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ON MINIMAL HOMOTHETICAL HYPERSURFACES

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Abstract. We give a classification of minimal homothetical hypersurfaces in an $(n+1)$-dimensional Euclidean space. In fact, when $n \geq 3$, a minimal homothetical hypersurface is a hyperplane, a quadratic cone, a cylinder on a quadratic cone or a cylinder on a helicoid.

1. Introduction. An n-dimensional hypersurface in Euclidean space E^{n+1} is called a *translation hypersurface* if it is the graph of a function $F(x_1, x_2, \ldots, x_n) = f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)$, where $f_1, f_2, \ldots, f_n$ are smooth functions. A hypersurface is said to be *minimal* if its mean curvature is zero identically. As is well known, a minimal translation surface in a 3-dimensional Euclidean space E^3 must be a plane or a Scherk surface which is the graph of the function

$$F(x_1, x_2) = \frac{1}{a} \ln \frac{\cos(ax_1)}{\cos(ax_2)},$$

where a is a non-zero constant. Some general results on translation hypersurfaces have been given in [1]–[5].

A hypersurface in an $(n+1)$-dimensional Euclidean space is called *homothetical* if it is given by

$$X(x_1, \ldots, x_n) = (x_1, \ldots, x_n, F(x_1, \ldots, x_n)),$$

where $F(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$ with smooth functions $f_1, f_2, \ldots, f_n$.

In [6], I. Van de Woestyne classified 2-dimensional minimal homothetical surfaces, and proved:

THEOREM 1. *A 2-dimensional minimal homothetical surface in a 3-dimensional Euclidean space E^3 must be a plane or a helicoid.*

In the present paper, we study n-dimensional minimal homothetical hypersurfaces in an $(n+1)$-dimensional Euclidean space, and give their classification. On the way, when $n = 2$, we give a new method to prove Theorem 1.

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We obtain the following result:

THEOREM 2. *Let M^n $(n \geq 3)$ be an n-dimensional minimal homothetical hypersurface in an $(n+1)$-dimensional Euclidean space E^{n+1}. Then the hypersurface is a hyperplane, a quadratic cone, a cylinder on a quadratic cone or a cylinder on a helicoid.*

2. Preliminaries. Let M^n be a hypersurface in Euclidean space E^{n+1}, given by

$$X = (x_1, \dots, x_n, F(x_1, \dots, x_n)).$$

So

$$\frac{\partial X}{\partial x_i} = \left(0, \dots, 1, \dots, 0, \frac{\partial F}{\partial x_i}\right), \qquad \frac{\partial^2 X}{\partial x_i \partial x_i} = \left(0, \dots, 0, \frac{\partial^2 F}{\partial x_j \partial x_i}\right).$$

Let $P_i = \partial F/\partial x_i$. We have

$$g_{ij} = \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \right\rangle = \delta_{ij} + \frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_j},$$

so $(g_{ij}) = I + P^T P$, where $P = (P_1, \dots, P_n)$. The inverse of the matrix (g_{ij}) is given by

$$(g^{ij}) = I - \frac{1}{1+|P|^2} P_i P_j,$$

where $|P|^2 = P^T P$. Let $g = \det(g_{ij}) = 1 + |P|^2$, and

$$W := \sqrt{g} = \sqrt{1+|P|^2}.$$

We have

$$g^{ij} = \delta_{ij} - \frac{1}{W^2} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j}.$$

The unit normal vector is given by

$$\nu = \frac{1}{W}\left(-\frac{\partial F}{\partial x_1}, \dots, -\frac{\partial F}{\partial x_n}, 1\right).$$

So the second fundamental form is given by

$$\begin{aligned} b_{ij} &= \left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \nu \right\rangle = \left\langle \left(0, \dots, 0, \frac{\partial^2 F}{\partial x_i \partial x_j}\right), \frac{1}{W}\left(-\frac{\partial F}{\partial x_1}, \dots, -\frac{\partial F}{\partial x_n}, 1\right) \right\rangle \\ &= \frac{1}{W} \frac{\partial^2 F}{\partial x_i \partial x_j}. \end{aligned}$$

Then we define the mean curvature by

$$nH = \sum_{i,j} g^{ij} b_{ij} = \sum_{i,j} g^{ij} \frac{1}{W} \frac{\partial^2 F}{\partial x_j \partial x_i} = \frac{1}{W} \sum_{i,j} \left(\delta_{ij} - \frac{1}{W^2}\right) \frac{\partial^2 F}{\partial x_j \partial x_i},$$

i.e.,

$$nH = \frac{1}{W}\left(\sum_i \frac{\partial^2 F}{\partial x_i^2} - \frac{1}{W^2}\sum_{i,j}\frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_j}\frac{\partial^2 F}{\partial x_j \partial x_i}\right). \tag{2.1}$$

We call a hypersurface *minimal* if $H \equiv 0$.

