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LARGE FREE SUBGROUPS OF AUTOMORPHISM GROUPS OF ULTRAHOMOGENEOUS SPACES

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Abstract. We consider the following notion of largeness for subgroups of S_∞. A group G is large if it contains a free subgroup on $\mathfrak{c}$ generators. We give a necessary condition for a countable structure A to have a large group $\mathrm{Aut}(A)$ of automorphisms. It turns out that any countable free subgroup of S_∞ can be extended to a large free subgroup of S_∞, and, under Martin's Axiom, any free subgroup of S_∞ of cardinality less than $\mathfrak{c}$ can also be extended to a large free subgroup of S_∞. Finally, if G_n are countable groups, then either $\prod_{n\in\mathbb{N}} G_n$ is large, or it does not contain any free subgroup on uncountably many generators.

1. Introduction. In this paper we study properties of the automorphism group $\mathrm{Aut}(A)$ of an ultragomogeneous countable structure A. An ultrahomogeneous structure A can be seen as the Fraïssé limit of its Fraïssé class, that is, the class $\mathcal{K}$ of all finitely generated substructures of A. A Fraïssé class has three properties: the hereditary property, the joint embedding property, and the amalgamation property. (For details see [H].) Some connections between properties of the Fraïssé classes $\mathcal{K}$ and the automorphism groups of their Fraïssé limits are given for example in [KPT], [KS].

We are going to search for large free subgroups of $\mathrm{Aut}(A)$, for countable structures A. Macpherson [M1] showed that if A is ω-categorical, then $\mathrm{Aut}(A)$ contains a dense free subgroup of rank ω, and the automorphism group of the random graph contains a dense free subgroup on two generators. Cameron [C, p. 84] proved that every closed oligomorphic subgroup of S_∞ contains $\mathrm{Aut}(\mathbb{Q},\le)$, and the latter group contains a free subgroup of rank continuum. Melles and Shelah [MS] proved that if A is a saturated model of a complete theory T with $|A|=\lambda>|T|$, then $\mathrm{Aut}(A)$ has a dense free subgroup of cardinality 2^λ. Gartside and Knight [GK] showed that if A is ω-categorical and $K_n=\{(g_1,\dots,g_n)\in \mathrm{Aut}(A)^n : g_1,\dots,g_n \text{ are free generators}\}$, then K_n is comeager in $\mathrm{Aut}(A)^n$ for every n. Some other results of this sort can be found in the survey paper [M2]. It was proved by Shelah [Sh1] that $\mathrm{Aut}(A)$ cannot be a free uncountable group when A is a

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countable structure. Recently, Shelah [Sh2] proved that even no uncountable Polish group can be free.

Let $(A, \mathcal{C}, \mathcal{F}, \mathcal{R})$ be a countable structure where $\mathcal{C}$ stands for the set of all constants, $\mathcal{F}$ for the set of functions and $\mathcal{R}$ for the set of relations. We will use one symbol A for both a structure and its underlying set. Recall that a structure A is *ultrahomogeneous* if every embedding of a finitely generated substructure can be extended to an automorphism of A. We denote by $\mathrm{gen}(X)$ the substructure of A generated by X, i.e., the intersection of all substructures containing X. In particular, $\mathrm{gen}(\emptyset) = \mathrm{gen}(\mathcal{C})$. Let $\mathrm{Aut}(A)$ denote the group of all automorphisms of A. Since A is countable, $\mathrm{Aut}(A)$ is isomorphic to a closed subgroup of the group S_∞ of all permutations of $\mathbb{N}$. With the topology inherited from S_∞, $\mathrm{Aut}(A)$ is a topological group. If $B_1, B_2 \subset A$ are finitely generated substructures and $g : B_1 \to B_2$ is an isomorphism, then g will be called a *partial isomorphism*. The set of all partial isomorphisms of A will be denoted by $\mathrm{Part}(A)$.

We denote by $\mathbb{P}$ the set of all pairs (n, p) where $p : \{0,1\}^n \to \mathrm{Part}(A)$ and $\mathrm{dom}(p(s))$ is an n-element substructure of A for every $s \in \{0,1\}^n$. The set $\mathbb{P}$ is ordered in the following way: $(n, p) \le (k, q)$ if and only if $k \le n$ and $q(t) \subset p(s)$ (i.e., $p(s)$ extends $q(t)$) provided $t \prec s$ (i.e., s is an extension of t). We will show that, under some reasonable assumption on A, the generic filter G on $\mathbb{P}$ produces a family of $\mathfrak{c}$ free generators in $\mathrm{Aut}(A)$. Note that the poset $\mathbb{P}$ is countable, and therefore it has the countable chain property. In Section 2 we will use the Rasiowa–Sikorski lemma to get a generic filter G that intersects countably many dense subsets of $\mathbb{P}$. In this way we will infer that $\mathrm{Aut}(A)$ contains a free subgroup on $\mathfrak{c}$ generators, and this result is valid in ZFC. In Section 3 it will be proved (by a similar argument and also under ZFC) that any countably generated free subgroup of S_∞ can be extended to a $\mathfrak{c}$-generated free subgroup of S_∞, and that under Martin's Axiom any $<\mathfrak{c}$-generated free subgroup of S_∞ can be extended to a $\mathfrak{c}$-generated free subgroup of S_∞. In Section 4 we prove the following dichotomy: the product $\prod_{n\in\mathbb{N}} G_n$ of countable groups G_n either contains a $\mathfrak{c}$-generated free subgroup, or contains no uncountably generated free subgroup. Section 5 brings final remarks and open questions.

2. $\mathfrak{c}$-generated free subgroups of $\mathrm{Aut}(A)$**.** In this section we will assume that every finitely generated substructure of A is finite, that is, its Fraïssé class consists of finite structures. The next lemma shows that a generic filter gives a family of functions which map A onto itself.

LEMMA 2.1. *For every $k \in A$, the set*

$$D_k := \{(n, p) \in \mathbb{P} : \forall s \in \{0,1\}^n \ k \in \mathrm{dom}(p(s)) \cap \mathrm{rng}(p(s))\}$$

is dense in $\mathbb{P}$.

Proof. Let $k \in A$ and $(n,p) \in \mathbb{P}$. For any $s \in \mathrm{dom}(p)$, let $\tilde{p}(s)$ be an automorphism of A such that $p(s) \subset \tilde{p}(s)$. Let (C_m) be an increasing sequence of finitely generated structures such that $A = \bigcup_{m\in\mathbb{N}} C_m$. Then there exists n_0 such that for any $s \in \mathrm{dom}(p)$, we have $\mathrm{dom}(p(s)) \subset C_{n_0}$ and

$$k \in \mathrm{dom}(\tilde{p}(s)\restriction C_{n_0}) \cap \mathrm{rng}(\tilde{p}(s)\restriction C_{n_0}).$$

Let $n' = |C_{n_0}|$, and for any $t \in \{0,1\}^{n'}$, set $p'(t) = \tilde{p}(t\restriction n)\restriction C_{n_0}$. Then $(n',p') \le (n,p)$ and $(n',p') \in D_k$. ∎

In the following reasoning, we will apply the above trick of using an increasing sequence (C_m) without any comments.

If $g \in \mathrm{Part}(A)$, then we set $V(g) := \{f \in \mathrm{Aut}(A) : g \in f\}$. It is well known that the family of all sets of the form $V(g)$ constitutes a basis of the natural topology on $\mathrm{Aut}(A)$.

Lemma 2.2. *Let F be a nowhere dense subset of* $\mathrm{Aut}(A)$. *Then the set*

$$D_F = \{(n,p) \in \mathbb{P} : \forall s \in \{0,1\}^n \ V(p(s)) \cap F = \emptyset\}$$

is dense in $\mathbb{P}$.

