{bmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} AAA^\dagger + BB^*A & 0 \\ B^*AA^\dagger + DB^*A^\dagger & 0 \end{bmatrix}.$$

Therefore, $A + BB^*A = AA^\dagger$ and $B^*AA^\dagger + DB^*A^\dagger = B^*A^\dagger$. ∎

By the previous theorem we state the reverse-order law in the special case of product operators.

THEOREM 2.8. *Let orthogonal projections $P, Q \in \mathcal{L}(\mathcal{X})$ be represented as in* (1.2) *and* (1.4), *and assume PQ, B and AB have closed ranges. Then*

(i) $(AB)^\dagger = B^\dagger A^\dagger$.

(ii) *ABB^* has closed range and $(ABB^*)^\dagger = (B^*)^\dagger B^\dagger A^\dagger$.*

Proof. To prove (i), note that

$$\begin{aligned}(B^* - B^*BB^*A^\dagger)(1 - BB^\dagger) &= B^*(1 - BB^\dagger) - B^*BB^*A^\dagger(1 - BB^\dagger)\\ &= B^* - B^*BB^\dagger - B^*BB^*A^\dagger + B^*BB^*A^\dagger BB^\dagger\\ \text{(by Theorem 2.7(iv))} &= B^* - B^*BB^\dagger - B^*A^\dagger BB^* + B^*A^\dagger BB^*BB^\dagger\\ &= B^* - B^* - B^*A^\dagger BB^* + B^*A^\dagger BB^* = 0.\end{aligned}$$

By Theorem 2.7(iii), we have $B^*A^2(1 - BB^\dagger) = 0$, so $B^*A^2 = B^*A^2BB^\dagger$. Taking adjoints we get $A^2B = BB^\dagger A^2B$. So, condition (ii) of [KA, Theorem 2.1] holds. By (ii)$\Rightarrow$(iii) of that theorem, $B^\dagger A^\dagger$ satisfies conditions (a)–(c) of Definition 1.1.

On the other hand, by Theorem 2.7(ii), we have

$$(B^\dagger A^\dagger(AB))^* = (B^\dagger(A^\dagger AB))^* = (B^\dagger B)^* = B^\dagger B = B^\dagger A^\dagger(AB).$$

Hence $B^\dagger A^\dagger$ satisfies condition (d) of Definition 1.1. Therefore, $B^\dagger A^\dagger$ is the Moore–Penrose inverse of AB.

To prove (ii), by Definition 1.1 we have

(a) $$\begin{aligned}(ABB^*)(B^*)^\dagger B^\dagger A^\dagger(ABB^*) &= AB(B^*(B^*)^\dagger B^\dagger)A^\dagger(ABB^*)\\ &= ABB^\dagger(A^\dagger AB)B^*\\ \text{(by Theorem 2.7(ii))} &= ABB^\dagger BB^* = AB(BB^\dagger B)^* = ABB^*;\end{aligned}$$

(b) $$\begin{aligned}(B^*)^\dagger B^\dagger A^\dagger(ABB^*)(B^*)^\dagger B^\dagger A^\dagger &= (B^*)^\dagger B^\dagger(A^\dagger AB)B^*(B^*)^\dagger B^\dagger A^\dagger\\ \text{(by Theorem 2.7(ii))} &= ((B^*)^\dagger B^\dagger B)(B^*(B^*)^\dagger B^\dagger)A^\dagger\\ &= (B^\dagger BB^\dagger)^*(B^\dagger BB^\dagger)^*A^\dagger = (B^*)^\dagger B^\dagger A^\dagger;\end{aligned}$$

(c) $$\begin{aligned}(B^*)^\dagger B^\dagger A^\dagger(ABB^*) &= (B^*)^\dagger B^\dagger(A^\dagger AB)B^*\\ \text{(by Theorem 2.7(ii))} &= ((B^*)^\dagger B^\dagger BB^*\\ &= (B^\dagger BB^\dagger)^*B^* = (B^*)^\dagger B^* = BB^\dagger;\end{aligned}$$

(d) $$(ABB^*)(B^*)^\dagger B^\dagger A^\dagger = A(B(B^\dagger B)^*B^\dagger)A^\dagger = ABB^\dagger A^\dagger.$$

By (i), we have $(AB)^\dagger = B^\dagger A^\dagger$, hence $AB(AB)^\dagger = ABB^\dagger A^\dagger$ is an orthogonal projection. Therefore, $(B^*)^\dagger B^\dagger A^\dagger$ is the Moore–Penrose inverse of ABB^*. Hence ABB^* has closed range and $(ABB^*)^\dagger = (B^*)^\dagger B^\dagger A^\dagger$. ∎

If $B \in \mathcal{L}(\mathcal{X})$ has closed range, then $\operatorname{ran}(B) = \operatorname{ran}(BB^*)$. In the following theorem we show that sometimes $\operatorname{ran}(B) = \operatorname{ran}(BB^*)$, even if the range of B is not necessarily closed.