3. Proof of Theorem 2. Before the proof, we introduce some notation to simplify the writing:

$$\begin{aligned}
F &= f_1(x_1)f_2(x_2)\cdots f_n(x_n),\\
F' &= f_1'(x_1)f_2'(x_2)\cdots f_n'(x_n),\\
F_i &= f_1(x_1)f_2(x_2)\cdots f_{i-1}(x_{i-1})f_i'(x_i)f_{i+1}(x_{i+1})\cdots f_n(x_n),\\
F_{i^2} &= f_1(x_1)f_2(x_2)\cdots f_{i-1}(x_{i-1})f_i''(x_i)f_{i+1}(x_{i+1})\cdots f_n(x_n),\\
F(i) &= f_1(x_1)f_2(x_2)\cdots f_{i-1}(x_{i-1})f_{i+1}(x_{i+1})\cdots f_n(x_n),\\
F'(i) &= f_1'(x_1)f_2'(x_2)\cdots f_{i-1}'(x_{i-1})f_{i+1}'(x_{i+1})\cdots f_n'(x_n),\\
F(ij) &= f_1(x_1)f_2(x_2)\cdots f_{i-1}(x_{i-1})f_{i+1}(x_{i+1})\\
&\qquad \cdots f_{j-1}(x_{j-1})f_{j+1}(x_{j+1})\cdots f_n(x_n),\\
F'(ij) &= f_1'(x_1)f_2'(x_2)\cdots f_{i-1}'(x_{i-1})f_{i+1}'(x_{i+1})\\
&\qquad \cdots f_{j-1}'(x_{j-1})f_{j+1}'(x_{j+1})\cdots f_n'(x_n).
\end{aligned}$$

Since $H \equiv 0$, from (2.1) we get

$$W^2\sum_i F_{i^2} - \sum_{i,j} F_iF_jF_{ij} = 0, \tag{3.1}$$

and so

$$\sum_i F_{i^2} + \sum_{i,j}(F_i^2F_{j^2} - F_iF_jF_{ij}) = 0. \tag{3.2}$$

Noting that $\sum_{i,j}(F_i^2F_{j^2} - F_iF_jF_{ij}) = 0$, when $i = j$, from (3.2) we get

$$\sum_i F_{i^2} + \sum_{i\neq j}(F_i^2F_{j^2} - F_iF_jF_{ij}) = 0, \tag{3.3}$$

and so

$$\sum_i F_{i^2} + \sum_{i\neq j} F(F_iF_{ij^2} - F_{ij}^2) = 0, \tag{3.4}$$

where

$$\begin{aligned}
F_{ij^2} = f_1(x_1)f_2(x_2)\cdots f_{i-1}(x_{i-1})f_i'(x_i)f_{i+1}(x_{i+1})&\\
\cdots f_{j-1}(x_{j-1})f_j''(x_j)f_{j+1}(x_{j+1})\cdots f_n(x_n),&
\end{aligned}$$

in accordance with the notation introduced above.

When $f_i(x_i) \neq 0$ and $f_i'(x_i) \neq 0$, from (3.4) we can get

$$\sum_i \frac{f_i''}{f_i} + \sum_{i\neq j} F(ij)^2 f_i'^2 (f_j f_j'' - f_j'^2) = 0. \tag{3.5}$$

Setting $T_{ij} = F^2(ij) f_i'^2 (f_j f_j'' - f_j'^2)$, $i \neq j$, we have

$$\frac{\partial T_{ij}}{\partial x_i} = 2F(ij)^2 f_i' f_i'' (f_j f_j'' - f_j'^2), \tag{3.6}$$

and then

$$\frac{\partial^2 T_{ij}}{\partial x_j \partial x_i} = 2F(ij)^2 f_i' f_i'' (f_j f_j''' - f_j' f_j''). \tag{3.7}$$