Proof. Let $(n,p) \in \mathbb{P}$. Since F is nowhere dense, for every $s \in \{0,1\}^n$ there exists an embedding $g_s : B_s \to A$ (B_s is a finitely generated substructure) such that g_s is an extension of $p(s)$ and $V(g_s) \cap F = \emptyset$. Let $C = \mathrm{gen}(\bigcup\{\mathrm{dom}(g_s) : s \in \mathrm{dom}(p)\})$. Let $n' = |C|$, and for every $t \in \{0,1\}^{n'}$ let $p'(t) : C \to A$ be an embedding and an extension of $g_{t\restriction n}$. Then $(n',p') \le (n,p)$ and $(n',p') \in D_F$ (because $V(p'(t)) \subset V(g_{t\restriction n})$). ∎

Consider the following example. Let $A = \mathbb{N}$, and define unary relations R_n on A, $n \in \mathbb{N}$, by $x \in R_n$ if and only if $x = 2n$ or $x = 2n+1$. Since $(A, \{R_n : n \in \mathbb{N}\})$ is a relational structure, any of its finitely generated substructures is finite. If $f \in \mathrm{Aut}(A)$, then $f(2n) = 2n$ and $f(2n+1) = 2n+1$, or $f(2n+1) = 2n$ and $f(2n) = 2n+1$. Clearly, A is ultrahomogeneous and $\mathrm{Aut}(A)$ is isomorphic to $\mathbb{Z}_2^{\mathbb{N}}$. Hence for any $f \in \mathrm{Aut}(A)$ we have $f \circ f = \mathrm{id}$, which means that $\mathrm{Aut}(A)$ does not even contain a free subgroup on one generator.

This example shows that to get a promised large free subgroup, we need an additional assumption.

Let us introduce the following definition. We say that a relational structure A is *ω-independent* if for any finitely generated substructures B_1, B_2 of A, and for any m, there is a set $D \subset A \setminus (B_1 \cup B_2)$ consisting of $m+1$ elements such that, for any embedding $f : B_1 \to B_2$ and any partial permutation g of D, the function $f \cup g$ is an embedding of $B_1 \cup \mathrm{dom}(g)$ into A.

Now we show that some natural examples of ultrahomogeneous structures are ω-independent and have the property that every finitely generated substructure is finite.

1. First consider $\mathbb{N}$ without any structure. Then every finite set is a finitely generated substructure, and the embeddings are exactly the one-to-one mappings. To see that $\mathbb{N}$ is ω-independent, fix two finite subsets $B_1, B_2 \subset \mathbb{N}$. Let $C = B_1 \cup B_2$ and let $x_0, \ldots, x_m$ be pairwise distinct elements of $\mathbb{N} \setminus C$. Then it is clear that the union of any one-to-one mapping $f : B_1 \to B_2$ and a partial permutation g of $x_0, \ldots, x_m$ is an embedding.

2. The next example is a rational Urysohn space $\mathbb{U}$. Recall that a metric space is a *rational Urysohn space* if it is countable and every finite rational space (i.e., with rational distances) has an isometric copy in $\mathbb{U}$. It is known that $\mathbb{U}$ is ultrahomogeneous in the sense that, for every finite rational metric space $C \subset \mathbb{U}$ and every isometric embedding $f : C \to \mathbb{U}$, there is an isometry $\tilde{f} : \mathbb{U} \to \mathbb{U}$ which extends f. The following is standard and well known:

CLAIM 2.3. *Assume that A is an ultrahomogeneous structure. Let Y be a structure which is isomorphic to a finitely generated substructure of A such that $Y = X \cup Z$, $X \cap Z = \emptyset$ and $X \subset A$, for some X, Z. Then there is $Z' \subset A$ and a partial isomorphism $g : Z \to Z'$ such that the mapping $h : Y \to X \cup Z'$ given by $h(x) = x$ for $x \in X$ and $h(x) = g(x)$ for $x \in Z$ is a partial isomorphism of Y and $X \cup Z'$.*

Now we prove that the Urysohn space is ω-independent. Let B_1, B_2 be two finite subspaces of $\mathbb{U}$, $C = B_1 \cup B_2$, let d be a metric on $\mathbb{U}$, and let $M = \max\{d(z, c) : z, c \in C\} + 1$. Define a finite rational metric space (Y, ρ) as follows. Let $Y = C \cup \{a_0, \ldots, a_m\}$ where $a_0, \ldots, a_m$ are distinct elements which do not belong to C. If $x, y \in C$, then set $\rho(x, y) = d(x, y)$; if $x \in C$ and $y = a_i$, then set $\rho(x, y) = M$; finally, if $x = a_i$ and $y = a_j$, then $\rho(x, y) = 1$ if $i \neq j$ and $\rho(x, y) = 0$ if $i = j$.

Then (Y, ρ) is a finite rational metric space. Moreover, by Claim 2.3, there are $x_0, \ldots, x_m \in \mathbb{U} \setminus C$ such that $d(x, x_i) = M$ for every $x \in C$ and $i = 0, \ldots, m$, and $d(x_i, x_j) = 1$ for $i \neq j$. If $f : B_1 \to B_2$ is an isometric embedding and g is partial permutation of $x_0, \ldots, x_m$, then it is easy to see that the union of f and g is an isometric embedding. Hence the rational Urysohn space $\mathbb{U}$ is ω-independent.

3. Let $\mathbb{G}$ be a random graph, that is, a countable infinite graph where for any finite disjoint sets X and Y we can find a vertex with edges going to every vertex in X but to no vertex in Y. We will show that $\mathbb{G}$ is ω-independent. Fix two finite graphs B_1 and B_2. Take any distinct $x_0, \ldots, x_m$, and define a graph $B_1 \cup B_2 \cup \{x_0, \ldots, x_m\}$ as an extension of $B_1 \cup B_2$ such that there are no edges between $x_0, \ldots, x_m$ and $B_1 \cup B_2$, and there is no edge between x_i and x_j for $i, j \leq m$. Using Claim 2.3 we may assume that $x_0, \ldots, x_m \in \mathbb{G} \setminus (B_1 \cup B_2)$. Let g be any partial permutation of $x_0, \ldots, x_m$ and $f : B_1 \to B_2$ be an embedding. Set $f_g = f \cup g : B_1 \cup \mathrm{dom}(g) \to \mathbb{G}$.

Let $a, b \in B_1 \cup \mathrm{dom}(g)$. If $a, b \in B_1$, then there is an edge between a and b if and only if there is an edge between $f_g(a)$ and $f_g(b)$. If a or b is among $x_0, \ldots, x_m$, then there is neither an edge between a and b nor one between $f_g(a)$ and $f_g(b)$. Thus f_g embeds $B_1 \cup \mathrm{dom}(g)$ into $\mathbb{G}$.

4. Let $\mathbb{G}^n$ be the random K_n-free graph, that is, the ultrahomogeneous countable graph which omits K_n, the complete graph on n vertices. Equivalently, $\mathbb{G}^n$ is the Fraïssé limit of the class of all finite K_n-free graphs. Using the same argument as for the random graph, one can see that $\mathbb{G}^n$ is ω-independent.

5. Let $\mathbb{E}$ be a countable equivalence relation with infinitely many infinite equivalence classes. Let $f : B_1 \to B_2$ be an embedding between two finite equivalence relations B_1 and B_2 (i.e., finite sets with equivalence classes induced from $\mathbb{E}$). Take a set $\{x_0, \ldots, x_m\}$ of elements from a fixed equivalence class such that $\{x_0, \ldots, x_m\} \cap (B_1 \cup B_2) = \emptyset$. Clearly for any partial permutation g of $\{x_0, \ldots, x_m\}$ the function $f \cup g$ is an embedding.

6. The same reasoning remains true if one considers $\mathbb{E}^n$, a countable equivalence relation with n infinite equivalence classes.

