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CALABI–YAU STABLE MODULE CATEGORIES OF FINITE TYPE

BY

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Abstract. We describe the stable module categories of the self-injective finite-dimensional algebras of finite representation type over an algebraically closed field which are Calabi–Yau (in the sense of Kontsevich).

Introduction. Throughout the paper, by an *algebra* we mean a finite-dimensional associative K-algebra with an identity over a fixed algebraically closed field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules and by D the standard duality $\operatorname{Hom}_K(-, K)$ on $\operatorname{mod} A$. An algebra A is said to be of *finite type* if $\operatorname{mod} A$ admits only finitely many isomorphism classes of indecomposable modules. Further, an algebra A is called *self-injective* if A_A is injective, or equivalently the projective A-modules are injective. An important class of self-injective algebras is formed by the symmetric algebras A for which A and $D(A)$ are isomorphic as A-A-bimodules.

Following Bondal and Kapranov [7], a triangulated K-linear category $\mathcal{A}$ is said to have a *Serre duality* if there is a triangle autoequivalence $S : \mathcal{A} \to \mathcal{A}$, called a *Serre functor*, such that there are natural K-linear automorphisms $\operatorname{Hom}_{\mathcal{A}}(A, B) \cong D \operatorname{Hom}_{\mathcal{A}}(B, S(A))$ for all objects A and B in $\mathcal{A}$, where $D = \operatorname{Hom}_K(-, K)$. Moreover, if S and S' are two Serre functors of $\mathcal{A}$, then they are naturally isomorphic (see [7], [20]). Further, following Kontsevich [19] (see also [18]), a triangulated K-linear category $\mathcal{A}$, with shift functor T, is said to be *Calabi–Yau* if an iterated shift functor T^m is a Serre duality of $\mathcal{A}$ for some integer $m \geq 0$.

An important class of triangulated K-linear categories of algebraic nature is formed by the *stable module categories* $\underline{\operatorname{mod}}\, A$ of self-injective algebras A, where the shift T is given by the inverse Ω_A^{-1} of Heller's syzygy functor (see [16]). Recall that the objects of $\underline{\operatorname{mod}}\, A$ are the objects of $\operatorname{mod} A$ without projective direct summands, and for any two objects M and N of $\underline{\operatorname{mod}}\, A$ the

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space of morphisms from M to N in $\underline{\mathrm{mod}}\,A$ is the quotient $\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/P(M,N)$, where $P(M,N)$ is the subspace of $\mathrm{Hom}_A(M,N)$ consisting of all A-homomorphisms which factorize through projective A-modules. Then we have two mutually inverse *Heller's syzygy functors* Ω_A, $\Omega_A^{-1} : \underline{\mathrm{mod}}\,A \xrightarrow{\sim} \underline{\mathrm{mod}}\,A$ which assign to an object M of $\underline{\mathrm{mod}}\,A$ respectively the kernel $\Omega_A(M)$ of its projective cover $P_A(M) \to M$ and the cokernel $\Omega_A^{-1}(M)$ of its injective envelope $M \to I_A(M)$ in $\mathrm{mod}\,A$. Further, denote by $\nu_A : \underline{\mathrm{mod}}\,A \xrightarrow{\sim} \underline{\mathrm{mod}}\,A$ the *Nakayama functor* $D\,\mathrm{Hom}_A(-,A)$. By general theory, the *Auslander–Reiten translation* $\tau_A = D\,\mathrm{Tr}$ is equivalent to $\Omega_A^2\nu_A = \nu_A\Omega_A^2$. In particular, $\tau_A = \Omega_A^2$ if A is symmetric. Two self-injective algebras A and Λ are called *stably equivalent* if the stable module categories $\underline{\mathrm{mod}}\,A$ and $\underline{\mathrm{mod}}\,\Lambda$ are equivalent.

