COLLOQUIUM MATHEMATICUM

VOL. 109 2007 NO. 2

A PRIDDY-TYPE KOSZULNESS CRITERION FOR NON-LOCALLY FINITE ALGEBRAS

BY

MAURIZIO BRUNETTI and ADRIANA CIAMPELLA (Napoli)

Abstract. A celebrated result by S. Priddy states the Koszulness of any locally finite homogeneous PBW-algebra, i.e. a homogeneous graded algebra having a Poincaré–Birkhoff–Witt basis. We find sufficient conditions for a non-locally finite homogeneous PBW-algebra to be Koszul, which allows us to completely determine the cohomology of the universal Steenrod algebra at any prime.

Introduction. The notion of Koszul algebra, introduced by S. Priddy in [15] in particular to construct resolutions for the Steenrod algebra, has led to remarkable achievements in the study of associative algebras defined by quadratic relations. The Koszulness condition provides decisive information to solve several basic problems in that context. [14] gives a beautiful and comprehensive account of the impact of Koszul algebras in several areas of mathematics. Such algebras arise in fact in algebraic geometry, representation theory, non-commutative geometry, number theory, and obviously algebraic topology.

In this paper we deal with homogeneous algebras A isomorphic to a quotient of the form $T(V)/J(R)$, where $T(V)=\bigoplus_i T_i$ is the tensor algebra over a $\mathbb{K}$-vector space V with basis $X=\{x_i \mid i\in\mathcal{I}\}$, $\mathcal{I}$ is a (not necessarily bounded) totally ordered set, and $J(R)$ is the two-sided ideal of relations generated by some $R\subset T_2=V\otimes V$.

Note that all the Koszulness criteria listed for example in [9] concerning the Hilbert series of A become meaningless if $\mathcal{I}$ is not finite; even Priddy's criterion, i.e. the existence of a Poincaré–Birkhoff–Witt basis [15], only holds if the algebra has an internal degree induced by a map $g:\bigcup\mathcal{I}^n\to\mathbb{Z}$ and it is locally finite with respect to length and g (see [15]). It follows that there are examples of homogeneous quadratic algebras whose Koszulness cannot be checked using directly the criteria listed in [9] and [14]: Poisson enveloping algebras of Poisson algebras with generators indexed by $\mathbb{Z}$ and quadratic brackets (see [11] for the definition), infinite quantum grassmannians (see

2000 *Mathematics Subject Classification*: 16S37, 18G15, 55S10.

Key words and phrases: Koszul algebras, cohomology of algebras.

Definition 3.3 and Example 3.4), and the universal Steenrod algebra $Q(p)$ at the prime p. We introduce a class $\mathcal{G}$ of PBW-algebras, and show that all the algebras in $\mathcal{G}$ are Koszul. Proposition 2.5 then gives a sufficient condition for an algebra A to be in $\mathcal{G}$, which is reasonably easy to check. Our tools resemble and generalize the methods used in [4] to show that the algebra $Q(2)$ is Koszul.

The paper is organized as follows. Section 1 contains the definitions of a Koszul algebra and a PBW-basis; in Section 2 we define the class $\mathcal{G}$ and prove that all the algebras in $\mathcal{G}$ are Koszul; in Section 3 we give a list of non-locally finite algebras in $\mathcal{G}$, and finally in Section 4 we give the solution to a problem left unsolved in [5] and in [6], which actually motivated this research: the identification of the target in a certain embedding of the E_2-term of the Adams spectral sequence. The paper ends with a short digression on the characteristics of this hard-to-find non-Koszul PBW algebra that probably does not exist.

1. Preliminaries. By a *quadratic algebra* A we shall always mean what A. Polishchuk and L. Positselski call a one-generated homogeneous quadratic associative algebra with unit 1_A (see [14, p. 6]). Such an algebra is determined by a vector space V with basis $X = \{x_i \mid i \in \mathcal{I}\}$, and a subspace of quadratic relations $R \subseteq V \otimes V$. As recalled in the Introduction, A is isomorphic to a quotient of the free associative algebra $T(V) = \bigoplus_i T_i$. The kernel of the quotient map $p : T(V) \to A$ is the two-sided ideal $J(R)$ generated by R. We shall always assume that $\mathcal{I}$ is a totally ordered set, without making any assumption on its boundedness.

A quadratic algebra is naturally augmented by $\varepsilon : A \to \mathbb{K}$ which maps the $p(x_i)$'s to zero. The algebra A is then decomposed as $\mathbb{K} \oplus A_+$, where $\mathbb{K}$ is the line spanned by 1_A and A_+ is the augmentation ideal $\operatorname{Ker} \varepsilon$. Unless otherwise stated, we always compute $\operatorname{Tor}^A(\mathbb{K}, \mathbb{K})$ and $\operatorname{Ext}_A(\mathbb{K}, \mathbb{K})$ with respect to the augmentation ε. In notations, the first degree of the cohomology algebra

$$H(A) = \bigoplus \operatorname{Ext}_A^{s,t}(\mathbb{K}, \mathbb{K})$$

is the homological degree and the second one denotes the length. The diagonal cohomology $D(A) = \bigoplus H^{q,q}(A)$ is in general a subalgebra of $H(A)$.

Definition 1.1. A homogeneous quadratic algebra A is said to be *Koszul* if

$$H(A) = D(A).$$

This definition can be found in [15] and [14]. The reader should be aware that the algebras studied in [15] are positively graded with respect to the internal degree (see the definition below), while the algebras we are going to introduce do not satisfy this condition.