From (3.6) and (3.7), we get

$$\frac{\partial^n T_{ij}}{\partial x_1 \ldots \partial x_n} = 2^{n-1} F(ij) F'(ij) f_i' f_i'' (f_j f_j''' - f_j' f_j''). \tag{3.8}$$

Differentiating (3.5) with respect to $x_1, \ldots, x_n$ yields

$$\sum_{i\neq j} \frac{\partial^n T_{ij}}{\partial x_1 \ldots \partial x_n} = 0. \tag{3.9}$$

From (3.8) and (3.9) we get

$$2^{n-1} \sum_{i\neq j} F(ij) F'(ij) f_i' f_i'' (f_j f_j''' - f_j' f_j'') = 0. \tag{3.10}$$

When $f_i' \neq 0$ for $i = 1, \ldots, n$, from (3.10) we get

$$\sum_{i\neq j} \frac{f_i''}{f_i} \left(\frac{f_j'''}{f_j'} - \frac{f_j''}{f_j} \right) = 0. \tag{3.11}$$

When $n = 2$, we assume that $f(x) = f_1(x_1)$ and $g(y) = f_2(x_2)$, so that (3.11) becomes

$$\frac{f''}{f} \left(\frac{g'''}{g'} - \frac{g''}{g} \right) + \frac{g''}{g} \left(\frac{f'''}{f'} - \frac{f''}{f} \right) = 0. \tag{3.12}$$

If $f'' g'' \neq 0$, from (3.12) we get

$$\frac{f f'''}{f' f''} + \frac{g g'''}{g' g''} = 2. \tag{3.13}$$

Set $\frac{f f'''}{f' f''} = a$ and $\frac{g g'''}{g' g''} = b$, where $a + b = 2$.

If $a \neq -1$ and $b \neq -1$, we easily get

$$f'' = a_1 f^a, \tag{3.14}$$

$$g'' = b_1 g^b, \tag{3.15}$$

$$f'^2 = \frac{2a_1}{a+1} f^{a+1} + a_2, \tag{3.16}$$

$$g'^2 = \frac{2b_1}{b+1} g^{b+1} + b_2, \tag{3.17}$$

where a_1, a_2, b_1, b_2 are constants. Substituting the four equations above into (3.1), we have

$$a_1 f^{a-1}\left(b_2 \frac{a-3}{a+1} f^2 + 1\right) + b_1 g^{b-1}\left(a_2 \frac{b-3}{b+1} g^2 + 1\right) - 2a_2 b_2 = 0. \tag{3.18}$$

From (3.18) we get

$$a_1 f^{a-1}\left(b_2 \frac{a-3}{a+1} f^2 + 1\right) = K = \text{const},$$
$$b_1 g^{b-1}\left(a_2 \frac{b-3}{b+1} g^2 + 1\right) = T = \text{const},$$

where $K + T = 2a_2 b_2$. Hence $a_1 = b_1 = a_2 b_2 = 0$, which implies

$$f'' = g'' = 0.$$

When $a = -1$ or $b = -1$, taking $a = -1$ and $b = 3$ for example, from (3.14)–(3.17) we get

$$f'' = \frac{a_1}{f}, \qquad f'^2 = 2a_1 \ln f + 2a_2,$$
$$g'' = b_1 g^3, \qquad g'^2 = \frac{b_1 g^4}{2} + 2b_2,$$

where a_1, a_2, b_1 and b_2 are constants. Substituting the four equations above into (3.1), we have

$$b_1 f g^3 + \frac{a_1 b_1 f g^5}{2} - 8a_1 b_2 f g \ln f + \frac{a_1 g}{f} - 8a_2 b_2 f g + 2a_1 b_2 f g = 0. \tag{3.19}$$

Noting that $f \neq \text{const}$ and $g \neq \text{const}$, from (3.19) we obtain

$$\left(b_1 g^2 + \frac{a_1 b_1}{2} g^4\right) + \left(\frac{a_1}{f} - 8a_1 b_2 \ln f\right) - 8a_2 b_2 + 2a_1 b_2 = 0,$$

which means that $a_1 = b_1 = a_2 b_2 + a_1 b_2 = 0$, i.e. $f'' = g'' = 0$. This contradicts the assumption that $f'' g'' \neq 0$. Therefore $f'' g'' = 0$.

