Abstract. We characterise the set on which an infinitely differentiable function can be locally polynomial.

1. Introduction. Donoghue [1] has shown that there exists a smooth non-polynomial function \( f: \mathbb{R} \to \mathbb{R} \) having the property that every interval contains a subinterval upon which \( f \) coincides with a polynomial. In this paper we characterise the sets where a smooth function can be locally polynomial in this manner. I have written this note so that it may be read independently of [1] but, as might be expected, the reader who consults that paper will find substantial overlaps. The reader must decide if the title of this paper is appropriate.

We make the following definitions.

**Definition 1.** A function \( f: \mathbb{R} \to \mathbb{R} \) is said to be **real-analytic** at a point \( x \) if we can find a \( \delta > 0 \) such that \( f \) has a power series expansion

\[
f(x + h) = \sum_{r=0}^{\infty} a_r h^r
\]

valid for \( |h| < \delta \). We say that \( f \) is **locally polynomial** at \( x \) if, in addition, we can find an \( N \) such that

\[
f(x + h) = \sum_{r=0}^{N} a_r h^r
\]

for all \( |h| < \delta \).

The following result goes back, effectively, to Du Bois-Reymond.

**Theorem 2.** If \( E \) is closed, we can find an infinitely differentiable function \( f: \mathbb{R} \to \mathbb{R} \) such that \( f \) is not real-analytic at each point \( E \) but is real-analytic at each point of its complement.

Note that the set of points where a function is real-analytic must be open. There is a substantial literature dealing with this phenomenon. The paper [2] provides a particularly deep account.

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The object of this note is to prove the following result.

**Theorem 3.** Given a closed subset $E$ of $\mathbb{R}$ with no isolated points, we can find an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ which is not real-analytic at each point of $E$ but is locally polynomial at each point of its complement.

The following observations explain why Theorem 3 takes the form it does.

**Lemma 4.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable. Let $E$ be the set where $f$ is not locally polynomial. Then:

(i) $E$ is closed.
(ii) $E$ contains no isolated points.
(iii) If $x$ is a frontier point of $E$ (that is to say, $x \in E \cap \text{Cl}(\mathbb{R} \setminus E)$), then $f$ is not real-analytic at $x$.
(iv) Suppose that $E$ has empty interior. Then, if $x \in E$, we can find $x_j \in E$ and $n_j \to \infty$ such that $f^{(n_j)}(x_j) \neq 0$ and $x_j \to x$ as $j \to \infty$.

**Proof.**

(i) Direct from definition.

(ii) Write $U = \mathbb{R} \setminus E$. Suppose that $f(t) = P(t)$ for some polynomial $P$ on an open interval $I$ and $f(t) = Q(t)$ for some polynomial $Q$ on an open interval $J$. If $I \cap J \neq \emptyset$ then, since $I \cap J$ is an open interval, $P = Q$ and $f(t) = P(t)$ on $I \cup J$. Thus, by standard arguments, if $f(t) = P(t)$ for some polynomial $P$ on an open interval $I$ and $L$ is an open interval with $I \subseteq L \subseteq U$, we have $f(t) = P(t)$ on $L$.

Suppose that $x$ does not lie in the closure of $E \setminus \{x\}$. Then we can find a $\delta > 0$ such that

$$(x - \delta, x), (x, x + \delta) \subseteq U$$

and polynomials $P$ and $Q$ such that $f(t) = P(t)$ for $t \in (x - \delta, x)$ and $f(t) = Q(t)$ for $t \in (x, x + \delta)$. Since $f$ is infinitely differentiable, all its derivatives are continuous and

$$P^{(r)}(x) = f^{(r)}(x) = Q^{(r)}(x)$$

for all $r$. Thus $P = Q$ and $f(t) = P(t)$ for $t \in (x - \delta, x + \delta)$.

(iii) Suppose that $x \in \text{Cl} U$ and there exists a $\delta > 0$ such that the power series

$$\sum_{r=0}^{\infty} a_r (t - x)^r$$

covers to $f(t)$ for all $|t - x| < \delta$. Choose $y \in U$ such that $|y - x| < \delta/2$. We can find an open interval $J$ containing $y$ and a polynomial $P$ such that $f = P$ on $J$. By the uniqueness of power series, $f = P$ on $(x - \delta, x + \delta)$ so $x \in U$. 