Any subset of the free monoid $\bigcup \mathcal{I}^k$ is totally ordered by the length first, and then by the natural lexicographical order. For any multi-indices or *labels*

$$I = (i_1, \ldots, i_k) \in \mathcal{I}^k \quad \text{and} \quad J = (j_1, \ldots, j_l) \in \mathcal{I}^l,$$

we set $x_I = x_{i_1} \cdots x_{i_k}$, $\ell(I) = k$, and $(I, J) = (i_1, \ldots, i_k, j_1, \ldots, j_l)$. By convention, the monomial $x_\emptyset$ associated to $\emptyset \in \mathcal{I}^0$ represents 1 in $T(V)$, hence $p(x_\emptyset) = 1_A$.

Let now $\mathcal{B}$ be a basis of monomials for A_+. We associate to $\mathcal{B}$ the following set of multi-indices:

$$S_{\mathcal{B}} = \{I \mid a_I \in \mathcal{B}\},$$

where $a_I = p(x_I)$.

Definition 1.2. A basis of monomials $\mathcal{B}$ for A_+ is a *Poincaré–Birkhoff–Witt* (*PBW*) *basis* if the following conditions hold.

1. For any I and J in $S_{\mathcal{B}}$ such that $a_I a_J \neq 0$, either $a_I a_J$ belongs to $\mathcal{B}$, or the label of each monomial which appears in the expression of $a_I a_J$ in terms of elements of $\mathcal{B}$ is greater than (I, J).
2. For each $k > 2$, $(i_1, \ldots, i_k) \in S_{\mathcal{B}}$ if and only if $(i_1, \ldots, i_j)$ and $(i_{j+1}, \ldots, i_k)$ are in $S_{\mathcal{B}}$ for each $j \in \{1, \ldots, k-1\}$.

Suppose now that the set R is homogeneous with respect to the *internal degree*

$$\deg x_{i_1} \cdots x_{i_n} := g(i_1, \ldots, i_n),$$

where $g : \bigcup \mathcal{I}^k \to \mathbb{Z}$ denotes a fixed monoid homomorphism. In this way A becomes a bigraded object. Typically and throughout the paper, when $\mathcal{I} \subseteq \mathbb{Z}$, the internal degree is given by the map $(i_1, \ldots, i_n) \mapsto i_1 + \cdots + i_n$. We shall say that an algebra A is *locally finite* if the $\mathbb{K}$-module

$$A^{t,p} = \{\text{elements of length } t \text{ and internal degree } p\}$$

is finitely generated for any $t \in \mathbb{N}$ and $p \in \mathbb{Z}$.

We shall also make use of the following filtration for $A = T(V)/J(R)$. Denote by A_i the subalgebra generated by all the $p(x_j)$'s with $j \leq i$. There are two families of inclusions,

$$\psi_k : A_k \to A \quad \text{and} \quad \phi_{i_1 i_2} : A_{i_1} \to A_{i_2}, \qquad \forall i_1 \leq i_2.$$

We have

$$A \cong \varinjlim \{A_\bullet, \phi_\bullet\}.$$

Proposition 1.3. *The homology of A is isomorphic to*

$$\varinjlim \{\mathrm{Tor}^{A_i}(\mathbb{K}, \mathbb{K}), (\phi_\bullet)_*\}.$$

Furthermore if each A_i is locally finite, the cohomology of A is isomorphic to

$$\varprojlim \{\mathrm{Ext}_{A_i}(\mathbb{K}, \mathbb{K}), \phi_\bullet^*\}. \tag{1.1}$$

Proof. The homology functor commutes with direct limits. The local finiteness of A_i's ensures that the inverse limit satisfies the Mittag-Leffler conditions: in this case the dimension of $\mathrm{Ext}^{s,t,p}_{A_i}(\mathbb{K},\mathbb{K})$ as a $\mathbb{K}$-module is finite in every fixed homological degree s, length t and internal degree p. Thus $\varprojlim^1\{\mathrm{Ext}_{A_i}(\mathbb{K},\mathbb{K}),\phi^*_\bullet\}=0$. ∎

REMARK 1.4. When $\mathcal{I}\subseteq\mathbb{Z}$, the cohomology of A is surely given by (1.1). In fact the subalgebra A_i is a quotient of $T(\mathrm{Span}\{x_j\mid j\le i\})$ which is locally finite (see Proposition 3.1 in [5]).

The next proposition states the famous Priddy Koszulness criterion.

PROPOSITION 1.5. *Every locally finite PBW-algebra A is a Koszul algebra.*

Proof. See Priddy's original proof in [15, Section 5], and note that the hypothesis on $\mathcal{I}$ to be bounded below (which is tacitly assumed by that author who usually thinks of $\mathcal{I}$ as the positive integers) can be replaced by local finiteness. ∎

Unfortunately, even when $\mathcal{I}\subseteq\mathbb{Z}$ Propositions 1.3 and 1.4 are not sufficient alone to prove the Koszulness of a non-locally finite algebra A: in general the algebras A_i are neither Koszul nor quadratic (see Example 3.5).

We end this section by recalling the notion of *reduction* introduced by G. M. Bergman in [3] to state the diamond lemma for associative algebras (see also [2]).