It is known that $S = \Omega_A\nu_A$ is a Serre duality of $\underline{\mathrm{mod}}\,A$ (see [12, (1.2)]). Therefore, the stable module category $\underline{\mathrm{mod}}\,A$ of a self-injective algebra A is Calabi–Yau if and only if $\nu_A \cong \Omega_A^{-m-1}$ (equivalently, $\tau_A \cong \Omega_A^{-m+1}$) for some integer $m \geq 0$. In that case, the smallest integer $m \geq 0$ with the property $\nu_A \cong \Omega_A^{-m+1}$ is called the *stable Calabi–Yau dimension* of A (see [12]). It is shown in [12, Propositions 2.1 and 2.2] that a connected self-injective algebra A is of stable Calabi–Yau dimension 0 (respectively, 1) if and only if A is Morita equivalent to a Nakayama algebra of Loewy length at most 2 (respectively, a local Nakayama algebra of Loewy length at least 3). In particular, every self-injective algebra A of stable Calabi–Yau dimension at most 1 is of finite representation type. Further, it follows from [6, Theorem 1.2] that every connected self-injective algebra A of stable Calabi–Yau dimension 2 is Morita equivalent to a deformed preprojective algebra of generalized Dynkin type. We mention that, with the exception of a few cases of small dimension, the deformed preprojective algebras of generalized Dynkin type are of wild representation type (see [12, Theorem 3.7]). Finally, we denote by Γ_A^{s} the stable Auslander–Reiten quiver of A, obtained from its Auslander–Reiten quiver Γ_A by removing the projective-injective vertices and the arrows attached to them. Then we also get the induced automorphisms Ω_A, Ω_A^{-1}, τ_A, τ_A^{-1}, ν_A, ν_A^{-1} of Γ_A^{s}.

It has been proved in [11] and [12] that the class of all connected symmetric algebras of tame representation type and with the Calabi–Yau stable module categories coincides with the class of algebras which are Morita equivalent to the algebras of the following three types: socle deformations of the symmetric algebras of Dynkin type, socle deformations of the symmetric algebras of tubular type, and algebras of quaternion type. We would like to mention that there are also wild self-injective algebras with the Calabi–Yau stable module categories: for example, most of the preprojective algebras of Dynkin type have this property (see [4], [6], [13]). Further, in [12, (4.5)], a class of self-injective algebras of finite representation type whose stable module categories are not Calabi–Yau is exhibited. We also refer to [1] for the structure of triangulated categories with finitely many indecomposable objects.

The aim of this note is to describe the Morita equivalence classes of all connected self-injective algebras of finite type whose stable module categories are Calabi–Yau.

From now on we use the term *algebra* for a basic, connected algebra over K. By a *Dynkin graph* we mean a graph of one of the Dynkin types: $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. For a Dynkin graph Δ, denote by h_Δ the *Coxeter number* of Δ. Recall that h_Δ is the order of the Coxeter element of the Coxeter (Weyl) group of Δ, so that

$$h_{\mathbb{A}_n} = n + 1, \quad h_{\mathbb{D}_n} = 2n - 2, \quad h_{\mathbb{E}_6} = 12, \quad h_{\mathbb{E}_7} = 18, \quad h_{\mathbb{E}_8} = 30.$$

Moreover, we define h^*_Δ to be $h_\Delta/2$ if $\Delta = \mathbb{A}_{4l-3}$ $(l \geq 1)$, $\mathbb{D}_n$ (n even), $\mathbb{E}_7$, $\mathbb{E}_8$, and h_Δ for the remaining Dynkin graphs Δ .

Let A be a non-simple self-injective algebra of finite type. By general theory, we may associate to A three combinatorial invariants: a Dynkin graph $\Delta(A)$ and integers $s(A) \geq 1$ and $t(A) \in \{1, 2, 3\}$ (see Section 1 for the details).

The following theorem is the main result of the paper.

THEOREM. *Let A be a non-simple self-injective algebra of finite type. Then $\underline{\mathrm{mod}}\, A$ is Calabi–Yau if and only if the following conditions are satisfied*:

(1) $\gcd(s(A), h^*_{\Delta(A)}) = 1$.
(2) $t(A) \leq 2$.
(3) *$s(A)$ and $t(A)$ have the same parity if $\Delta(A) = \mathbb{A}_{4l-3}$ for some $l \geq 2$.*

It would be interesting to determine the stable Calabi–Yau dimension of self-injective algebras of finite representation type. For the symmetric algebras of finite representation type it is given in [12, Theorem 4.3].