(iv) Suppose that $x$ is such that we cannot find $x_j \in E$ and $n_j \to \infty$ with $f^{(n_j)}(x_j) \neq 0$ and $x_j \to x$ as $j \to \infty$. Then we can find a $\delta > 0$ and $N$ such that, if $t \not\in E$ and $|t - x| < \delta$, we have $f^{(n)}(t) = 0$ for all $n \geq N$. Since $E$ has empty interior, it follows, by continuity, that $f^{(n)}(t) = 0$ for all $n \geq N$ and $|t - x| < \delta$. By repeated use of the mean value theorem, there is a polynomial $P$ of degree at most $N - 1$ such that $f(t) = P(t)$ for $|t - x| < \delta$ and so $x \not\in E$. \hfill \blacksquare

We shall also prove the following result.

**Theorem 5.** If $U$ is a non-empty open subset of $\mathbb{R}$, we can find an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \not\in U$ and a set $H \subset U$ with the following properties:

(i) $U \setminus H$ has Lebesgue measure zero.

(ii) If $x \in H$, then we can find an integer $N(x)$ with $f^{(n)}(x) = 0$ for all $n \geq N(x)$.

(iii) $f$ is not locally polynomial at any point of $U$.

This gives another proof of Theorem 2.

In order to make the proof of Theorem 5 as different as possible from the usual proof, we avoid the use of functions like $\exp(-1/x^2)$ and use instead a “stitching method” based on Lemma 7.

**2. Main proof.** In this section we prove the following version of Theorem 3. We use the notation $g|A$ to mean the restriction of the function $g$ to a set $A$.

**Theorem 6.** Given a non-trivial closed subset $E$ of $[0, 1]$ with no isolated points and empty interior, we can find an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = 0$ for $x \not\in [0, 1]$, which is not real-analytic at each point of $E$ but is locally polynomial at each point of its complement.

The main point of difference between Theorem 3 and Theorem 6 is that, in Theorem 6, we suppose that $E$ has empty interior. However, this is the interesting case and it should be fairly clear that there must be a number of ad hoc ways of getting from Theorem 6 to Theorem 3. We shall sketch one of them in the final section.

We need the following lemma which the reader may quite properly dismiss as trivial.

**Lemma 7.**

(i) Given an integer $n \geq 0$ and an interval $[a, b]$, we can find a constant $K$ with the following property: Given $\alpha_j, \beta_j \in \mathbb{R}$ with $|\alpha_j|, |\beta_j| \leq 1$, we can find a real-polynomial $P$ of degree at most $2n + 1$ such that

$$P^{(j)}(a) = \alpha_j, \quad P^{(j)}(b) = \beta_j \quad \text{and} \quad |P^{(j)}(t)| \leq K$$

for all $t \in [a, b]$ and $0 \leq j \leq n$.  


(ii) Given an integer \( n \geq 0 \), \( \eta > 0 \), an interval \([a, b]\) and \( \alpha, \beta \in \mathbb{R} \), we can find a real polynomial \( Q \) such that
\[
Q^{(n)}(a) = \alpha, \quad Q^{(n)}(b) = \beta
\]
but
\[
Q^{(j)}(a) = Q^{(j)}(b) = 0 \quad \text{and} \quad |Q^{(j)}(t)| \leq \eta
\]
for all \( t \in [a, b] \) and \( 0 \leq j \leq n - 1 \).

(iii) Given an integer \( n \geq 0 \), \( \eta > 0 \), and an interval \([a, b]\) we can find a real polynomial \( R \) of degree exactly \( 2n + 2 \) such that
\[
R^{(j)}(a) = R^{(j)}(b) = 0 \quad \text{and} \quad |R^{(j)}(t)| \leq \eta
\]
for all \( t \in [a, b] \) and \( 0 \leq j \leq n \).

Proof. By translation and rescaling we may take \( a = 0 \) and \( b = 1 \).