Let $\mathcal{M}$ be the set of monomials in $T(V)$. A *reduction system* is by definition a subset $\mathcal{S}\subset\mathcal{M}\times T(V)$; we assume here that every monomial appears at the first place in at most one pair in $\mathcal{S}$, hence it makes sense to denote its generic element by (x_J,y_J), where y_J is a suitable non-commutative polynomial in $T(V)$. Every reduction system determines a family of $\mathbb{K}$-linear endomorphisms in $T(V)$. Namely, an element $\sigma_J=(x_J,y_J)\in\mathcal{S}$ determines the maps

$$r_{I'\sigma_J I''}:x_I\to\begin{cases}x_{I'}y_Jx_{I''} & \text{if } I=(I',J,I''),\\ x_I & \text{otherwise.}\end{cases}$$

Such maps are called *simple reductions*; any finite composition of simple reductions is called a (*general*) *reduction*. An element on which all reductions act trivially (i.e. as the identity) is said to be *irreducible*. Let now $A=T(V)/J(R)$ be a PBW-algebra, and $\mathcal{B}$ a PBW-basis for A. The subspace R of quadratic relations determines a reduction system S_R and a set $\mathcal{R}$ of related reductions in the following way. Surely there exists in R a subset of independent generators whose elements have the form $x_{i_1}x_{i_2}-F_{i_1i_2}$, where

$$F_{i_1i_2}=\sum_{(j_1,j_2)\in S_{\mathcal{B}}}c^{j_1j_2}_{i_1i_2}x_{(j_1,j_2)}.$$

The elements in S_R are precisely the pairs $(x_{(i_1,i_2)}, F_{i_1 i_2})$. By definition $p(x_I) = p(r(x_I))$ for any reduction $r \in \mathcal{R}$ and for any $x_I \in \mathcal{M}$. Since $\mathcal{B}$ is a PBW-basis the reductions satisfy a *confluence condition*: for any $y \in T(V)$ there exists $r \in \mathcal{R}$ such that $r(y)$ is irreducible; furthermore if $r(y)$ and $r'(y)$ are both irreducible, then $r(y) = r'(y)$.

2. A class of PBW-algebras. Let A be a PBW-algebra as above. Fixing a label I, we consider the subset S_I of S_R corresponding to all those reductions $r = r_k \circ \cdots \circ r_1$ such that r_{s+1} does not act trivially on $r_s \circ \cdots \circ r_1(x_I)$. Note that S_I is empty if and only if $p(x_I) \in \mathcal{B}$.

Definition 2.1. We say that an algebra A with a PBW-basis $\mathcal{B}$ is *good* if the following two conditions hold:

(i) the subalgebra A_i is locally finite for all $i \in \mathcal{I}$;
(ii) the map

$$\vartheta_A : I \in \bigcup \mathcal{I}^n \mapsto \max(\{i_1, \ldots, i_n\} \cup \{\text{indices appearing in } S_I\}) \in \mathcal{I} \tag{2.1}$$

is well defined.

Sometimes we shall write just "ϑ", omitting the subscript, when it is clear which algebra we are referring to, and denote by $\mathcal{G}$ the class of good PBW-algebras. To prove the Koszulness of all the algebras in $\mathcal{G}$, we shall give a refinement of Priddy's argument in homology. A variant has been used in [4].

Definition 2.2. Let $B_*(A)$ denote the normalized bar complex of an algebra A with basis $\mathcal{B}$. The *irreducibility index* of a generating chain

$$c = [a_{I_1} \mid \cdots \mid a_{I_s}]$$

in $B_s(A) = A_+^{\otimes s}$ is the integer

$$\mathrm{ai}(c) = \begin{cases} s & \text{if } \mathcal{U} = \emptyset, \\ \min \mathcal{U} & \text{if } \mathcal{U} \neq \emptyset, \end{cases}$$

where $\mathcal{U} = \{h \mid a_{I_h} \cdot a_{I_{h+1}} \in \mathcal{B}\}$.

Lemma 2.3. *Both the chains* $c = [a_{I_1} \mid \cdots \mid a_{I_s}] \in B_*(A)$ *and*

$$\partial(c) = \sum_{j=1}^{s-1} (-1)^{j-1} [a_{I_1} \mid \cdots \mid a_{I_j} a_{I_{j+1}} \mid \cdots \mid a_{I_s}] \in B_{s-1}(A)$$

belong to $(\psi_{\vartheta(I)})_\sharp B_*(A_{\vartheta(I)})$, *where* $I = (I_1, \ldots, I_s)$.

Proof. The lemma immediately follows from the definition (2.1) of ϑ, once we note that all the simple reductions needed to express $x_{I_j} x_{I_{j+1}}$ as a sum of irreducible elements are in S_I. ∎

Theorem 2.4. *All the good PBW-algebras are Koszul.*

Proof. Let $\mathcal{B}$ be a PBW-basis of an algebra $A \in \mathcal{G}$, and consider a $\mathbb{K}$-linear map $\Phi : B_s(A) \to B_{s+1}(A)$ which acts on the generating chains as follows. When $I_1 = (i_1, \ldots, i_{l_1})$, and $\ell(I_1) > 1$,

$$\Phi([a_{I_1} \,|\, \cdots \,|\, a_{I_s}]) = [a_{i_1} \,|\, a_{i_2} \cdots a_{l_1} \,|\, a_{I_2} \,|\, \cdots \,|\, a_{I_s}].$$

If, on the other hand, $c = [a_{i_1} \,|\, \cdots \,|\, a_{i_k} \,|\, a_{i_{k+1}} \cdots a_{i_{k+r}} \,|\, \cdots \,|\, a_{I_s}]$ with $r > 1$, then

$$\Phi(c) = \begin{cases} (-1)^k [a_{i_1} \,|\, \cdots \,|\, a_{i_k} \,|\, a_{i_{k+1}} \,|\, a_{i_{k+2}} \cdots a_{i_{k+r}} \,|\, \cdots \,|\, a_{I_s}] & \text{if } \mathrm{ai}(c) > k, \\ 0 & \text{if } \mathrm{ai}(c) \leq k. \end{cases}$$