When $f'' = g'' = 0$, we infer from (3.1) that $2 f f'^2 g g'^2 = 0$, which means $f = 0$, $f' = 0$, $g = 0$ or $g' = 0$, and then f and g must be polynomials of degree no more than one.

When only one of f'' and g'' is zero, without loss of generality, we set $f'' = 0$ and $g'' \neq 0$. Then we have $f(x) = ax + b$, where a and b are constant. Thus from (3.1) we get

$$f(g'' - 2a^2 g g'^2 + a^2 g^2 g'') = 0. \tag{3.20}$$

Since $g'' \neq 0$, from (3.20) we deduce

$$\frac{2gg'^2}{g''} - g^2 = \frac{1}{a^2} = \text{const.} \tag{3.21}$$

Hence

$$g = \frac{1}{a}\tan\left(\frac{c}{a}y + \frac{d}{a}\right), \tag{3.22}$$

where c and d are constants. This completes the proof of Theorem 1.

When $n \geq 3$, without loss of generality we assume that $f_i''/f_i \neq$ const, $i = 1, \dots, r$, while $f_i''/f_i = \eta_i =$ const, $i = r+1, \dots, n$, and we set $\eta = \sum_{k=r+1}^n \eta_k$.

Case 1: $r = 0$. In this case, from $f_i''/f_i = \eta_i$, $i = r+1, \dots, n$, we get

$$f_i'^2 = \eta_i f_i^2 + \eta_i', \tag{3.23}$$

where η_i' is a constant. Then from (3.5), (3.23), we have

$$\sum_i \eta_i + \sum_{i\neq j} F^2(ij)(\eta_i f_i^2 + \eta_i')(-\eta_j') = 0,$$

i.e.,

$$F^2 \sum_{i\neq j}\left(\eta_i + \frac{\eta_i'}{f_i^2}\right)\frac{\eta_j'}{f_j^2} = \sum_i \eta_i. \tag{3.24}$$

Since $F \neq 0$, from (3.24) we have

$$\sum_{i\neq j}\left(\eta_i + \frac{\eta_i'}{f_i^2}\right)\frac{\eta_j'}{f_j^2} = \frac{\sum_i \eta_i}{F^2}. \tag{3.25}$$

For all $k = 1, \dots, n$, from (3.25) we obtain

$$\frac{\eta_k'}{f_k^2}\sum_{i\neq k}\left(\eta_i + \frac{2\eta_i'}{f_i^2}\right) + \Omega(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = \frac{\sum_i \eta_i}{F^2} \tag{3.26}$$

for some Ω. Differentiating (3.26) with respect to x_k, we get

$$\frac{2\eta_k' f_k'}{f_k^3}\sum_{i\neq k}\left(\eta_i + \frac{2\eta_i'}{f_i^2}\right) = \frac{2f_k'\sum_i \eta_i}{F^2 f_k}. \tag{3.27}$$

Since $f_i' \neq 0$, from (3.27) we find that

$$\eta_k'\sum_{i\neq k}\left(\eta_i + \frac{2\eta_i'}{f_i^2}\right) = \frac{f_k^2}{F^2}\sum_i \eta_i. \tag{3.28}$$

This yields

$$\eta_k'\sum_{i\neq k}[(F(k)_{x_i}')^2 + \eta_i'[F(ik)]^2] = \sum_i \eta_i,$$

i.e.,

$$\eta_k' \sum_{i\neq k} (f_i'^2 + \eta_i') F(ik)^2 = \sum_i \eta_i. \tag{3.29}$$

Since for $n \geq 3$, $F(ik)$ is not a constant, from (3.23) we see that $f_i'^2 + \eta_i' \neq 0$. Noting that $F(ik)$ and $f_i'^2 + \eta_i'$ are functions of different independent variables, we see that

$$\sum_{i\neq k} (f_i'^2 + \eta_i') F(ik)^2 \neq \text{const.}$$

Hence

$$\eta_k' = \sum_i \eta_i = 0.$$

Thus $f_i'^2 = \eta_i f_i^2$, $i = 1, \ldots, n$, $\eta_i \neq 0$. Further, from $f_i''/f_i = \eta_i$ we get $f_i(x_i) = a_i e^{\sqrt{\eta_i} x_i} + a_i' e^{-\sqrt{\eta_i} x_i}$ when $\eta_i > 0$, and $f_i = b_i \cos\sqrt{-\eta_i} x_i + b_i' \sin\sqrt{-\eta_i} x_i$ when $\eta_i < 0$, where a_i, a_i', b_i and b_i' are constants.