7. Let $(\mathbb{D}, \leq)$ be the universal countable ultrahomogeneous partially ordered set. This is the Fraïssé limit of all finite partially ordered sets—see [Sch] and [So] for more information. Let $f : B_1 \to B_2$ be an embedding between two finite suborders B_1 and B_2 of $\mathbb{D}$. Take a set $\{x_0, \ldots, x_m\} \subset \mathbb{D}$ such that

$$\forall i, j\ (i \neq j \ \Rightarrow\ \neg(x_i \leq x_j)) \text{ and } \forall y \in B_1 \cup B_2\ \forall i\ (\neg(x_i \leq y) \text{ and } \neg(y \leq x_i)).$$

Then for any partial permutation g of $\{x_0, \ldots, x_m\}$, the function $f \cup g$ is an embedding.

Let $x_0, \ldots, x_m$ be pairwise distinct elements of A. A *shift* on $\{x_0, \ldots, x_m\}$ is a partial function $\varphi : \{x_0, \ldots, x_m\} \to A$ such that $\varphi(x_i) = x_{i-1}$ for $i = 1, \ldots, m$ (φ is a *left-shift*) or $\varphi(x_i) = x_{i+1}$ for $i = 0, \ldots, m-1$ (φ is a *right-shift*). Note that φ is undefined either at x_0 or at x_m, so φ is actually a partial mapping on $\{x_0, \ldots, x_m\}$. An $(x_0, \ldots, x_m)$-*function*, where $x_0, \ldots, x_m$ are pairwise distinct, is a partial function $g : \bigcup_{i=1}^k I_i \to A$ such that:

(i) $I_1, \ldots, I_k$ are pairwise disjoint;
(ii) each I_i is of the form $\{x_p, x_{p+1}, \ldots, x_q\}$ for some $0 \leq p < q \leq m$;
(iii) each restriction $g \restriction I_i$ is a shift.

We will consider the following condition:

($*$) For any finitely generated substructures $B_1, B_2 \subset A$ and any $m \in \mathbb{N}$, there exist pairwise distinct $x_0, \ldots, x_m \in A \setminus (B_1 \cup B_2)$ such that for any embedding $f : B_1 \to B_2$ and any $(x_0, \ldots, x_m)$-function g, there exists an embedding $f_g : \mathrm{gen}(B_1 \cup \mathrm{dom}(g)) \to A$ with $f, g \subset f_g$.

Since every $(x_0,\dots,x_m)$-function g is a partial permutation of $\{x_0,\dots,x_m\}$, condition $(*)$ is weaker than ω-independence.

Assume that A is the Fraïssé limit of a class $\mathcal{K}_0$. Let

$$\mathcal{K} = \mathcal{K}_0 \star \mathcal{LO} := \{\langle B,\le\rangle : B \in \mathcal{K}_0 \text{ and } \le \text{ is a linear ordering on } B\}.$$

The class $\mathcal{K}_0$ has the *strong amalgamation property* if for any $A,B,C \in \mathcal{K}_0$ and embeddings $f : A \to B$ and $g : A \to C$, there is $D \in \mathcal{K}_0$ and embeddings $r : B \to D$ and $s : C \to D$ with $r \circ f = s \circ g$ such that $r(B) \cap s(C) = r(f(A)) = s(g(A))$. In [KPT] it was proved that if $\mathcal{K}_0$ is a Fraïssé class with strong amalgamation property, then so is $\mathcal{K}$. We will denote the Fraïssé limit of $\mathcal{K}$ by $A_\le$.

LEMMA 2.4. *Let A be an ω-independent ultrahomogeneous relational countable structure. Then $A_\le$ satisfies $(*)$.*

Proof. Let $B_1, B_2 \subset A$ and let $m \in \mathbb{N}$. Since A is ω-independent, there is a set $\{y_0,\dots,y_m\} \subset A \setminus (B_1 \cup B_2)$ such that, for any embedding $f : B_1 \to B_2$ and any partial permutation g of $y_0,\dots,y_m$, the function $f \cup g$ is an embedding. We define a linear order $\preceq$ on $B_1 \cup B_2 \cup \{y_0,\dots,y_m\}$ as follows: $\preceq$ on $B_1 \cup B_2$ equals $\le$, $y_i \preceq y_k$ provided $i \le k$, and $x \preceq y_i$ for every $x \in B_1 \cup B_2$ and $i = 0,\dots,m$. Since $B_1 \cup B_2 \cup \{y_0,\dots,y_m\}$ is a substructure of A, and $\preceq$ is a linear order on it, the structure $\langle B_1 \cup B_2 \cup \{y_0,\dots,y_m\}, \preceq\rangle$ can be embedded into $A_\le$. By Claim 2.3 we can find $x_0,\dots,x_m \in A$ such that $\langle B_1 \cup B_2 \cup \{x_0,\dots,x_m\}, \le\rangle$ is a substructure of $A_\le$ isomorphic to $\langle B_1 \cup B_2 \cup \{y_0,\dots,y_m\}, \preceq\rangle$.

Take any $A_\le$-embedding $f : B_1 \to B_2$ and any $(x_0,\dots,x_m)$-function g. Then $f \cup g$ is an A-embedding. Note that both f and g preserve $\le$. Since each element of $B_1 \cup B_2$ is in relation $\le$ to each x_i, the function $f \cup g$ is an $A_\le$-embedding. ∎

8. Consider the structure $(\mathbb{Q},\le)$ of all rational numbers. If $\mathbb{N}$ stands for the natural numbers without any structure, then $(\mathbb{Q},\le)$ is isomorphic to $\mathbb{N}_\le$. By Lemma 2.4, $(\mathbb{Q},\le)$ has $(*)$.

9. Let $(\mathbb{B},\vee,\wedge,\neg,0,1)$ be a countable atomless Boolean algebra. Let $B_1, B_2 \subset \mathbb{B}$ be finite subalgebras and let $f : B_1 \to B_2$ be an embedding. Let $C = \mathrm{gen}(B_1 \cup B_2)$ be the smallest subalgebra of $\mathbb{B}$ containing B_1 and B_2. Let $\{c_i : i \in I\}$ be the set of all atoms of C. We say that a finite subalgebra X of $\mathbb{B}$ is *independent of* C provided there is a finite set $\{x_j : j \in J\}$ with $\mathrm{gen}(\{x_j : j \in J\}) = X$ and

$$\bigwedge_{j\in J_1} x_j \wedge \bigwedge_{j\in J_2} \neg x_j \wedge c_i \ne 0$$

for every $i \in I$ and every partition J_1, J_2 of J. Clearly, such an algebra X exists and any one-to-one self-mapping of $\{x_j : j \in J\}$ can be extended to an automorphism of X.

Claim 2.5. *Let X be a finite algebra independent of $X_1 \cup X_2$, and let g be an automorphism of X. Then $f \cup g$ can be extended to an embedding $f_g : \mathrm{gen}(B_1 \cup X) \to \mathbb{B}$.*

Proof. Let $\{a_k : k \in K\}$ be the set of all atoms of B_1, and $\{b_k : k \in K\} \subset B_2$ be such that $f(a_k) = b_k$. The atoms of $\mathrm{gen}(B_1 \cup X)$ are of the form

$$\bigwedge_{j \in J_1} x_j \wedge \bigwedge_{j \in J_2} \neg x_j \wedge a_k$$

for every $k \in K$ and every partition J_1, J_2 of J. Define f_g on atoms as follows:

$$f_g\Big(\bigwedge_{j \in J_1} x_j \wedge \bigwedge_{j \in J_2} \neg x_j \wedge a_k\Big) = g\Big(\bigwedge_{j \in J_1} x_j \wedge \bigwedge_{j \in J_2} \neg x_j\Big) \wedge f(a_k).$$