We now filter the reduced bar construction $B_*(A)$ as follows. Let $F_I B_{s,t,p}(A)$ and $F_{I+1} B_{s,t,p}(A)$ be the submodules of $B_s(A)$ generated by

$$\Big\{ [a_{I_1} \,|\, \cdots \,|\, a_{I_s}] \in B_s(A) : \sum_{j=1}^{s} \ell(I_j) = t, \; \sum_{j=1}^{s} g(I_j) = p, \; (I_1, \ldots, I_s) \geq I \Big\}$$

and

$$\Big\{ [a_{I_1} \,|\, \cdots \,|\, a_{I_s}] \in B_s(A) : \sum_{j=1}^{s} \ell(I_j) = t, \; \sum_{j=1}^{s} g(I_j) = p, \; (I_1, \ldots, I_s) > I \Big\}$$

respectively. The restriction of the map ∂ respects this decreasing filtration and induces a map

$$\partial' : \frac{F_I B_s(A)}{F_{I+1} B_s(A)} \;\to\; \frac{F_I B_{s-1}(A)}{F_{I+1} B_{s-1}(A)},$$

which acts on the generators as follows:

$$\partial' : [a_{I_1} \,|\, \cdots \,|\, a_{I_s}] \to \sum_{j=1}^{s-1} (-1)^{j-1} [a_{I_1} \,|\, \cdots \,|\, a_{I_j} a_{I_{j+1}} \,|\, \cdots \,|\, a_{I_s}].$$

The summands on the right side which are zero in $F_I B_{s-1}(A)/F_{I+1} B_{s-1}(A)$ are precisely those such that $a_{I_j} a_{I_{j+1}} \notin \mathcal{B}$. The reader can now verify Φ induces a contracting homotopy

$$\Phi' : \frac{F_I B_{s,s+r}(A)}{F_{I+1} B_{s,s+r}(A)} \to \frac{F_I B_{s+1,s+r}(A)}{F_{I+1} B_{s+1,s+r}(A)} \quad (r > 0)$$

by evaluating Φ on a chain $c = [a_{I_1} \,|\, \cdots \,|\, a_{I_s}]$ such that $(I_1, \ldots, I_s) = I$ to avoid trivial cases. The equality

$$(\partial \Phi + \Phi \partial)(c) \equiv c \bmod F_{I+1} B_{s,s+r}(A)$$

depends in particular on the fact that $\max\{\ell(I_1), \ldots, \ell(I_s)\} > 1$, and that

$$[a_{I_1} \,|\, \cdots \,|\, a_{I_j} a_{I_{j+1}} \,|\, \cdots \,|\, a_{I_s}]$$

belongs to $F_{I+1} B_{s-1,s+r}(A)$ when $j < \mathrm{ai}(c)$.

Suppose now that $c = \sum_{h=1}^{m} [y_{I_1^h} \mid \cdots \mid y_{I_s^h}]$ represents a cycle in $B_{s,t,p}(A)$ with $s \neq t$; the index

$$\theta(c) = \max\{\vartheta(I_1^h, \ldots, I_s^h) \mid h = 1, \ldots, m\}$$

is well defined since $A \in \mathcal{G}$. We also know that $A_{\theta(c)}$ is locally finite.

When $\Phi(c)$ is not zero, it involves the same indices of c "split" in a different way, hence by Lemma 2.3 the element $c_1 = c - \partial\Phi(c)$, and similarly every

$$c_{i+1} = c_i - \partial\Phi(c_i),$$

lives in $(\psi_{\theta(c)})_\sharp B_{s,t,p}(A_{\theta(c)})$, which is finite-dimensional since A is good. In particular, once you fix s, t, p and c there are only a finite number of different submodules $(\psi_{\theta(c)})_\sharp F_I B_{s,t,p}(A_{\theta(c)})$; it follows that there exists a c_t on which $\partial\Phi + \Phi\partial$ acts trivially, showing that c is a boundary, i.e.

$$\mathrm{Tor}_{s,t}^{A}(\mathbb{K}, \mathbb{K}) = 0 \quad \text{for any } s \neq t. \ \blacksquare$$

Obviously all finitely generated or locally finite PBW-algebras are good. In the other cases condition (ii) of Definition 2.1 could be difficult to check. For this reason, at least when the indices are in $\mathbb{Z}$, we give a sufficient condition for an algebra to be in $\mathcal{G}$. According to the notations introduced at the end of Section 1, we denote by $h_{i_1 i_2}$ the maximal index appearing in the polynomial $p_{i_1 i_2} = x_{i_1} x_{i_2} - F_{i_1 i_2}$.

PROPOSITION 2.5. *Let A be a PBW-algebra with $\mathcal{I} \subseteq \mathbb{Z}$. If the set*

$$\mathcal{M}_{\bar{j}} = \{\bar{j}\} \cup \{h_{i\bar{j}} \mid i \in \mathcal{I},\ (i, \bar{j}) \notin S_{\mathcal{B}}\} \tag{2.2}$$

admits a maximum for any $\bar{j}$, and the function

$$\omega : j \in \mathcal{I} \mapsto \max \mathcal{M}_j$$

is non-decreasing, then A is good.