(i) It is sufficient to prove the result (with a different value of \( K \)) when \( \beta_j = 0 \) for all \( 0 \leq j \leq n \). Set \( P_r(x) = x^r(1 - x)^{n+1} \) for \( 0 \leq r \leq n \) and observe that the matrix \((P_r^{(s)}(0))_{0 \leq r \leq n, 0 \leq s \leq n}\) is triangular with non-zero diagonal elements. It follows that there exists a \( \widetilde{K} \) such that, if \( |\alpha_j| \leq 1 \) for \( 0 \leq j \leq n \), we can find \( A_j \) with \( |A_j| \leq \widetilde{K} \) and
\[
\sum_{r=0}^{n} A_r P_r^{(s)}(0) = \alpha_s
\]
for \( 0 \leq s \leq n \). Setting \( P = \sum_{r=0}^{n} A_r P_r \) we see that
\[
P^{(j)}(0) = \alpha_j, \quad P^{(j)}(1) = 0
\]
and
\[
|P^{(j)}(t)| \leq (n + 1)\widetilde{K} \sup_{0 \leq r \leq n} \sup_{x \in [0, 1]} |P_r^{(j)}(x)|
\]
for all \( t \in [0, 1] \) and all \( 0 \leq j \leq n \).

(ii) Let \( N \) be a large integer to be chosen later. Let \( h(x) = \sin(Nx - n\pi/2) \) and set \( g(x) = N^{-n} \alpha(1 - x)^{n+1} h(x) \). Then
\[
g^{(n)}(0) = \alpha, \quad g^{(j)}(1) = 0 \quad \text{for} \quad 0 \leq j \leq n
\]
and there is a constant \( A \) independent of \( N \) such that
\[
|g^{(j)}(0)| \leq AN^{-1} \quad \text{for} \quad 0 \leq j \leq n - 1.
\]
By considering the Taylor expansion of \( g \) we know that there is a polynomial \( G \) such that
\[
|g^{(j)}(t) - G^{(j)}(t)| \leq AN^{-1}
\]
for all \( t \in [0, 1] \) and all \( 0 \leq j \leq n \).
Thus

\[ |G^{(n)}(0) - \alpha| \leq AN^{-1}, \]
\[ |G^{(j)}(0)| \leq 2AN^{-1} \quad \text{for } 0 \leq j \leq n - 1, \]
\[ |G^{(j)}(1)| \leq AN^{-1} \quad \text{for } 0 \leq j \leq n. \]

By part (i) we can find a polynomial \( P \) with

\[ Q^{(n)}(0) = G^{(n)}(0) - \alpha, \]
\[ P^{(j)}(0) = G^{(j)}(0) \quad \text{for } 0 \leq j \leq n - 1, \]
\[ Q^{(j)}(1) = G^{(j)}(1) \quad \text{for } 0 \leq j \leq n \]

and

\[ |P^{(j)}(t)| \leq 2KAN^{-1} \]

for all \( t \in [0, 1] \) and all \( 0 \leq j \leq n \). If we set \( Q = G - P \) and take \( N \) large enough, the required result follows.

(iii) Just set \( R(t) = \varepsilon t^{n+1} (1 - t)^{n+1} \) with \( \varepsilon \) sufficiently small but non-zero.

Proof of Theorem 6. By rescaling, we may suppose \( 0, 1 \in E \). Standard results on topology show that \([0, 1] \setminus E\) is the countable union \( U \) of disjoint open intervals \( U_1, U_2, \ldots \). Since \( E \) has no isolated points and empty interior, the \( U_r \) cannot share endpoints and cannot have 0 or 1 as endpoints. Thus

\[ [0, 1] \setminus \bigcup_{r=1}^{n} U_r = \bigcup_{r=0}^{n} J_{n,r} \]

where \( J_{n,r} = [a_{n,r}, b_{n,r}] \) and

\[ 0 = a_{n,0} < b_{n,0} < a_{n,1} < b_{n,1} < a_{n,2} < \cdots < b_{n,n-1} < a_{n,n} < b_{n,n} = 1. \]

We take \( f_0 = 0 \) and \( J_{0,0} = [0, 1] \). We construct inductively functions \( f_n : \mathbb{R} \to \mathbb{R} \) such that

(i) \( f_n|U_r \) is a polynomial for all \( 1 \leq r \leq n \), \( f_n|J_{n,r} \) is a polynomial for all \( 0 \leq r \leq n \) and \( f(x) = 0 \) for all \( x \notin [0, 1] \),

(ii) \( f_n \) has a continuous \( n \)th derivative.