Clearly, f_g can be uniquely extended to a homomorphism $f_g : \mathrm{gen}(B_1 \cup X) \to \mathbb{B}$. We need only prove that f_g is one-to-one. Suppose that

$$f_g\Big(\bigwedge_{j \in J_1} x_j \wedge \bigwedge_{j \in J_2} \neg x_j \wedge a_k\Big) = f_g\Big(\bigwedge_{j \in J_1'} x_j \wedge \bigwedge_{j \in J_2'} \neg x_j \wedge a_{k'}\Big).$$

Then

$$g\Big(\bigwedge_{j \in J_1} x_j \wedge \bigwedge_{j \in J_2} \neg x_j\Big) \wedge f(a_k) = g\Big(\bigwedge_{j \in J_1'} x_j \wedge \bigwedge_{j \in J_2'} \neg x_j\Big) \wedge f(a_{k'}).$$

Since X is independent of B_2, both sides of the above equality are nonzero. As f is embedding, we have $a_k = a_{k'}$. Moreover, g is an isomorphism of X, so $J_1 = J_1'$ and $J_2 = J_2'$. ■

Let $B_1, B_2 \subset \mathbb{B}$ be finite subalgebras and let $f : B_1 \to B_2$ be an embedding. For any $m \in \mathbb{N}$ one can find $x_0, \ldots, x_m$ witnessing that $X = \mathrm{gen}(\{x_0, \ldots, x_m\})$ is independent of $C = \mathrm{gen}(B_1 \cup B_2)$. Let g be any partial permutation of $x_0, \ldots, x_m$. We extend g to an isomorphism of X, and using Claim 2.5, we find an embedding f_g extending $f \cup g$. This shows that $\mathbb{B}$ is ω-independent (in particular, it satisfies $(*)$).

Note that $\mathbb{B}$ is not a relational structure, so we cannot apply Lemma 2.4.

It is folklore that $\mathbb{U}$, $\mathbb{G}$, $\mathbb{G}^n$, $\mathbb{E}$ and $\mathbb{E}^n$ have the strong amalgamation property, and there exist their ordered counterparts: the ordered rational Urysohn space $\mathbb{U}_\leq$, the ordered random graph $\mathbb{G}_\leq$, the ordered K_n-free random graph $\mathbb{G}^n_\leq$, and the ordered relations $\mathbb{E}_\leq$ and $\mathbb{E}^n_\leq$. All of those structures are relational and ω-independent, so we can apply Lemma 2.4 to conclude that each of them satisfies condition $(*)$.

Now we will show how $(*)$ implies the existence of a large free subgroup of $\mathrm{Aut}(A)$.

Let $m \in \mathbb{N}$ and let $r_1, \ldots, r_k \in \{1, \ldots, m\}$ be such that $r_i \neq r_{i+1}$ for $i \in \{1, \ldots, k-1\}$, and let $n_1, \ldots, n_k \in \mathbb{Z} \setminus \{0\}$. Then

$$w(y_1, \ldots, y_m) = y_{r_1}^{n_1} \ldots y_{r_k}^{n_k} \tag{2.1}$$

will be called a *word of length* n where $n = |n_1| + \dots + |n_k|$. If additionally, $f_1, \dots, f_m$ are functions or partial functions defined on A, then we denote by $w(f_1, \dots, f_m)$ the function defined in a natural way: the operation is the composition and the domain of $w(f_1, \dots, f_m)$ is the natural domain. It is possible that $w(f_1, \dots, f_m) = \emptyset$, and if all f_i are elements of $\mathrm{Aut}(A)$, then so is $w(f_1, \dots, f_m)$. We also consider the empty set $\emptyset$ as a word of length zero. In that case we also define $w(f_1, \dots, f_k) = \mathrm{id}$, the identity function. Clearly, $f_1, \dots, f_m \in \mathrm{Aut}(A)$ are *free generators*, i.e., they generate a free subgroup of $\mathrm{Aut}(A)$, if $w(f_1, \dots, f_m) \neq \mathrm{id}$ for every nonempty word $w(y_1 \dots, y_m)$.

LEMMA 2.6. *For every nonempty word $w(y_1, \dots, y_m)$ of length n, and for distinct $x_0, \dots, x_n$, there exist $(x_0, \dots, x_n)$-functions $g_1, \dots, g_m$ such that $w(g_1, \dots, g_m)(x_0) = x_n$.*

Proof. Assume that w is given by (2.1). We will define $g_{r_k}, g_{r_{k-1}}, \dots, g_{r_1}$ step by step. Since it is possible that $r_i = r_j$ for $i \neq j$, some of the functions $g_1, \dots, g_m$ may be defined in more than one step.

If $n_k < 0$, then set $g_{r_k}(x_i) = x_{i-1}$ for $i = 1, \dots, |n_k|$, and if $n_k > 0$, then set $g_{r_k}(x_i) = x_{i+1}$ for $i = 0, \dots, |n_k| - 1$.

If $n_{k-1} < 0$, then set $g_{r_{k-1}}(x_i) = x_{i-1}$ for $i = |n_k| + 1, \dots, |n_k| + |n_{k-1}|$, and if $n_{k-1} > 0$, then set $g_{r_{k-1}}(x_i) = x_{i+1}$ for $i = |n_k|, \dots, |n_k| + |n_{k-1}| - 1$.

We continue this procedure, and finally, if $n_1 < 0$, we set $g_{r_1}(x_i) = x_{i-1}$ for $i = |n_k| + \dots + |n_2| + 1, \dots, |n_k| + \dots + |n_1|$, and if $n_k > 0$, we set $g_{r_1}(x_i) = x_{i+1}$ for $i = |n_k| + \dots + |n_2|, \dots, |n_k| + \dots + |n_1| - 1$.

To illustrate the reasoning consider the following example. Let $w(y_1, y_2) = y_1^{-2} y_2 y_1^3$. Then $r_1 = 1$, $r_2 = 2$, $r_3 = 1$, $n_1 = -2$, $n_2 = 1$, $n_3 = 3$ and we define g_1 as the right-shift on $\{x_0, x_1, x_2, x_3\}$, g_2 as the right-shift on $\{x_3, x_4\}$, and finally g_1 as the left-shift on $\{x_4, x_5, x_6\}$. Then g_1 is a union of two shifts. ■

LEMMA 2.7. *Assume that A has property $(*)$. For any nonempty word $w(y_1, \dots, y_m)$ and any pairwise distinct finite sequences $s_1, \dots, s_m$ of 0's and 1's of the same length, the set*

$$D_w^{s_1,\dots,s_m} = \Big\{(n,p) : |s_1| \le n \text{ and for every } t_1, \dots, t_m \in \{0,1\}^n \text{ with } s_i \prec t_i \\ \text{we have } w(p(t_1), \dots, p(t_m)) \neq \mathrm{id}\Big\}$$

is dense in $\mathbb{P}$.

Proof. Choose any $(n,p) \in \mathbb{P}$ and let B_1 be a finitely generated substructure of A such that $\bigcup\{\mathrm{dom}(p(s)) : s \in \mathrm{dom}(p)\} \subset B_1$ and $|B_1| \ge |s_1|$. Set $n' = |B_1|$ and for every $s \in \{0,1\}^{n'}$ let $p'(s) : B_1 \to A$ be an embedding which extends $p(s{\restriction}n)$. Then $(n', p') \le (n,p)$.