Proof. By Remark 1.4, we have just to see that $\vartheta_A(I)$ is a well defined integer for any label $I = (i_1, \ldots, i_k)$. This is in fact true, and depends on the following inequality:

$$\vartheta_A(I) \leq \Omega(I) := \max\{i_1, \ldots, i_k, \omega(i_2), \ldots, \omega^{k-1}(i_k)\}. \tag{2.3}$$

If x_I is irreducible, i.e. $I \in S_{\mathcal{B}}$, we have $\max\{i_1, \ldots, i_k\}$ on both sides of (2.3). The equality also occurs in (2.3) when $I \notin S_{\mathcal{B}}$ and $F_{i_h i_{h+1}} = 0$ for all $(i_h, i_{h+1}) \notin S_{\mathcal{B}}$.

Suppose now that $J_h := (i_h, i_{h+1}) \notin S_{\mathcal{B}}$, and $F_{i_h i_{h+1}} \neq 0$. We have

$$\begin{aligned} r_{(i_1,\ldots,i_{h-1})\sigma_{J_h}(i_{h+2},\ldots,i_k)}(x_I) &= x_{(i_1,\ldots,i_{h-1})} \cdot F_{i_h i_{h+1}} \cdot x_{(i_{h+2},\ldots,i_k)} \\ &= x_{(i_1,\ldots,i_{h-1})} \cdot \sum_{(j_1,j_2)\in S_{\mathcal{B}}} c_{i_h,i_{h+1}}^{j_h j_{h+1}} x_{(j_h,j_{h+1})} \cdot x_{(i_{h+2},\ldots,i_k)}. \end{aligned} \tag{2.4}$$

First of all we show that if $c_{i_h,i_{h+1}}^{j_h j_{h+1}} \neq 0$, then

$$\Omega(i_1,\dots,i_{h-1},j_h,j_{h+1},i_{h+2},\dots,i_k) \leq \Omega(I). \tag{2.5}$$

The two labels just differ for two integers. Note first that

$$\begin{aligned} j_h, j_{h+1} &\leq \omega(i_{h+1}) && \text{(by definition of the map } \omega), && (2.6)\\ &\leq \omega^h(i_{h+1}) && \text{(since } \omega \text{ is non-decreasing)}, && (2.7)\\ &\leq \Omega(I) && \text{(by definition of the map } \Omega). && (2.8) \end{aligned}$$

Applying ω^{h-1} to both sides of (2.6) we get in particular $\omega^{h-1}(j_h) \leq \omega^h(i_{h+1})$. Finally, since $j_h > i_h$ by condition 1 in Definition 1.2, it follows that $j_{h+1} < i_{h+1}$, and hence $\omega^h(j_{h+1}) \leq \omega^h(i_{h+1})$.

From (2.5) we see in particular that no indices in (2.4) are greater than $\Omega(I)$. The inequality (2.5) also provides the inductive argument to show that for any composition $r = r_s \circ \cdots \circ r_1$ of simple reductions, no indices appearing in the polynomials $r_i \circ \cdots \circ r_1(x_I)$ with $i = 1,\dots,s$ are greater than $\Omega(I)$. ∎

The following example shows that the existence of a map like ω in Proposition 2.5 is not necessary for a $\mathbb{Z}$-indexed PBW-algebra to be good.

EXAMPLE 2.6. Let A be the algebra over a field $\mathbb{K}$ with $\operatorname{char}\mathbb{K} \neq 2$ generated by $\{y_i \mid i \in \mathbb{Z}\}$ subject to the following generating relations:

$$y_i y_j = \begin{cases} 0 & \text{if } i \leq j \text{ and } j \neq 3,\\ i(3-i)y_{3^{|i|}+1}y_{2-i-3^{|i|}} & \text{if } i \leq j = 3. \end{cases} \tag{2.9}$$

The elements

$$y_{i_1}\cdots y_{i_n} \quad \text{with} \quad i_1 > \cdots > i_n$$

are all distinct and form a PBW-basis $\mathcal{B}$. In fact any dependence relation among its elements would depend on non-trivial equalities between monomials, which actually do not occur. In fact, a non-zero monomial not in $\mathcal{B}$ contains $h > 0$ non-consecutive y_3's, and it is equal to exactly one element in $\mathcal{B}$. The algebra A is good since the map required in Definition 2.1 is

$$\vartheta : (i_1,\dots,i_n) \mapsto \max\{i_1,\dots,i_n, 3^{|i_h|}+1 \mid i_{h+1} = 3\},$$

but the set $\mathcal{M}_3$ defined in (2.2) is not upper bounded.

3. Some operations on good PBW-algebras and examples. Let $A' = T(V')/J(R')$ and $A'' = T(V'')/J(R'')$ be two quadratic $\mathbb{K}$-algebras. For the following definition we adopt notations of [14].

DEFINITION 3.1.

(i) The *free product* $A' \sqcup A''$ is the algebra freely generated by A' and A'', i.e. $T(V' \oplus V'')$ quotiented by $J(R' \oplus \{\underline{0}''\} + \{\underline{0}'\} \oplus R'')$. We equip the free product with an internal degree inherited by A' and A''.

(ii) The *direct sum* $A' \sqcap A''$ is the quotient of $A' \sqcup A''$ obtained by setting

$$A'_+ A''_+ = A''_+ A'_+ = 0.$$

(iii) The *q-tensor product* $A' \otimes^q A''$ with $q \in \mathbb{P}^1_{\mathbb{K}}$ is the quotient of $A' \sqcup A''$ by the ideal

$$J(\mathrm{Span}\{a''a' - q^{\deg a'' \deg a'} a'a''\}). \tag{3.1}$$

For $q = \infty$, equation (3.1) has to be read $a'a'' = 0$.