For basic background on the representation theory applied here we refer to [3], [5], [23], [24], [25], [26], [29].

1. Self-injective algebras of finite type. By general theory (see [10], [17], [21], [22], [26], [27], [28]), the class of self-injective algebras of finite type may be divided into two disjoint classes: the standard algebras which admit simply connected Galois coverings, and the remaining non-standard algebras. The non-standard self-injective algebras of finite type occur only in characteristic 2 and are symmetric (see [22], [27], [28]), and consequently their stable module categories are Calabi–Yau. Therefore, in order to describe the

Calabi–Yau stable module categories of finite type, we may restrict ourselves to the standard self-injective algebras of finite type. We will now present a suitable description of these algebras and relevant facts.

An important class of self-injective algebras is formed by the orbit algebras $\widehat{B}/G$, where $\widehat{B}$ is the *repetitive algebra* of the algebra B (see [17]), and G is an admissible group of automorphisms of $\widehat{B}$. Recall that

$$\widehat{B} = \bigoplus_{k\in\mathbb{Z}} (B_k \oplus D(B)_k)$$

with $B_k = B$ and $D(B)_k = D(B)$ for all $k \in \mathbb{Z}$, and the multiplication in $\widehat{B}$ is defined by

$$(a_k, f_k) \cdot (b_k, g_k) = (a_k b_k, a_k g_k + f_k b_{k+1})_k$$

for $a_k, b_k \in B_k$, $f_k, g_k \in D(B)_k$. For a fixed set $\mathcal{E} = \{e_i \mid 1 \le i \le n\}$ of orthogonal primitive idempotents of B with $1_B = e_1 + \cdots + e_n$, consider the canonical set $\widehat{\mathcal{E}} = \{e_{j,k} \mid 1 \le j \le n,\ k \in \mathbb{Z}\}$ of orthogonal primitive idempotents of $\widehat{B}$ such that $1_{B_k} = e_{1,k} + \cdots + e_{n,k}$. By an *automorphism* of $\widehat{B}$ we mean a K-algebra automorphism of $\widehat{B}$ which fixes the chosen set $\widehat{\mathcal{E}}$ of orthogonal primitive idempotents of $\widehat{B}$. A group G of automorphisms of $\widehat{B}$ is said to be *admissible* if the induced action of G on $\widehat{\mathcal{E}}$ is free and has finitely many orbits. Then the *orbit algebra* $\widehat{B}/G$ (see [15]) is a self-injective algebra and the G-orbits in $\widehat{\mathcal{E}}$ form a canonical set of orthogonal primitive idempotents of $\widehat{B}/G$ whose sum is the identity of $\widehat{B}/G$. We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ whose restriction to each copy $B_k \oplus D(B)_k$ is the identity map $B_k \oplus D(B)_k \to B_{k+1} \oplus D(B)_{k+1}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension $\mathrm{T}(B) = B \ltimes D(B)$ of B by $D(B)$, and it is a symmetric algebra. An automorphism φ of $\widehat{B}$ is said to be *positive* (respectively, *rigid*) if $\varphi(B_k) \subseteq \sum_{i\ge k}(B_i)$ (respectively, $\varphi(B_k) = B_k$) for any $k \in \mathbb{Z}$. Moreover, φ is said to be *strictly positive* if φ is positive but not rigid.

Let Δ be a Dynkin graph $\mathbb{A}_n$ $(n \ge 1)$, $\mathbb{D}_n$ $(n \ge 4)$, $\mathbb{E}_n$ $(n = 6, 7, 8)$. By a *tilted algebra* of Dynkin type Δ we mean an algebra $B = \mathrm{End}_H(T)$ where H is the path algebra $K\vec{\Delta}$ of a quiver $\vec{\Delta}$ with the underlying graph Δ and T is a multiplicity-free tilting H-module (see [3], [16], [23]).

The following theorem gives a description of the non-simple standard self-injective algebras of finite type (see [10], [21], [22], [28], and [26, Theorem 3.5]).