Let $B_2 = \mathrm{gen}(\bigcup\{\mathrm{rng}(p'(s)) : s \in \mathrm{dom}(p')\})$, and let $(x_0, \dots, x_{|w|})$, where $|w|$ stands for the length of w, be chosen as in $(*)$. Then choose $(x_0, \dots, x_{|w|})$-

functions $g_1, \dots, g_m$ as in Lemma 2.6. Now, for every $i = 1, \dots, m$ and every $s \in \{0,1\}^{n'}$ with $s_i \prec s$, let $f_s : \mathrm{gen}(B_1 \cup \mathrm{dom}(g_i)) \to A$ be chosen for $p'(s)$ and g_i, according to $(*)$. Let $E = \mathrm{gen}(\bigcup\{\mathrm{dom}(f_s) : s_i \prec s\})$ and $n'' = |E|$. Finally, for every $t \in \{0,1\}^{n''}$, let $p''(s) : E \to A$ be defined in the following way. If $s_i \prec t$ for some $i = 1, \dots, m$, then $p''(t)$ is an extension of $f_{t\restriction n'}$; otherwise, let $p''(t)$ be any extension of $p'(t\restriction n')$. Then $(n'', p'') \le (n', p')$, and consequently $(n'', p'') \le (n, p)$.

We need to show that $(n'', p'') \in D_w^{s_1,\dots,s_m}$. If $t_1, \dots, t_m \in \{0,1\}^{n''}$ are such that $s_i \prec t_i$, then $p''(t_1), \dots, p''(t_m)$ are extensions of $g_1, \dots, g_m$, respectively. Thus by Lemma 2.6 we obtain

$$w(p''(t_1), \dots, p''(t_m))(x_0) = w(g_1, \dots, g_m)(x_0) = x_{|w|}. \ \blacksquare$$

THEOREM 2.8. *Assume that A satisfies $(*)$. Then for every residual set $Z \subset \mathrm{Aut}(A)$, there is a family $\mathcal{F} \subset Z$ of $\mathfrak{c}$ free generators.*

Proof. Let Z be a residual subgroup of $\mathrm{Aut}(A)$. By Lemmas 2.1, 2.2, 2.7 and the Rasiowa–Sikorski lemma, there exists a filter G on $\mathbb{P}$, which has nonempty intersection with all sets $D_k, D_w^{s_1,\dots,s_l}$ and D_{F_n}, where (F_n) is a sequence of nowhere dense sets such that $\mathrm{Aut}(A) \setminus Z = \bigcup F_n$.

Let $g : \{0,1\}^{\mathbb{N}} \to \mathrm{Aut}(A)$ be defined in the following way. If $\alpha \in \{0,1\}^{\mathbb{N}}$, then

$$g(\alpha) = \bigcup\{p(\alpha\restriction n) : (n, p) \in G\}.$$

First we show that $g(\alpha)$ is well defined. If $(n, p), (n', p') \in G$, then there is $(m, q) \in G$ below (n, p) and (n', p'). This ensures that if $x \in \mathrm{dom}(p(\alpha\restriction n)) \cap \mathrm{dom}(p'(\alpha\restriction n'))$, then $p(\alpha\restriction n)(x) = p'(\alpha\restriction n')(x)$.

Now, we show that $\mathrm{dom}(g(\alpha)) = \mathrm{rng}(g(\alpha)) = A$. Let $k \in A$. Since D_k is dense, there is $(n, p) \in D_k \cap G$. Then

$$k \in \mathrm{dom}(p(\alpha\restriction n)) \cap \mathrm{rng}(p(\alpha\restriction n)) \subset \mathrm{dom}(g(\alpha)) \cap \mathrm{rng}(g(\alpha)).$$

Now we show that $g(\alpha) \in \mathrm{Aut}(A)$. It is enough to show that for any finitely generated substructure C, $g(\alpha)\restriction C$ is an embedding. Assume $C = \{x_1, \dots, x_k\}$. Since $C \subset \mathrm{dom}(g(\alpha))$, there are $(p_1, n_1), \dots, (p_k, n_k) \in G$ such that $x_i \in \mathrm{dom}(p_i(\alpha\restriction n_i))$. Since G is a filter, there is $(m, q) \in G$ below each (n_i, p_i). This shows that $g(\alpha)(x_i) = q(\alpha\restriction m)(x_i)$ for every $i = 1, \dots, k$. Thus $g(\alpha)\restriction C = q(\alpha\restriction m)\restriction C$, which shows that it is an embedding.

Now we will show that $g(\alpha) \in Z$. Let $k \in \mathbb{N}$ and let $(n, p) \in G \cap D_{F_k}$. Then $g(\alpha) \in V(p(\alpha\restriction n)) \subset \mathrm{Aut}(A) \setminus F_k$. Since k has been taken arbitrarily, $g(\alpha) \in \mathrm{Aut}(A) \setminus \bigcup_{n\in\mathbb{N}} F_n = Z$.

It remains to show that $\{g(\alpha) : \alpha \in \{0,1\}^{\mathbb{N}}\}$ is a family of free generators. Let $w(y_1, \dots, y_m)$ be any word and $\alpha_1, \dots, \alpha_m$ be distinct elements of $\{0,1\}^{\mathbb{N}}$. Let $k \in \mathbb{N}$ be such that $\alpha_i\restriction k \ne \alpha_j\restriction k$ for $i \ne j$, and let $(n, p) \in D_w^{\alpha_1\restriction k,\dots,\alpha_m\restriction k} \cap G$. Since $\alpha_i\restriction k \prec \alpha_i\restriction n$ for $i = 1, \dots, m$, for some

$x \in A$ we have

$$w(g(\alpha_1), \dots, g(\alpha_m))(x) = w(p(\alpha_1 \restriction n), \dots, p(\alpha_m \restriction n))(x) \neq x.$$

This ends the proof. ∎

Let us note that condition $(*)$ does not imply that $\mathrm{Aut}(A)$ is oligomorphic (e.g. let A be the rational Urysohn space), therefore our result is different from that of Cameron mentioned in the Introduction.

3. Large free subgroups of S_∞. Now we show that, in the case of S_∞, the automorphism group of $\mathbb{N}$ without any structure, we can strengthen the conclusion of Theorem 2.8. Clearly, S_∞ is simply the group of all bijections of $\mathbb{N}$. We say that a bijection $f \in S_\infty$ is *proper* (or *has infinite support*) if for every finite set $B \subset \mathbb{N}$, there is $x \notin B$ such that $f(x) \neq x$.

LEMMA 3.1. *Assume $f_1, \dots, f_m$ are free generators and $w(y_1, \dots, y_m)$ is any nonempty word. Then $w(f_1, \dots, f_m)$ is proper.*

Proof. This follows from the fact that each $f \in S_\infty$ with $f^n \neq \mathrm{id}$ for every $n > 0$ (which clearly holds for the function $w(f_1, \dots, f_m)$) is automatically proper. Indeed, otherwise f would correspond to a bijection of a finite set (that is, $f = g \cup \mathrm{id}_{\mathbb{N}\setminus A}$ for some finite $A \subset \mathbb{N}$, where g is a permutation of A), and hence $f^n = \mathrm{id}$ where $n = |A|!$. ∎

LEMMA 3.2. *Let A be a relational structure which is ω-independent. For any bijections $f_1, \dots, f_k \in \mathrm{Aut}(A)$, $k \geq 2$, such that $f_2, \dots, f_{k-1}$ are proper, any nonzero numbers $n_1, \dots, n_{k-1}$ and every finite structure $C \subset A$, there exist $x \in A \setminus C$, finite structures $B_1, B_2 \subset A \setminus C$ and a bijection $g : B_1 \to B_2$ such that $x \in \mathrm{dom}(f_k \circ g^{n_{k-1}} \circ f_{k-1} \circ g^{n_{k-2}} \circ \dots \circ g^{n_1} \circ f_1)$ and*

$$f_k \circ g^{n_{k-1}} \circ f_{k-1} \circ g^{n_{k-2}} \circ \dots \circ g^{n_1} \circ f_1(x) \neq x.$$

Proof. We assume $k > 2$ (the case $k = 2$ is much simpler and will be obvious after considering the case $k > 2$). Since A is ω-independent, there exist $y_0, \dots, y_t$, $t > 2|C| + 5k$, such that for any isomorphism $h : C \to C$ and any partial permutation h' of $y_0, \dots, y_t$, the function $h \cup h'$ is an embedding.