Proposition 3.2. *Let $\{\star_i\}_{i\in\mathbb{N}}$ and $\{A(i) \mid i \in \mathbb{N}\}$ be any sequence of operators in $\{\sqcup, \sqcap, \otimes^q\}$ and of good PBW-algebras respectively. The algebras*

$$\mathop{\star}_{i=1}^{n} A(i) \tag{3.2}$$

are all Koszul.

Proof. Since good algebras are Koszul by Theorem 2.4, the statement essentially follows from [1] where it is proved that the operators $\sqcup, \sqcap, \otimes^q$ preserve Koszulness. The assumption on A to be finitely generated is not really relevant there. A proof of this result is also sketched in [14, p. 58]. A third proof, when A' and A'' are good, could use the arguments given along the proof of Theorem 2.4, noticing that $A' \star A''$ has a PBW-basis, and can be filtered by the locally finite algebras $A'_i \star A''_j$. For instance, a PBW-basis $\mathcal{B}_{A' \sqcup A''}$ for $A' \sqcup A''$ is given by

$$\{a'_I a''_{J_1} a'_{I_1} \cdots a'_{I_n} a''_J \mid I \in S_{\mathcal{B}_{A'}} \cup \{\emptyset\},\, I_i \in S_{\mathcal{B}_{A'}},\, J_i \in S_{\mathcal{B}_{A''}},\, J \in S_{\mathcal{B}_{A''}} \cup \{\emptyset\}\}$$

where by convention all the indices in A' are greater than those in A''. ∎

We now list several types of interesting good PBW-algebras.

Definition 3.3.

(i) An algebra A is said to be *monomial* if the subspace R of relations is generated by monomials.

(ii) A *skew-polynomial algebra* is a quotient $T(V)/J(R)$ where R is generated by

$$x_i x_j - q_{ij} x_j x_i \quad (i < j)$$

with $q_{ij} \in \mathbb{K}^*$.

When $\mathcal{I} \subseteq \mathbb{Z}$ such algebras—and quotients of skew-polynomials algebras by monomial relations—are all good; in fact the map required in Definition 2.1 is

$$\vartheta : I \in \bigcup \mathcal{I}^n \mapsto \max\{i_1, \ldots, i_n\} \in \mathcal{I} \tag{3.3}$$

for all of them. The algebra of Example 3.4, whose generators are not indexed by $\mathbb{Z}$, is related to the coordinate ring of quantum $n \times n$-matrices presented for instance in [10].

EXAMPLE 3.4. Let N be a fixed integer. After choosing the lexicographical order on $\mathcal{I} = \mathbb{N} \times \{j \in \mathbb{Z} \mid j \leq N\}$ and considering the internal degree induced by the map $g : (i,j) \in \mathcal{I} \mapsto i + j \in \mathbb{Z}$, we consider the graded algebra $A = T(V)/J(R)$ where R is generated by

$$\begin{aligned} x_{ij}x_{il} &= q x_{il}x_{ij}, \\ x_{ij}x_{kj} &= q x_{kj}x_{ij}, \\ x_{ij}x_{kl} &= x_{kl}x_{ij} + (q - q^{-1})x_{il}x_{kj} \end{aligned}$$

for $j < l$, $i < k$ and $q \in \mathbb{K}^*$. A PBW-basis is given by the monomials $x_{i_1 j_1} \cdots x_{i_n j_n}$ satisfying the following two properties:

- if $i_h < i_{h+1}$ then $j_h > j_{h+1}$;
- if $i_h = i_{h+1}$ then $j_h \geq j_{h+1}$.

The algebra A is not locally finite since

$$x_{10}, x_{2,-1}, x_{3,-2}, \dots$$

are infinite independent monomials of internal degree 1. In any case the subalgebras A_{ij} are locally finite. In fact, for every

$$x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_h j_h} \in T(V_{ij})^{h,t}$$

we have

$$t - hi \leq j_1 + \cdots + j_h \leq hN,$$

hence

$$t - hi - (h-1)N \leq j_1, \dots, j_h \leq N.$$

It follows that

$$\dim_K T(V_{ij})^{h,t} \leq \binom{h(i+1)}{h} (hN + hi - t)^h.$$

Again the map ϑ of (3.3) is suitable to say that A is good.

The Koszulness of the algebras above can also be proved without Theorem 2.5, since the A_i's are locally finite Koszul algebras and we could just use Proposition 1.3. This is not the case for the examples below.

EXAMPLE 3.5. Let A be an algebra generated by $\{y_i \mid i \in \mathbb{Z}\}$ subject to the following generating relations:

$$y_i y_{i+1} = 0 \quad \forall i \neq 0, 1, \quad \text{and} \quad y_1 y_i = y_{i+1} y_0 \quad \forall i > 1. \tag{3.4}$$

A PBW-basis is given by the monomials $y_{i_1} \cdots y_{i_n}$ with $i_j \neq i_{j+1} - 1$ for all $i \neq 0, 1$, and $i_{j+1} \leq 1$ if $i_j = 1$. The independence of such monomials is due to the following fact: for each $I = (i_1, \dots, i_n)$ only one of

$$r_{(i_1,\dots,i_{j-2})\sigma_{(i_{j-1},i_j)}(i_{j+1},\dots,i_n)} \quad \text{and} \quad r_{(i_1,\dots,i_{j-1})\sigma_{(i_j,i_{j+1})}(i_{j+2},\dots,i_n)}$$

can possibly stay in $\mathcal{R}$, hence no monomials are *ambiguous* in the sense of [2] and [3].