THEOREM 2.9. *Let orthogonal projections $P, Q \in \mathcal{L}(\mathcal{X})$ be represented as in* (1.2) *and* (1.4)*, and suppose PQ has closed range. If $\|A\| < 1$, then* $\operatorname{ran}(B) = \operatorname{ran}(BB^*)$.

Proof. It is trivial that $\operatorname{ran}(BB^*) \subseteq \operatorname{ran}(B)$. To show the opposite inclusion, let $y \in \operatorname{ran}(B)$, so there is $x \in \mathcal{X}$ such that $y = Bx$. Theorem 2.7(iii) shows that $B = A^2B + A^\dagger BB^*B$. By (1.5), we have

$$B = (A - BB^*)B + A^\dagger BB^*B = AB - BB^*B + A^\dagger BB^*B.$$

Now Theorem 2.7(iv) yields $B = AB + BB^*(-B + A^\dagger B)$. Therefore

$$(1 - A)Bx = BB^*(-Bx + A^\dagger Bx). \tag{2.3}$$

From $\|A\| < 1$ we know that $1 - A$ is invertible and $(1 - A)^{-1} = \sum_{n=0}^{\infty} A^n$. It follows from this relation, (2.3) and Theorem 2.7(iv) that BB^* commutes with A^n for all $n \geq 1$. Continuity of BB^* implies that

$$\begin{aligned}(1 - A)^{-1}(1 - A)Bx &= \sum_{n=0}^{\infty} A^n BB^*(-Bx + A^\dagger Bx)\\ &= BB^*\Big(\sum_{n=0}^{\infty} A^n(-Bx + A^\dagger Bx)\Big).\end{aligned}$$

Hence $y = Bx \in \operatorname{ran}(BB^*)$ and $\operatorname{ran}(B) = \operatorname{ran}(BB^*)$. ∎

Now, we show that there is a relation between the entries of the associated operator matrix for the composition of three special operators.

THEOREM 2.10. *Suppose that Q is an orthogonal projection in $\mathcal{L}(\mathcal{X})$, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and P is an orthogonal projection in $\mathcal{L}(\mathcal{Y})$. If T and PTQ have closed ranges, and $(QT^*PTQ)^\dagger$ commutes with $T^\dagger T$ and $T = \left[\begin{smallmatrix} T_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where T_1 is unitary, and*

$$P = \begin{bmatrix} A_1 & B_1 \\ B_1^* & D_1 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T^*) \end{bmatrix} \mapsto \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

$$Q = \begin{bmatrix} A_2 & B_2 \\ B_2^* & D_2 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T^*) \\ \ker(T) \end{bmatrix} \mapsto \begin{bmatrix} \operatorname{ran}(T^*) \\ \ker(T) \end{bmatrix},$$

then:

(i) $A_1T_1A_2^2T_1^{-1}A_1T_1A_2 = A_1T_1A_2T_1^{-1}A_1T_1A_2$,
(ii) $B_1^*T_1A_2^2T_1^{-1}A_1T_1A_2 = B_1^*T_1A_2T_1^{-1}A_1T_1A_2$,
(iii) $A_1T_1B_2B_2^*T_1^{-1}A_1T_1A_2 = 0$,
(iv) $B_1^*T_1B_2B_2^*T_1^{-1}A_1T_1A_2 = 0$.

Proof. A straightforward computation shows that

$$PTQ = \begin{bmatrix} A_1 & B_1 \\ B_1^* & D_1 \end{bmatrix}\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ B_2^* & D_2 \end{bmatrix} = \begin{bmatrix} A_1T_1A_2 & A_1T_1B_2 \\ B_1^*T_1A_2 & B_1^*T_1B_2 \end{bmatrix}.$$

By assumption $T_1 \in \mathcal{L}(\operatorname{ran}(T^*), \operatorname{ran}(T))$ is unitary, and from [LA, p. 25] T_1 is invertible and $T_1^{-1} = T_1^*$. Set

$$S = \begin{bmatrix} A_1T_1A_2 & A_1T_1B_2 \\ B_1^*T_1A_2 & B_1^*T_1B_2 \end{bmatrix}.$$

Then

$$T^*T = \begin{bmatrix} 1_{\operatorname{ran}(T^*)} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \operatorname{ran}(T^*TS^\dagger) \subseteq \operatorname{ran}(S^\dagger) = \operatorname{ran}(S^*).$$

We know that $S^\dagger S$ is a projection on $\operatorname{ran}(S^*)$. Therefore $S^\dagger ST^*TS^\dagger = T^*TS^\dagger$. Hence, condition (ii) of [KA, Theorem 2.1] holds, and by (ii)$\Rightarrow$(iii) of that theorem, $ST^\dagger$ satisfies conditions (a)–(c) of Definition 1.1.