We first show that there are elements $x_0, \dots, x_{2k-1}$ such that:

(a) $x_i \notin C$ for $i = 0, \dots, 2k-1$;
(b) $f_i(x_{2i-2}) = x_{2i-1}$ for $i = 1, \dots, k$;
(c) $x_1, \dots, x_{2k-2}$ are distinct;
(d) $x_0 \neq x_{2k-1}$.

First, take

$$x_1 \in \{y_0, \dots, y_t\} \setminus (f_1^{-1}(C) \cup C)$$

and set $x_0 = f_1^{-1}(x_1)$. Then take

$$x_2 \in \mathbb{N} \setminus \big(f_2^{-1}(C) \cup C \cup f_2^{-1}(\{x_0, x_1\}) \cup \{x_0, x_1\}\big)$$

such that $f_2(x_2) \neq x_2$ and set $x_3 = f_2(x_2)$. It is easy to see that (a) holds for $i = 0, 1, 2, 3$; (b) holds for $i = 1, 2$; and x_1, x_2, x_3 are distinct. In the next step we take

$$x_4 \in \mathbb{N} \setminus \big(f_3^{-1}(C) \cup C \cup f_3^{-1}(\{x_0, x_1, x_2, x_3\}) \cup \{x_0, x_1, x_2, x_3\}\big)$$

such that $f_3(x_4) \neq x_4$ and set $x_5 = f_3(x_4)$. We continue this procedure, and finally we take

$$x_{2k-2} \in \mathbb{N} \setminus \big(f_k^{-1}(C) \cup C \cup f_k^{-1}(\{x_0, \ldots, x_{2k-3}\}) \cup \{x_0, \ldots, x_{2k-3}\}\big)$$

and $x_{2k-1} = f_k(x_{2k-2})$. Then (a)–(d) are satisfied.

Now take elements $y_0^1, \ldots, y_{|n_1|}^1, y_0^2, \ldots, y_{|n_2|}^2, \ldots, y_0^{k-1}, \ldots, y_{|n_{k-1}|}^{k-1}$ such that

(i) $y_0^i = x_{2i-1}$ and $y_{|n_i|}^i = x_{2i}$ for $i = 1, \ldots, k-1$;
(ii) $y_j^i \notin C$ for all i, j;
(iii) $y_0^1, \ldots, y_{|n_1|}^1, y_0^2, \ldots, y_{|n_2|}^2, \ldots, y_0^{k-1}, \ldots, y_{|n_{k-1}|}^{k-1}$ are distinct.

By (a) and (c), we can choose such elements. For every $i = 1, \ldots, k-1$, let

$$D_i = \begin{cases} \{y_0^i, \ldots, y_{|n_i|-1}^i\} & \text{if } n_i > 0, \\ \{y_1^i, \ldots, y_{|n_i|}^i\} & \text{if } n_i < 0. \end{cases}$$

Now we define a function g on $B = D_1 \cup \cdots \cup D_{k-1}$ in the following way. For every $i = 1, \ldots, k-1$, set

$$g(y_l^i) = \begin{cases} y_{l+1}^i & \text{if } n_i > 0,\ l = 0, \ldots, |n_i| - 1, \\ y_{l-1}^i & \text{if } n_i < 0,\ l = 1, \ldots, |n_i|. \end{cases}$$

By (iii), the function g is well defined, one-to-one, and $B \cup g(B) \subset \mathbb{N} \setminus C$. Also, for every $i = 1, \ldots, k-1$, by (i), we have

$$g^{n_i}(x_{2i-1}) = g^{n_i}(y_0^i) = y_{|n_i|}^i = x_{2i}.$$

Together with (b) and (d), this gives the assertion. ∎

Lemma 3.3. *Assume that $f_1, \ldots, f_m \in S^\infty$ are pairwise distinct free generators. Then there is $g \in S_\infty \setminus \{f_1, \ldots, f_m\}$ such that $f_1, \ldots, f_m, g$ are free generators.*

Proof. It is enough to show that there exists $g \in S_\infty$ such that for any word $w = w(y_1, \ldots, y_{m+1})$ such that y_{m+1} appears in w, $w(f_1, \ldots, f_m, g) \neq \mathrm{id}$. The family of such words is countable; enumerate it as $W = \{w_n : n \in \mathbb{N}\}$. We will define sequences (C_n) and (C_n') of pairwise disjoint, finite subsets of $\mathbb{N}$, and a sequence of partial functions (g_n), such that for every $n \in \mathbb{N}$,

1. $C_n' \subset C_n$;
2. $C_n \setminus C_n' \neq \emptyset$;

3. $g_n : C'_n \to C_n$ is one-to-one;
4. there is $x_n \in C_n$ such that $x_n \in \mathrm{dom}(w_n(f_1, \dots, f_m, g_n))$ and $w_n(f_1, \dots, f_m, g_n)(x_n) \neq x_n$.

Then any bijective extension of $g = \bigcup_{n\in\mathbb{N}} g_n$ will satisfy our needs. Such an extension exists, since by 1–3, the sets $\mathrm{dom}(g)$, $\mathbb{N} \setminus \mathrm{dom}(g)$, $\mathrm{rng}(g)$ and $\mathbb{N} \setminus \mathrm{rng}(g)$ are infinite.

Let $n = 1$. Write y instead of y_{m+1}. Then

$$w_1 = u_k \cdot y^{n_{k-1}} \cdot u_{k-1} \cdot y^{n_{k-2}} \cdots y^{n_1} \cdot u_1$$

for some words $u_1, \dots, u_k$ in which y does not appear (it is possible that u_1 or u_k are empty words, but for $i \notin \{1, k\}$, u_i is nonempty). By Lemma 3.2 applied to the functions $f_i = u_i(f_1, \dots, f_m)$ (if u_i is empty, then $f_i = \mathrm{id}$) and $C = \emptyset$, there are finite sets B_1, B_2, an element x_1 and a bijective map $g_1 : B_1 \to B_2$ such that $x_1 \in \mathrm{dom}(w_1(f_1, \dots, f_m, g_1))$ and $w_1(f_1, \dots, f_m, g_1)(x_1) \neq x_1$. Let $C_1 = B_1 \cup B_2 \cup \{x_1, y_1\}$, where y_1 is not in $B_1 \cup B_2 \cup \{x_1\}$, and $C'_1 = B_1$.

Assume that we have already made the construction up to step n. Then we proceed exactly as in the first step, but for the word w_{n+1}, and we use Lemma 3.2 for $C = C_1 \cup \dots \cup C_n$. ∎

If w, w' are words, then we write $w' \leq w$ whenever w' is created from w by erasing some symbols on the left side. In particular,

$$y_{r_2}^{n_2} \dots y_{r_k}^{n_k} \leq y_{r_1}^{n_1} y_{r_2}^{n_2} \dots y_{r_k}^{n_k},$$

and if $n_1 > 0$, then

$$y_{r_1}^{n_1-1} y_{r_2}^{n_2} \dots y_{r_k}^{n_k} \leq y_{r_1}^{n_1} y_{r_2}^{n_2} \dots y_{r_k}^{n_k}.$$

Also, we assume that $\emptyset \leq w$ for any w.

LEMMA 3.4. *For any $k, l \in \mathbb{N}$, any word $w(y_1, \dots, y_m)$ with $k + l = m$, any free generators $f_1, \dots, f_k \in S^\infty$, and any pairwise different sequences $s_1, \dots, s_l$ of 0's and 1's of the same length, the set*

$$D_{w,f_1,\dots,f_k}^{s_1,\dots,s_l} = \{(n, p) : n \geq |s_1| \text{ and if } t_1, \dots, t_l \in \{0,1\}^n \text{ with } s_i \prec t_i \\ \text{then } w(f_1, \dots, f_k, p(t_1), \dots, p(t_l)) \neq \mathrm{id}\}$$

is dense in $\mathbb{P}$.