The algebra A is good since the map

$$\omega : i \in \mathbb{Z} \mapsto \begin{cases} 1, & i \leq 1, \\ i+1, & i > 1, \end{cases}$$

satisfies the conditions of Proposition 2.5. Note also that none of the algebras A_i with $i > 1$ is Koszul, in fact $y_i y_1 y_i$ gives a non-quadratic relation in A_i and $[y_i \,|\, y_1 y_i]$ represents a non-trivial cycle in $\mathrm{Tor}^{A_i}_{2,3}(\mathbb{K},\mathbb{K})$.

The algebra in Example 3.5 also shows that the inequality (2.3) may be strict. In fact the monomial $x_{-1}x_{-1}x_2$ belongs to the PBW-basis, hence $\vartheta(-1,-1,2) = 2$ while $\Omega(-1,-1,2) = 4$.

EXAMPLE 3.6. The (mod 2) *universal Steenrod algebra* $Q(2)$ is an $\mathbb{F}_2$-algebra with a countable set of generators $\{y_i \mid i \in \mathbb{Z}\}$ subject to the so-called generalized Adem relations:

$$y_{2k-1-n}\, y_k = \sum_j \binom{n-1-j}{j} y_{2k-1-j}\, y_{k+j-n} \qquad (k \in \mathbb{Z},\, n \in \mathbb{N}_0).$$

This algebra is also known as the *algebra of all generalized Steenrod operations* (see [13]) or the *extended Steenrod algebra* (see [7]).

The subset

$$\{y_{i_1} \cdots y_{i_h} \mid i_j \geq 2i_{j+1} \text{ for each } j = 1, \dots, h-1\}$$

forms a PBW-basis for $Q(2)$ (see [12]). By Proposition 2.5, $Q(2)$ is good. The required map is

$$\omega : j \in \mathbb{Z} \mapsto \begin{cases} j & \text{for } j \leq 0, \\ 2j-1 & \text{for } j \geq 1. \end{cases}$$

Our last example is the $\pmod p$ *universal Steenrod algebra* $Q(p)$ at odd primes. It is generated as an $\mathbb{F}_p$-algebra by

$$\{z_{\varepsilon,i} \mid \varepsilon \in \{0,1\},\, i \in \mathbb{Z}\} \quad \text{with} \quad \deg z_{\varepsilon,i} = 2i(p-1) + \varepsilon,$$

subject to the following generalized Adem relations:

$$z_{\varepsilon,pk-1-n} z_{0,k} = \sum_j A_{(n,j)} z_{\varepsilon,pk-1-j} z_{0,k-n+j},$$

$$z_{1-\varepsilon,pk-n} z_{1,k} = \sum_j A_{(n,j)} z_{1-\varepsilon,pk-j} z_{1,k-n+j} + \varepsilon \sum_j B_{(n,j)} z_{1,pk-j} z_{0,k-n+j},$$

for each $(k,n) \in \mathbb{Z} \times \mathbb{N}_0$, where $A_{(n,j)}$ and $B_{(n,j)}$ are respectively equal to

$$(-1)^{j+1}\binom{(p-1)(n-j)-1}{j} \quad \text{and} \quad (-1)^j \binom{(p-1)(n-j)}{j}.$$

This presentation already appeared in [8]. In Proposition 11.2 of [13], the author claims that

$$\mathcal{B} = \{z_{\varepsilon_1,i_1} \cdots z_{\varepsilon_h,i_h} \mid i_j \geq p i_{j+1} + \varepsilon_{j+1} \text{ for each } j = 1, \dots, h-1\}$$

is a basis, but the argument to show that $\mathcal{B}$ is a generating set does not seem to be complete. Nguyen Sum gives a more detailed proof in [17].

Proposition 3.7. *The universal Steenrod algebra $Q(p)$ is good at any prime.*

Proof. The argument for $p = 2$ has already been given in Example 3.6. When p is an odd prime, we set $\mathcal{I} = \{2pk+\varepsilon \mid k \in \mathbb{Z},\ \varepsilon \in \{0,1\}\}$. The algebra $Q(p)$ is then a quotient of $T(V)$ where V is generated by $\{x_i \mid i \in \mathcal{I}\}$, the quotient map being $p : x_{2pk+\varepsilon} \mapsto z_{\varepsilon,k}$. Proposition 3.7 now follows from Remark 1.4 and Proposition 2.5. The required map is

$$\omega : i \in \mathcal{I} \mapsto \begin{cases} i & \text{for } i \leq 0, \\ 2p(pk-1+\varepsilon)+1 & \text{for } i = 2pk+\varepsilon > 0. \end{cases} \blacksquare$$

4. Final remarks. In [6] for $p = 2$, and in [5] when p is odd, an algebra $Q(p)'$ has been introduced by changing the augmentation in $Q(p)$ as follows: the elements

$$\overline{y}_i = \begin{cases} y_i & \text{if } i \neq 0, \\ 1 + y_0 & \text{if } i = 0, \end{cases}$$

for $p = 2$, and

$$x_{\varepsilon,i} = \begin{cases} z_{\varepsilon,i} & \text{if } (\varepsilon,i) \neq (0,0), \\ z_{0,0} - 1 & \text{if } (\varepsilon,i) = (0,0), \end{cases}$$

for odd primes became the generators of the *new* augmentation ideal.