Suppose \( f_n \) has been constructed. Using Lemma 7 applied to the various intervals \( U_r \) and \( J_{n+1,s} \) we can find a function \( f_{n+1} : \mathbb{R} \to \mathbb{R} \) such that

(i) \( f_{n+1} \) \( f_{n+1}|U_r \) is a polynomial for all \( 1 \leq r \leq n+1 \), \( f_{n+1}|J_{n,r} \) is a polynomial for all \( 0 \leq r \leq n + 1 \) and \( f_{n+1}(x) = 0 \) for all \( x \notin [0, 1] \),

(ii) \( f_{n+1} \) has a continuous \( (n + 1) \)st derivative,

and, in addition,

(iii) \( f_{n+1} \) \( f_{n+1}|U_r = f_n|U_r \) for \( 1 \leq r \leq n \),

(iv) \( f_{n+1} \) \( f_{n+1}|U_{n+1} \) is a polynomial of degree at least \( n + 1 \),
whilst
\[(v)_{n+1} |f^{(r)}_{n+1}(x) - f^{(r)}_n(x)| \leq 2^{-n} \text{ for all } x \in \mathbb{R}, \ 0 \leq r \leq n.\]

Now condition \((v)\) tells us that \(f^{(r)}_n\) converges uniformly for each \(r\) and so \(f_n\) converges to an infinitely differentiable function \(f\). Condition \((iii)_r\) combined with condition \((iii)_n\) tells us that \(f|U_r\) is a polynomial of degree at least \(r\).

If \(x \in E\), then, since \(E\) has no interior and no isolated points, it follows that given any \(\delta > 0\) and \(N\) we can find an \(n \geq N\) and a \(U_n \subseteq (x - \delta, x + \delta)\). Since \(F|U_n\) is a polynomial of degree at least \(n\), our standard arguments show that \(f\) cannot be real-analytic at \(x\).

\[\text{(v)}_{n+1} |f^{(r)}_{n+1}(x) - f^{(r)}_n(x)| \leq 2^{-n} \text{ for all } x \in \mathbb{R}, \ 0 \leq r \leq n.\]

3. Final remarks. We note an immediate consequence of Theorem 3.

**Lemma 8.** Given \(a \in \mathbb{R}, \ N \geq 0\) and \(\delta > 0\) we can find an infinitely differentiable function \(g : \mathbb{R} \to \mathbb{R}\) and a set \(E\) of Lebesgue measure 0 with the following properties:

\(\begin{enumerate}
  \item \(g(x) = 0\) whenever \(|x - a| \geq \delta\).
  \item If \(x \notin E\), then there exists an \(M(x)\) such that \(g^{(m)}(x) = 0\) for all \(m \geq M(x)\).
  \item There exists an \(m \geq N\) such that \(g^{(m)}(a) \neq 0\).
\end{enumerate}\)

**Proof.** Choose a non-empty closed set \(\tilde{E}\) of Lebesgue measure zero (so, automatically, with empty interior) with no isolated points lying in \([0, 1]\). By Theorem 6, we can find an infinitely differentiable function \(\tilde{g}\) with the following properties:

\(\begin{enumerate}
  \item \(\tilde{g}(x) = 0\) for \(x \notin [0, 1]\).
  \item If \(x \notin \tilde{E}\) then \(\tilde{g}\) is locally polynomial at \(x\) and so in particular there exists an \(M(x)\) such that \(g^{(m)}(x) = 0\) for all \(m \geq M(x)\).
\end{enumerate}\)

Since \(\tilde{E}\) is non-empty, Lemma 4(iv) tells us that there exists a \(b \in [0, 1]\) and an \(m \geq N\) such that \(g^{(m)}(b) \neq 0\). The required result follows by translation and dilation.