THEOREM 1.1. *Let A be a non-simple standard self-injective algebra. The following conditions are equivalent*:

(i) *A is of finite type.*

(ii) $A \cong \widehat{B}/(\sigma\varphi^s)$, *where* B *is a tilted algebra of Dynkin type* Δ, φ *is a strictly positive primitive root of the Nakayama automorphism* $\nu_{\widehat{B}}$ *and* σ *is a rigid automorphism of* $\widehat{B}$ *of finite order.*

Therefore, we may associate to any non-simple standard self-injective algebra $A = \widehat{B}/(\sigma\varphi^s)$ of finite type the numerical data: the *Dynkin type* $\Delta(A) = \Delta(B)$ of the tilted algebra B, the *degree* $e(A)$ of the primitive root φ of $\nu_{\widehat{B}}$, the (*torsion*) *order* $t(A)$ of the automorphism σ, and the *power* $s(A) = s$ of φ. We also note that $f(A) = s(A)/e(A)$ is the *frequency* of A and $\mathrm{typ}(A) = (\Delta(A), f(A), t(A))$ is the *type* of A, defined in [2, (2.1)].

The following theorem proved by Asashiba [2, Theorem 2.2] describes the stable equivalence classes of all non-simple, basic, connected, standard, self-injective algebras of finite type.

THEOREM 1.2. *Let* A *and* A' *be two non-simple*, *standard*, *self-injective algebras of finite type. Then the following conditions are equivalent*:

(i) A *and* A' *are stably equivalent.*
(ii) $\mathrm{typ}(A) = \mathrm{typ}(A')$.

Further, we have the following proposition.

PROPOSITION 1.3. *The set of types of non-simple standard self-injective algebras of finite type is the disjoint union of the following sets*:

$\{(\mathbb{A}_n, s/n, 1) \mid n, s \in \mathbb{N}\}$; $\{(\mathbb{D}_n, s, 2) \mid n, s \in \mathbb{N},\ n \geq 4\}$;

$\{(\mathbb{A}_{2p+1}, s, 2) \mid p, s \in \mathbb{N}\}$; $\{(\mathbb{D}_4, s, 3) \mid s \in \mathbb{N}\}$;

$\{(\mathbb{D}_n, s, 1) \mid n, s \in \mathbb{N},\ n \geq 4\}$; $\{(\mathbb{E}_n, s, 1) \mid n = 6, 7, 8,\ s \in \mathbb{N}\}$;

$\{(\mathbb{D}_{3m}, s/3, 1) \mid m, s \in \mathbb{N},\ m \geq 2, 3 \nmid s\}$; $\{(\mathbb{E}_6, s, 2) \mid s \in \mathbb{N}\}$.

For a Dynkin graph Δ, we put $m_\Delta = h_\Delta - 1$.

The following proposition describes the action of the syzygy functors $\Omega_{\widehat{B}}$ on the stable module categories $\underline{\mathrm{mod}}\,\widehat{B}$ of the repetitive algebras $\widehat{B}$ of tilted algebras B of Dynkin types (see [9], [12, Proposition 4.2] and [14, Section 6]).

PROPOSITION 1.4. *Let* B *be a tilted algebra of Dynkin type* Δ. *Then we have equivalences of functors on the category* $\underline{\mathrm{mod}}\,\widehat{B}$:

(i) $\Omega_{\widehat{B}} \cong \tau_{\widehat{B}}^{h_\Delta/2}$ *for* $\Delta = \mathbb{A}_1$, $\mathbb{D}_n$ (n *even*), $\mathbb{E}_7$, $\mathbb{E}_8$.
(ii) $\Omega_{\widehat{B}} \cong \sigma\tau_{\widehat{B}}^{h_\Delta/2}$ *for* $\Delta = \mathbb{A}_n$ ($n \geq 3$ *odd*), $\mathbb{D}_n$ (n *odd*), $\mathbb{E}_6$ *and an automorphism* σ *of order* 2, *commuting with* $\tau_{\widehat{B}}$.
(iii) $\Omega_{\widehat{B}} \cong \varrho\tau_{\widehat{B}}^{m_\Delta/2}$ *for* $\Delta = \mathbb{A}_n$ (n *even*) *and an automorphism* ϱ *with* $\varrho^2 = \tau_{\widehat{B}}$.