On the other hand, $TS^\dagger(S^\dagger)^* = TS^\dagger(S^\dagger)^*T^\dagger T$, or equivalently $T(S^*S)^\dagger = T(S^*S)^\dagger T^\dagger T$. We observe that $S = PTQ$, and $(QT^*PTQ)^\dagger$ commutes with $T^\dagger T$. Hence condition (ii) of [KA, Theorem 2.2] holds. By (ii)$\Rightarrow$(iii) of that theorem, $ST^\dagger$ satisfies conditions (a), (b), (d) of Definition 1.1. Therefore, $ST^\dagger$ is the Moore–Penrose inverse of $TS^\dagger$ and $TS^\dagger = (ST^\dagger)^\dagger$. In addition

$$\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1T_1A_2 & A_1T_1B_2 \\ B_1^*T_1A_2 & B_1^*T_1B_2 \end{bmatrix}^\dagger = \left(\begin{bmatrix} A_1T_1A_2 & A_1T_1B_2 \\ B_1^*T_1A_2 & B_1^*T_1B_2 \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \right)^\dagger$$
$$= \begin{bmatrix} A_1T_1A_2T_1^{-1} & 0 \\ B_1^*T_1A_2T_1^{-1} & 0 \end{bmatrix}^\dagger.$$

Now, Theorem 2.1 implies that

$$TS^\dagger = \begin{bmatrix} A_1T_1A_2T_1^{-1} & 0 \\ B_1^*T_1A_2T_1^{-1} & 0 \end{bmatrix}^\dagger$$

is an idempotent operator, so by Theorem 2.3(ii), we can write

$$(TS^\dagger)(TS^\dagger)^*(TS^\dagger) = (TS^\dagger)^2$$

and

$$\begin{bmatrix} A_1T_1A_2T_1^{-1} & 0 \\ B_1^*T_1A_2T_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} T_1A_2T_1^{-1}A_1 & T_1A_2T_1^{-1}B_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1T_1A_2T_1^{-1} & 0 \\ B_1^*T_1A_2T_1^{-1} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A_1T_1A_2T_1^{-1} & 0 \\ B_1^*T_1A_2T_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} A_1T_1A_2T_1^{-1} & 0 \\ B_1^*T_1A_2T_1^{-1} & 0 \end{bmatrix}.$$

In fact

$$\tag{2.4} \begin{bmatrix} A_1T_1A_2^2T_1^{-1}A_1^2T_1A_2T_1^{-1} + A_1T_1A_2^2T_1^{-1}B_1B_1^*T_1A_2T_1^{-1} & 0 \\ B_1^*T_1A_2^2T_1^{-1}A_1^2T_1A_2T_1^{-1} + B_1^*T_1A_2^2T_1^{-1}B_1B_1^*T_1A_2T_1^{-1} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A_1T_1A_2T_1^{-1}A_1T_1A_2T_1^{-1} & 0 \\ B_1^*T_1A_2T_1^{-1}B_1^*T_1A_2T_1^{-1} & 0 \end{bmatrix}.$$

Therefore, by (1.5), we have

$$A_1T_1A_2^2T_1^{-1}(A_1^2+B_1B_1^*)T_1A_2T_1^{-1} = A_1T_1A_2^2T_1^{-1}(A_1)T_1A_2T_1^{-1},$$
$$B_1^*T_1A_2^2T_1^{-1}(A_1^2+B_1B_1^*)T_1A_2T_1^{-1} = B_1^*T_1A_2^2T_1^{-1}(A_1)T_1A_2T_1^{-1}.$$

By (2.4), we have

$$A_1T_1A_2^2T_1^{-1}A_1T_1A_2 = A_1T_1A_2T_1^{-1}A_1T_1A_2,$$
$$B_1^*T_1A_2^2T_1^{-1}A_1T_1A_2 = B_1^*T_1A_2T_1^{-1}A_1T_1A_2.$$

Hence, (i) and (ii) are obtained. By (i) and (ii) and (1.5),

$$A_1T_1(A_2^2-A_2)T_1^{-1}A_1T_1A_2 = A_1T_1B_2B_2^*T_1^{-1}A_1T_1A_2 = 0,$$
$$B_1^*T_1(A_2^2-A_2)T_1^{-1}A_1T_1A_2 = B_1^*T_1B_2B_2^*T_1^{-1}A_1T_1A_2 = 0.$$

Hence, (iii) and (iv) hold. ■

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