Proof. Take any $(n, p) \in \mathbb{P}$ and set $D = \bigcup\{\mathrm{dom}(p(s)) \cup \mathrm{rng}(p(s)) : s \in \mathrm{dom}(p)\}$. Let $g_1, \dots, g_l \in S_\infty \setminus \{f_1, \dots, f_k\}$ be pairwise distinct and such that $f_1, \dots, f_k, g_1, \dots, g_l$ are free generators; we can find such g_i's by Lemma 3.3. Set $B = \bigcup\{w'(f_1, \dots, f_k, g_1, \dots, g_l)^{-1}(D) : w' \leq w\}$, where $w'(f_1, \dots, g_l)^{-1}(D)$ denotes the preimage of D under $w'(f_1, \dots, f_k, g_1, \dots, g_l)$; in particular, $D \subset B$. Since $f_1, \dots, f_k, g_1, \dots, g_l$ are free and B is finite, by Lemma 3.1 there exists $x \in \mathbb{N} \setminus B$ such that $w(f_1, \dots, f_k, g_1, \dots, g_l)(x) \neq x$.

For every $i = 1, \dots, l$, let

$$E^i = \{w'(f_1, \dots, f_k, g_1, \dots, g_l)(x) : w' \le w \text{ and } w' \text{ begins with } y_{k+i}\},$$
$$E_i = \{w'(f_1, \dots, f_k, g_1, \dots, g_l)(x) : y_{k+i}w' \le w\}.$$

Since $x \in \mathbb{N} \setminus B$, we have $E_i \cap D = \emptyset$ and $E^i \cap D = \emptyset$. Now for every $i = 1, \dots, n$ set $h_i = g_i{\restriction}E_i$. Then h_i is a bijection between E_i and E^i.

We are ready to define (n', p'). Let

$$n' = n + |s_1| + \max\{|E_1|, \dots, |E_n|\}.$$

For $i = 1, \dots, l$, let $G_i \subset \mathbb{N} \setminus (B \cup E_i \cup E^i)$ be such that $|G_i| + n + |E_i| = n'$.

Now, for $t \in \{0,1\}^{n'}$ with $s_i \prec t$, set

$$p'(t) = p(t{\restriction}n) \cup h_i \cup \mathrm{id}_{G_i}.$$

For the remaining $t \in \{0,1\}^{n'}$, let $p'(t)$ be any bijective extension of $p(t{\restriction}n)$ with $|\mathrm{dom}(p'(t))| = n'$. Clearly, $(n', p') \in \mathbb{P}$ and $(n', p') \le (n, p)$. If $t_1, \dots, t_l$ are in $\{0,1\}^{n'}$ and $s_i \prec t_i$ for $i = 1, \dots, l$, then

$$\begin{aligned} w(f_1, \dots, f_k, p'(t_1), \dots, p'(t_l))(x) &= w(f_1, \dots, f_k, h_1, \dots, h_l)(x) \\ &= w(f_1, \dots, f_k, g_1, \dots, g_l)(x) \ne x. \end{aligned}$$

Hence $(n', p') \in D^{g_1,\dots,g_l}_{w,f_1,\dots,f_k}$. ∎

Now we extend Theorem 2.8 and Lemma 3.3.

THEOREM 3.5. *For any residual set $Z \subset S_\infty$ and any countable family $\mathcal{F}$ of free generators, there is a family $\mathcal{F}' \subset Z$ of free generators of cardinality $\mathfrak{c}$ such that $\mathcal{F} \cup \mathcal{F}'$ is a family of free generators.*

Proof. The proof is very similar to that of Theorem 2.8: using the Rasiowa–Sikorski lemma, we choose a generic filter G which has nonempty intersection with all sets D_k, D_{F_k}, $D^{s_1,\dots,s_l}_w$ and $D^{s_1,\dots,s_l}_{w,f_1,\dots,f_k}$ (where $f_1, \dots, f_k$ are elements of $\mathcal{F}$). Again, for every $\alpha \in \{0,1\}^{\mathbb{N}}$, we set

$$g(\alpha) = \bigcup\{p(\alpha{\restriction}n) : (n, p) \in G\}.$$

In view of the proof of Theorem 2.8, we only have to show that $\mathcal{F} \cup \{g(\alpha) : \alpha \in \{0,1\}^{\mathbb{N}}\}$ is a family of free generators. Let $w = w(y_1, \dots, y_n)$ be any word, let $k, l \in \mathbb{N}$ be such that $k + l = n$, and let $f_1, \dots, f_k \in \mathcal{F}$ be distinct. Let $\alpha_1, \dots, \alpha_l$ be different elements of $\{0,1\}^{\mathbb{N}}$, and let $r \in \mathbb{N}$ be such that $\alpha_i{\restriction}r \ne \alpha_j{\restriction}r$ for $i \ne j$. Let $(n, p) \in D^{\alpha_1{\restriction}r,\dots,\alpha_l{\restriction}r}_{w,f_1,\dots,f_k} \cap G$. Since $\alpha_i{\restriction}r \prec \alpha_i{\restriction}n$ for $i = 1, \dots, l$, there is $x \in \mathbb{N}$ such that

$$\begin{aligned} &w(f_1, \dots, f_k, g(\alpha_1), \dots, g(\alpha_l))(x) \\ &\qquad = w(f_1, \dots, f_k, p(\alpha_1{\restriction}n), \dots, p(\alpha_l{\restriction}n))(x) \ne x. \end{aligned}$$

This ends the proof. ∎

Let $\mathcal{M}$ stand for the σ-ideal of meager subsets of $\mathbb{R}$. Let $\mathfrak{m}_{\text{countable}} = \min\{\kappa$: "MA(κ) for countable posets" fails$\}$ (MA stands for Martin's Axiom). It is well known (see [V]) that

$$\mathfrak{m}_{\text{countable}} = \operatorname{cov}(\mathcal{M}) := \min\{|\mathcal{F}| : \bigcup \mathcal{F} = \mathbb{R},\ \mathcal{F} \subset \mathcal{M}\}.$$

Since the poset $\mathbb{P}$ is countable, we obtain the following.

THEOREM 3.6. *For any residual set $Z \subset S_\infty$ and any family $\mathcal{F}$ of free generators of cardinality less than* $\operatorname{cov}(\mathcal{M})$, *there is a family $\mathcal{F}' \subset Z$ of free generators of cardinality $\mathfrak{c}$ such that $\mathcal{F} \cup \mathcal{F}'$ is a family of free generators.*

4. Products of countable groups. In this section we will give a necessary and sufficient condition on a sequence of countable groups $G_1, G_2, \ldots$ for the existence of a free subgroup of $\prod G_n$ of $\mathfrak{c}$ generators. A family $\{X_s : s \in S\}$ of subsets of $\mathbb{N}$ is *independent* if $\bigcap_{s\in E} X_s \cap \bigcap_{s\in F} (\mathbb{N} \setminus X_s) \neq \emptyset$ for every finite $F, E \subset S$ with $E \cap F = \emptyset$. It is well known that there is an independent family of cardinality $\mathfrak{c}$.

LEMMA 4.1. *Let $n \geq 2$. There exists a family $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{c}\}$ of functions from $\{0, \ldots, n-1\}^{\mathbb{N}}$ such that for any $\alpha_0 < \cdots < \alpha_{n-1} < \mathfrak{c}$ there is $k \in \mathbb{N}$ such that $f_{\alpha_i}(k) = i$.*

Proof. Let $\{p_k : k \in \mathbb{N}\}$ be an enumeration of all subsets of $\mathbb{N}$ of cardinality n. Enumerate each p_k as $\{p_k(0), \ldots, p_k(n-1)\}$. Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ be an independent family of $\mathbb{N}$. For any α we define $f_\alpha : \mathbb{N} \to \{0, \ldots, n-1\}$ as follows. Fix $k \in \mathbb{N}$. If there is $i < n$ such that $p_k(i) \in U_\alpha$ and $p_k(j) \notin U_\alpha$ for every $j \neq i$, then set $f_\alpha(k) = i$; otherwise set $f_\alpha(k) = 0$.