Noting that $f_i'^2 = \eta_i f_i^2$, we have $a_i a_i' = 0$, which means $f_i(x_i) = a_i e^{\sqrt{\eta_i} x_i}$ or $f_i(x_i) = a_i' e^{-\sqrt{\eta_i} x_i}$ when $\eta_i > 0$; and $b_i^2 + b_i'^2 = 0$, which means $f_i(x_i) \equiv 0$ when $\eta_i < 0$.

By the assumption that $F \neq 0$, we see that $\eta_i > 0$ for $i = 1, \ldots, n$, while $\sum_i \eta_i = 0$, which is impossible. Thus $r \neq 0$.

Case 2: $3 \leq r \leq n$. In this case, from (3.11) we infer that for $i = 1, \ldots, r$,

$$\frac{f_i''}{f_i} \sum_{j\neq i}^n \frac{f_j'''}{f_j'} + \left(\frac{f_i'''}{f_i'} - \frac{2f_i''}{f_i} \right) \sum_{j\neq i}^n \frac{f_j''}{f_j} + \phi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = 0 \tag{3.30}$$

for some ϕ. Differentiating this with respect to x_i gives

$$\frac{\sum_{j\neq i}^n \frac{f_j'''}{f_j'}}{\sum_{j\neq i}^n \frac{f_j''}{f_j}} = -\frac{\left(\frac{f_i'''}{f_i'} - \frac{2f_i''}{f_i} \right)'_{x_i}}{\left(\frac{f_i''}{f_i} \right)'_{x_i}} = \lambda_i, \tag{3.31}$$

where $\lambda_i = \text{const}$. Then for $j \neq i$, from (3.31) we get

$$\frac{f_j'''}{f_j'} - \lambda_i \frac{f_j''}{f_j} = a_{ij} = \text{const}, \tag{3.32}$$

where

$$\sum_{j\neq i}^n a_{ij} = 0. \tag{3.33}$$

From (3.32) we see that for all $i \neq k$,

$$(\lambda_k - \lambda_i) \frac{f_j''}{f_j} = a_{kj} - a_{ij}, \tag{3.34}$$

which implies that $\lambda_k - \lambda_i = 0$ and $a_{kj} - a_{ij} = 0$. Therefore, we assume

$$\lambda_1 = \cdots = \lambda_r = \lambda \quad \text{and} \quad a_{kj} = a_{ij} = a_j.$$

Noting $f_j''/f_j = \eta_j$, $j = r+1, \dots, n$, for $i = 1, \dots, r$ from (3.33) we have

$$\sum_{j=r+1}^{n} (\lambda - 1)\eta_j + \sum_{k=1}^{r} a_k - a_i = 0, \tag{3.35}$$

which implies $a_1 = \cdots = a_r$, where $a_i = f_j'''/f_j' - \lambda f_j''/f_j$.

On the other hand, from (3.31) we get

$$\frac{\left(\frac{f_i'''}{f_i'} - \frac{2f_i''}{f_i}\right)'_{x_i}}{\left(\frac{f_i''}{f_i}\right)'_{x_i}} = \lambda. \tag{3.36}$$

From (3.36) we have

$$\frac{f_j'''}{f_j'} = (2 - \lambda)\,\frac{f_j''}{f_j} + b_j, \tag{3.37}$$

where b_j is a constant.

Comparing (3.32) with (3.37), we get $\lambda = 1$ and $a_i = b_i$ immediately. Thus (3.35) becomes

$$\sum_{k=1}^{r} a_k - a_i = 0.$$

This yields $a_i = 0$, $i = 1, \dots, r$, and so $f_i''/f_i = \eta_i = \text{const}$, $i = 1, \dots, n$. This is a contradiction.