The purpose was to make the epimorphisms

$$\overline{\pi} : \overline{y}_i \in Q(2)' \mapsto \begin{cases} \mathrm{Sq}^i \in \mathcal{A}_2 & \text{if } i > 0, \\ 0 & \text{if } i \leq 0, \end{cases}$$

and

$$\overline{\pi} : x_{\varepsilon,i} \in Q(p)' \mapsto \begin{cases} \beta^{\varepsilon} P^i \in \mathcal{A}_p & \text{if } \varepsilon + i > 0, \\ 0 & \text{if } \varepsilon + i \leq 0, \end{cases} \qquad (p \text{ odd})$$

maps of augmented algebras, where $\mathcal{A}_p$ denotes the ordinary Steenrod algebra at the prime p. The maps $\overline{\pi}^*$ defined on $\mathrm{Ext}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ turned out to be injective in both cases. In any case an explicit description of the target was still to come. Now we are able to state the following result.

Proposition 4.1. *The cohomology of $Q(p)'$ is isomorphic to*

$$\mathrm{Ext}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \otimes \Lambda[\alpha_0]$$

for $p = 2$, and

$$\mathrm{Ext}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p) \otimes \Lambda[\xi_{0,0}]$$

for p odd, where α_0 and $\xi_{0,0}$ are the duals of $\overline{y}_0$ and $x_{0,0}$ respectively.

Proof. In Section 3 we have seen that the algebra $Q(p)$ is good, hence it is Koszul by Theorem 2.4. Use now Proposition 0.2 in [6] when $p = 2$ and Proposition 0.2 in [5] when p is odd. ∎

Definition 2.1 suggests that a non-Koszul PBW-algebra should be really bad! In fact the map ϑ_A becomes meaningless only if there exist reductions

$$r = r_{I_1'\sigma_{J_1}I_1''} \circ \cdots \circ r_{I_s'\sigma_{J_s}I_s''}$$

acting non-trivially on a fixed x_I with arbitrarily high J_i's. When A is not locally finite, the use of *moments* in the sense of [16] is not necessarily profitable. In principle, it could happen that a certain r acting non-trivially on x_I with

$$a_I = \sum_{J \in S_{\mathcal{B}}} c_J a_J,$$

involves indices which are greater than those appearing in the J's such that $c_J \neq 0$. In this case all monomials containing such indices would cancel out in $r(x_I)$. In any case the authors have not succeeded in finding any concrete example of a PBW-algebra not of the form (3.2).

Acknowledgements. The authors are especially indebted to Luciano A. Lomonaco for his useful remarks and for pointing out the importance of the universal Steenrod algebra and its cohomology in algebraic topology.

REFERENCES

[1] J. Backelin and R. Fröberg, *Koszul algebras, Veronese subrings and rings with linear resolutions*, Rev. Roumaine Math. Pures Appl. 30 (1985), 85–97.
[2] R. Berger, *Confluence and Koszulity*, J. Algebra 201 (1998), 243–283.
[3] G. M. Bergman, *The diamond lemma for ring theory*, Adv. Math. 29 (1978), 178–218.
[4] M. Brunetti, A. Ciampella and L. A. Lomonaco, *The cohomology of the universal Steenrod algebra*, Manuscripta Math. 118 (2005), 271–282.
[5] —, —, —, *An embedding for the E_2-term of the Adams spectral sequence at odd primes*, Acta Math. Sinica 22 (2006), 1657–1666.
[6] M. Brunetti M. and L. A. Lomonaco, *An embedding for the E_2-term of the Adams spectral sequence*, Ricerche Mat. 54 (2005), 185–200.
[7] D. Chataur and M. Livernet, *Adem–Cartan operads*, Comm. Algebra 33 (2005), 4337–4360.
[8] A. Ciampella and L. A. Lomonaco, *The universal Steenrod algebra at odd primes*, ibid. 32 (2004), 2589–2607.
[9] R. Fröberg, *Koszul algebras*, in: Advances in Commutative Ring Theory (Fez, 1997), Lecture Notes in Pure Appl. Math. 205, Dekker, New York, 1999, 337–350.

[10] A. C. Kelly, T. H. Lenagan and L. Rigal, *Ring theoretic properties of quantum Grassmannians*, J. Algebra Appl. 3 (2004), 9–30.

[11] S.-Q. Oh, C.-G. Park and Y.-Y. Shin., *A Poincaré–Birkhoff–Witt theorem for Poisson enveloping algebras*, Comm. Algebra 30 (2002), 4867–4887.

[12] L. A. Lomonaco, *A basis of admissible monomials for the universal Steenrod algebra*, Ricerche Mat. 40 (1991), 137–147.

[13] M. A. Mandell, *E_∞ algebras and p-adic homotopy theory*, Topology 40 (2001), 43–94.

[14] A. Polishchuk and L. Positselski, *Quadratic Algebras*, Univ. Lecture Ser. 37, Amer. Math. Soc., Providence, RI, 2005.

[15] S. B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. 152 (1970), 39–60.

[16] N. E. Steenrod, *Cohomology Operations*, lectures written and revised by D. B. A. Epstein, Ann. of Math. Stud. 50, Princeton Univ. Press, Princeton, NJ, 1962.

[17] N. Sum, *On an invariant-theoretic description of the lambda algebra*, Vietnam J. Math. 33 (2005), 19–32.

Dipartimento di Matematica e Applicazioni
Università di Napoli "Federico II"
Via Claudio 21
I-80125 Napoli, Italy
E-mail: mbrunett@unina.it
ciampell@unina.it

Received 24 October 2006 (4808)