We can now prove Theorem 5.

**Proof of Theorem 5.** Choose a countable dense subset \(q_1, q_2, \ldots\) of \(U\) (without repeating points). Choose \(\delta_j > 0\) so that \(q_k \notin (q_j - 2\delta_j, q_j - 2\delta_j)\) for \(1 \leq k \leq j - 1\), \((q_j - 2\delta_j, q_j - 2\delta_j) \subseteq U\) and \(\delta_j < 2^{-j}\). We now take \(f_0 = 0\) and define \(f_j\) inductively as follows: If \(f^{(m)}_{j-1}(q_j) \neq 0\) for some \(m_j \geq j\) set \(f_j = f_{j-1}\). If \(f^{(m)}_{j-1}(q_j) = 0\) for all \(m \geq j\) then, by Lemma 8, we can find a smooth function \(g_j : \mathbb{R} \to \mathbb{R}\) and a set \(E_j\) of Lebesgue measure 0 such that:
(i) $g(x) = 0$ whenever $|x - q_j| \geq \delta_j$.
(ii) If $x \notin E$ then there exists an $M(x)$ such that $g^{(m)}(x) = 0$ for all $m \geq M(x)$.
(iii) There exists an $m_j \geq j$ such that $g^{(m_j)}(q_j) \neq 0$.

Now choose an $\varepsilon_j > 0$ with

$$\varepsilon_j |g_j^{(k)}(t)| \leq 2^{-j}$$

for all $t \in \mathbb{R}$ and all $0 \leq k \leq j$ and set $f_j = f_{j-1} + \varepsilon_j g_j$.

By the general principle of uniform convergence, all the derivatives of $f_j$ converge uniformly and $f_j$ converges uniformly to an infinitely differentiable function $f$. We note that $f_j(x) = 0$ and so $f(x) = 0$ for all $x \notin U$. Since $f^{(m_j)}_k(q_j) = f^{(m_j)}_j(q_j) \neq 0$ for all $k \geq j$, we have $f^{(m_j)}(q_j) \neq 0$. Since the $q_j$ are dense and $m_j \to \infty$, $f$ cannot be locally polynomial at any point of $U$.

Suppose now that $x$ is a point such that there does not exist an $M$ such that $g^{(m)}(x) = 0$ for all $m \geq M$. If $x \notin \bigcup_{j=1}^{\infty} E_j = E$, say, then we know that for each $j$ there exists an $N(j)$ such that $f^{(m)}_j(x) = 0$ for all $m \geq N(j)$. Thus $x \in \text{supp}(f_j - f_{j-1})$ for infinitely many $j$ and so

$$x \in \bigcap_{r=1}^{\infty} \bigcup_{s=r}^{\infty} [q_s - \delta_s, q_s + \delta_s] = F,$$

say. Elementary measure theory tells us that $E$ and $F$ have measure zero, so we are done. ■

Theorem 3 can be proved in a similar manner:

**Sketch proof of Theorem 3.** If $E$ is a closed set without isolated points we can write $E = E_0 \cup U$ where $U$ is open and $E_0$ is a closed set without isolated points and with empty interior. (Note that $E_0$ may not be disjoint from $U$.) By Theorem 6 we can find an infinitely differentiable function $f$ which is locally polynomial at each $x \notin E_0$ and is not locally polynomial at each $x \in E_0$. An inductive construction along the lines of the proof of Theorem 5 followed by a limiting argument produces a function $f$ with the required properties. ■

Our results generalise to higher dimensions though the proofs now seem to require the use of smooth partitions of unity.

**Lemma 9.** Suppose that $E$ is a closed subset of $\mathbb{R}^m$ whose complement has connected open components $U_1, U_2, \ldots$ with the property that

$$\text{Cl}(U_j) \cap \text{Cl}(U_k) = \emptyset$$
for $j \neq k$. Then we can find an infinitely differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f|U_j = P_j|U_j$ for some multinomial $P_j [j \geq 1]$ and $P_j \neq P_k$ when $j \neq k$.

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