Case 3: $r = 2$. In this case, (3.11) becomes

$$\left(\frac{f_1''}{f_1} + \eta\right)\left(\frac{f_2'''}{f_2'} - \frac{f_2''}{f_2}\right) + \left(\frac{f_2''}{f_2} + \eta\right)\left(\frac{f_1'''}{f_1'} - \frac{f_1''}{f_1}\right) = 0, \tag{3.38}$$

where $\eta = \sum_{k=3}^{n} \eta_k$. Hence

$$\frac{\frac{f_2'''}{f_2'} - \frac{f_2''}{f_2}}{\frac{f_2''}{f_2} + \eta} = -\frac{\frac{f_1'''}{f_1'} - \frac{f_1''}{f_1}}{\frac{f_1''}{f_1} + \eta} = K = \text{const},$$

which implies

$$\frac{f_2'''}{f_2'} = (K + 1)\,\frac{f_2''}{f_2} + \eta K, \tag{3.39}$$

$$\frac{f_1'''}{f_1'} = (1 - K)\,\frac{f_1''}{f_1} - \eta K. \tag{3.40}$$

As mentioned above, for $i = 3, \dots, n$, $f_i'' = \eta_i f$ and $f_i'^2 = \eta_i f_i^2 + \eta_i'$, and from (3.11) we get

$$\begin{aligned}(3.41)\qquad \frac{f_1''}{f_1}+\frac{f_2''}{f_2}&+f_1^2f_2^2\sum_{i\neq j}^{n}G(ij)f_i'^2(f_jf_j''-f_j'^2)\\ &+(f_1'^2f_2^2+f_1^2f_2'^2)\sum_{i=3}^{n}G(j)^2(f_jf_j''-f_j'^2)\\ &+[f_1^2(f_2f_2''-f_2'^2)+f_2^2(f_1f_1''-f_1'^2)]\sum_{i=1}^{n}G(i)^2f_i'^2\\ &+[f_1'^2(f_2f_2''-f_2'^2)+f_2'^2(f_1f_1''-f_1'^2)]G=0,\end{aligned}$$

where $G=f_3\dots f_n$ and

$$G(i)=f_3\dots f_{i-1}f_{i+1}\dots f_n,\qquad G(ij)=f_3\dots f_{i-1}f_{i+1}\dots f_{j-1}f_{j+1}\dots f_n.$$

Differentiating (3.41) with respect to x_1 and x_2 and substituting $f_i''/f_i=\eta_i$, (3.23), (3.39) and (3.40) into (3.11), after simplifying we get

$$(3.42)\qquad (2-K)\,\frac{f_1''}{f_1}\sum_{i=3}^{n}\frac{\eta_i'}{f_i^2}+(2+K)\,\frac{f_2''}{f_2}\sum_{i=3}^{n}\frac{\eta_i'}{f_i^2}+2\sum_{i,j=3,\,i\neq j}^{n}\left(\eta_i+\frac{\eta_i'}{f_i^2}\right)\frac{\eta_j'}{f_j^2}=0.$$

Differentiating with respect to x_1 and x_2, we get respectively

$$(3.43)\qquad (2-K)\left(\frac{f_1''}{f_1}\right)'_{x_1}\sum_{i=3}^{n}\frac{\eta_i'}{f_i^2}=0,$$

and

$$(3.44)\qquad (2+K)\left(\frac{f_2''}{f_3}\right)'_{x_1}\sum_{i=3}^{n}\frac{\eta_i'}{f_i^2}=0.$$

Since $2+K=2-K=0$ is impossible, we see that $\eta_i'=0$, $i=3,\dots,n$. Then (3.11) becomes

$$\begin{aligned}(3.45)\qquad \eta+\frac{f_1''}{f_1}+\frac{f_2''}{f_2}&+G^2(\eta[f_1^2(f_2f_2''-f_2'^2)+f_2^2(f_1f_1''-f_1'^2)]\\ &+[f_1'^2(f_2f_2''-f_2'^2)+f_2'^2(f_1f_1''-f_1'^2)])=0.\end{aligned}$$

Since $\eta+f_1''/f_1+f_2''/f_2\neq 0$, we see that $G^2=\text{const}$, which means that $f_i'=0$ for $i\geq 3$, contrary to assumption. So we get $r\neq 2$.