Let $\alpha_0 < \cdots < \alpha_{n-1}$. Pick $m_i \in U_{\alpha_i} \setminus \bigcup_{j\neq i} U_{\alpha_j}$ and set $p(i) = m_i$ for $i < n$. There is $k \in \mathbb{N}$ with $p = p_k$. Then $f_{\alpha_i}(k) = i$. ∎

Recall that if a word w is of the form $w = w(y_1, \ldots, y_m)$, then we assume that all variables of w are in $y_1, \ldots, y_m$, but not necessarily all y_i's must appear in w.

THEOREM 4.2. *Let G_n, $n \in \mathbb{N}$, be a family of groups.*

(i) *If for any nonempty word $w(y_1, \ldots, y_m)$ there are infinitely many n's for which there are $g_{n,1}, \ldots, g_{n,m} \in G_n$ with $w(g_{n,1}, \ldots, g_{n,m}) \neq e_n$ where e_n is a neutral element of G_n, then $\prod_{n=1}^\infty G_n$ contains a free group on $\mathfrak{c}$ generators.*

(ii) *If every G_n is countable, and for some nonempty word $w(y_1, \ldots, y_m)$, almost every n and all $g_{n,1}, \ldots, g_{n,m} \in G_n$ we have $w(g_{n,1}, \ldots, g_{n,m}) = e_n$, then $\prod_{n=1}^\infty G_n$ does not contain any free group on uncountably many generators.*

Proof. Assume that for any word $w(y_1, \ldots, y_m)$ there are infinitely many n's for which there are $g^w_{n,1}, \ldots, g^w_{n,m} \in G_n$ with $w(g^w_{n,1}, \ldots, g^w_{n,m}) \neq e_n$. For any nonempty word $w = w(y_1, \dots, y_m)$, set

$$E_w = \{n \in \mathbb{N} : \text{there are } g^w_{n,1}, \dots, g^w_{n,m} \in G_n \text{ with } w(g^w_{n,1}, \dots, g^w_{n,m}) \neq e_n\}.$$

Then $\{E_w : w = w(y_1, \dots, y_m)$ is a nonempty word$\}$ is a countable family of infinite sets. Let $\{E'_w : w = w(y_1, \dots, y_m)$ is a nonempty word$\}$ be a disjoint refinement of this family, i.e., a family of pairwise disjoint infinite sets with $E'_w \subset E_w$ for any nonempty word w. For any $\alpha < \mathfrak{c}$, define $f_\alpha \in \prod G_n$ as follows. Let w be a word. Consider two cases.

1. If $w = w(y_k)$ is a word with one variable y_k, then let $\{f^w_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the set $\prod_{n \in E'_w} \{e_n, g^w_{n,k}\} \setminus \prod_{n \in E'_w} \{e_n\}$.

2. If $w = w(y_1, \dots, y_m)$, then using Lemma 4.1 we can find a family $\{f^w_\alpha : \alpha < \mathfrak{c}\}$ such that for any $\alpha_1 < \dots < \alpha_m$ there is $n \in E'_w$ with $f^w_{\alpha_i}(n) = g^w_{n,k_i}$ for $i \leq m$. Finally, let $f_\alpha(n) = f^w_\alpha(n)$ if $n \in E'_w$, and $f_\alpha(n) = e_n$ otherwise. Clearly, in both cases, $\{f_\alpha : \alpha < \mathfrak{c}\}$ consists of free generators.

Assume now the G_n are countable, and let $w(y_1, \dots, y_m)$ be a word such that there is N with $w(g_{n,1}, \dots, g_{n,m}) = e_n$ for $n \geq N$ and all $g_{n,1}, \dots, g_{n,m}$ in G_n. Suppose $\prod_{n=1}^{\infty} G_n$ contains a free group on uncountably many generators, say $\{f_\alpha : \alpha < \omega_1\}$. Then for any distinct $\alpha_1, \dots, \alpha_m < \omega_1$ there is $n < N$, depending on α_i's, with $w(f_{\alpha_1}(n), \dots, f_{\alpha_m}(n)) \neq e_n$. As the groups G_n are countable, one can find two distinct m-element sets $\{\alpha_1, \dots, \alpha_m\}$ and $\{\beta_1, \dots, \beta_m\}$ of ordinals less than ω_1 such that

$$w(f_{\alpha_1}(n), \dots, f_{\alpha_m}(n)) = w(f_{\beta_1}(n), \dots, f_{\beta_m}(n))$$

for every $n < N$. Then

$$w(f_{\alpha_1}(n), \dots, f_{\alpha_m}(n)) w^{-1}(f_{\beta_1}(n), \dots, f_{\beta_m}(n)) = e_n$$

for every $n \in \mathbb{N}$. This contradicts the fact that $\{f_\alpha : \alpha < \omega_1\}$ are free generators. ∎

From Theorem 4.2 we immediately obtain the following dichotomy.

COROLLARY 4.3. *Let G_n, $n \in \mathbb{N}$, be countable groups. Then $\prod_{n \in \mathbb{N}} G_n$ either contains free subgroups on $\mathfrak{c}$ generators, or does not contain free subgroups on uncountably many generators.*

5. Final remarks and open questions. The results of Section 2 can be deduced from those of Section 3 for some class of structures. We say that a subset X of A is *independent* if any bijection $f : X \to X$ can be extended to an automorphism of A. If A contains an infinite independent set X, then take a set $\mathcal{F} \subset S_\infty(X)$ of $\mathfrak{c}$ free generators, and extend every $f \in \mathcal{F}$ to an automorphism f' of A. Then $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is a set of free generators in $\mathrm{Aut}(A)$.

Let X be an infinite independent, in the sense of Boolean algebras, set in $\mathbb{B}$. Then X is independent in the above sense. Now, let $X \subset \mathbb{U}$ be an isometric copy of $\mathbb{N}$ with the metric d given by $d(x, y) = 1 \Leftrightarrow x \neq y$. Then X is an independent subset of $\mathbb{U}$. However, $\mathbb{Q}$ does not contain an independent subset of cardinality greater than 2. The direct sum of countably many copies of $(\mathbb{Q}, +)$ is a countable ultrahomogeneous structure, and any of its finitely generated substructures is a torsion free Abelian group. Note that all of its finitely generated substructures are infinite and each of them contains an infinite independent subset. Hence the automorphism group of such a substructure contains a large free subgroup, and this cannot be proved by our method.

We are interested in extending small free subgroups of $\mathrm{Aut}(A)$ to large free groups. We introduce the cardinal number

$$\mathfrak{f}_A = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a maximal set of free generators in } \mathrm{Aut}(A)\}$$

where "maximal" means that $\mathcal{F}$ cannot be extended to a larger set of free generators. In Section 3 we proved that $\mathfrak{f} := \mathfrak{f}_{\mathbb{N}}$ is an uncountable cardinal $\geq \mathrm{cov}(\mathcal{M})$.

We end with a list of open questions:

1. Can one prove a similar result to that in Section 2 for structures whose finitely generated substructures are infinite?
2. Does $(*)$ imply that $\mathfrak{f}_A$ is uncountable? Does Martin's Axiom imply that $\mathfrak{f}_A = \mathfrak{c}$?
3. Is it true that $\mathfrak{f} = \mathrm{cov}(\mathcal{M})$?
4. Is it true that either $\mathrm{Aut}(A)$ does not contain an uncountably (infinitely) generated free subgroup, or it contains a free subgroup on $\mathfrak{c}$ generators?

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