Case 4: $r=1$. In this case, (3.11) becomes

$$(3.46)\qquad \eta\left(\frac{f_1'''}{f_1'}-\frac{f_1''}{f_1}\right)=0,$$

which means $\eta=\sum_{i=2}^{n}\eta_i=0$. Substituting (3.23) into (3.11) and using

$f_i''/f_i = \eta_i$, after simplifying we get

$$\frac{f_1''}{f_1^3} + \overline{G}^2 \sum_{i,j=2,\, i\neq j}^{n} \left(\eta_i + \frac{\eta_i'}{f_i^2}\right) \frac{(-\eta_j')}{f_j^2} + \left(\frac{f_1''}{f_1} - 2\,\frac{f_1'^2}{f_1^2}\right) \overline{G}^2 \sum_{i=2}^{n} \frac{\eta_i'}{f_i^2} = 0, \tag{3.47}$$

where $\overline{G} = f_2 \dots f_n$. Because $f_1''/f_1 \neq$ const, differentiating (3.47) with respect to x_1, we get

$$\left(\frac{f_1''}{f_1^3}\right)'_{x_1} + \left(\frac{f_1''}{f_1} - 2\,\frac{f_1'^2}{f_1^2}\right)'_{x_1} \overline{G}^2 \sum_{i=2}^{n} \frac{\eta_i'}{f_i^2} = 0. \tag{3.48}$$

If $\eta_i'/f_i^2 = 0$, we have $\eta_i' = 0$ $(i \geq 2)$ and $f_1'' = 0$, which means $f_1''/f_1 = 0$ $=$ const. This is a contradiction.

If $(f_i''/f_i^3)'_{x_i} = 0$ and $(f_1''/f_1 - 2f_1'^2/f_1^2)'_{x_i} = 0$, we set

$$\frac{f_1''}{f_1^3} = K_1 = \text{const}, \tag{3.49}$$

$$\frac{f_1''}{f_1} - 2\,\frac{f_1'^2}{f_1^2} = K_2 = \text{const}. \tag{3.50}$$

Substituting (3.49) into (3.50), we get

$$K_1 f_1^4 - 2f_1^2 = K_2 f_1^2. \tag{3.51}$$

By differentiating with respect to x_1, from (3.51) we have

$$2K_2 f_1^2 = 0, \tag{3.52}$$

which means $K_2 \equiv 0$. Then (3.47) becomes

$$\overline{G}^2 \sum_{i,j=2,\, i\neq j}^{n} \left(\eta_i + \frac{\eta_i'}{f_i^2}\right) \frac{\eta_j'}{f_j^2} = K_1. \tag{3.53}$$

Similarly to the passage from (3.24) to (3.29) we get

$$\eta_k' \sum_{i\neq k} G(ik)^2 ({f_i'}^2 + \eta_i') = K_1. \tag{3.54}$$

When $n \geq 4$, similarly to (3.29) we have

$$\sum_{i\neq k} G(ik)^2 ({f_i'}^2 + \eta_i') \neq \text{const}, \tag{3.55}$$

which means $\eta_k' = K_1 = 0$. Thus we have $f_1'' = 0$ and $f_1''/f_1 = 0 =$ const, which is a contradiction.

When $n = 3$, (3.54) yields

$$\eta_2'({f_3'}^2 + \eta_3') = K_1, \tag{3.56}$$

$$\eta_3'({f_2'}^2 + \eta_2') = K_1, \tag{3.57}$$

which means that $f_3' = \sqrt{\eta_3'} = \text{const}$ and $f_2' = \sqrt{\eta_2'} = \text{const}$. Then from (3.50), (3.51) and $K_2 = 0$, we get

$$f_1(x_1) = \pm \frac{1}{\sqrt{K_1/2\, x_1 + C_1}}, \tag{3.58}$$

where C_1 is a constant. From (3.56) and (3.57) we see that $\eta_2 = \eta_3 = 0$ and $\eta_2'\eta_3' = K_1/2$, and further

$$f_2(x_2) = \sqrt{\eta_2'}\, x_2 + C_2, \qquad f_3(x_3) = \sqrt{\eta_3'}\, x_3 + C_3,$$

where C_2 and C_3 are constants. This shows that the hypersurface is a quadratic cone.

From the proof above, we see that among the $f_i'(x_i)$, $i = 1, \dots, n$, no more than three are non-zero, while others satisfy $f_i'(x_i) = 0$ so that $f_i(x_i) = \text{const}$. Thus we complete the proof of Theorem 2.

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