In particular, *in all cases*, *we have* $\Omega_{\widehat{B}}^2 \cong \tau_{\widehat{B}}^{h_\Delta}$.

The following proposition (see [10], [17]) describes the induced action of the Nakayama automorphism $\nu_{\widehat{B}}$ on the stable module category $\underline{\mathrm{mod}}\,\widehat{B}$ of the repetitive algebra $\widehat{B}$ of a tilted algebra B of Dynkin type.

PROPOSITION 1.5. *Let B be a tilted algebra of Dynkin type Δ. Then $\nu_{\widehat{B}} \cong \tau_{\widehat{B}}^{-m_\Delta}$ as endofunctors on $\underline{\mathrm{mod}}\,\widehat{B}$.*

Let B be a tilted algebra of Dynkin type and G an admissible infinite cyclic group of automorphisms of $\widehat{B}$. Then there are a canonical *Galois covering* $F : \widehat{B} \to \widehat{B}/G$ with Galois group G (see [15, (3.1)]) and the associated *push-down functor*

$$F_\lambda : \mathrm{mod}\,\widehat{B} \to \widehat{B}/G.$$

Moreover, $\widehat{B}$ is locally representation-finite, by [17]. Therefore, applying [8, Section 3] and [15, Section 3], we obtain the following proposition.

PROPOSITION 1.6. *Let B be a tilted algebra of Dynkin type, G an admissible infinite cyclic group of automorphisms of $\widehat{B}$ and $A = \widehat{B}/G$. Then the following statements hold*:

(i) $F_\lambda : \mathrm{mod}\,\widehat{B} \to \mathrm{mod}\,A$ *is exact*, *dense*, *and induces a bijection between the G-orbits of isomorphism classes of indecomposable finite-dimensional $\widehat{B}$-modules and the isomorphism classes of indecomposable finite-dimensional A-modules.*
(ii) $F_\lambda \Omega_{\widehat{B}} \cong \Omega_A F_\lambda$.
(iii) $F_\lambda \tau_{\widehat{B}} \cong \tau_A F_\lambda$.

It is known (see [26, Theorem 3.10]) that, if A is a non-simple standard self-injective algebra of finite type and $e(A) \geq 2$, then one of the following cases holds:

(1) $\Delta(A) = \mathbb{A}_n$ and $e(A) = n$ for some $n \geq 2$,
(2) $\Delta(A) = \mathbb{D}_{3m}$ and $e(A) = 3$ for some $m \geq 2$

(see also Proposition 1.3). Moreover, $m_{\mathbb{A}_n} = n$ and $m_{\mathbb{D}_{3m}} = 2(3m) - 3 = 3(2m-1)$. Therefore, we may assign to any non-simple standard self-injective algebra A the natural number

$$r(A) = s(A)t(A)m_{\Delta(A)}/e(A).$$

We have the following consequence of Theorem 1.1 and Propositions 1.5 and 1.6 (see also [15, Theorem 3.6]).

COROLLARY 1.7. *Let A be a non-simple standard self-injective algebra of finite type. Then $r(A)$ is the order of the Auslander–Reiten translation τ_A.*

We end this section with a description of non-standard self-injective algebras of finite type. Recall that two self-injective algebras Λ and A are called *socle equivalent* if the factor algebras $\Lambda/\mathrm{soc}\,\Lambda$ and $A/\mathrm{soc}\,A$ are isomorphic.

Then we have the following version of a result from [22], [27], [28] (see [26, Theorem 3.8 and Proposition 3.9]).

THEOREM 1.8. *Let A be a self-injective algebra. Then A is non-standard of finite type if and only if* $\operatorname{char} K = 2$ *and A is socle equivalent to a standard algebra A' of the form $\widehat{B}/(\varphi)$, where B is a tilted algebra of Dynkin type $\mathbb{D}_{3m}$ and φ is a strictly positive 3-root of $\nu_{\widehat{B}}$.*

For a non-standard self-injective algebra A of finite type, the standard algebra A' socle equivalent to A is uniquely determined (up to isomorphism) by A, and is called the *standard form* of A. Moreover, the stable Auslander–Reiten quivers Γ_A^{s} and $\Gamma_{A'}^{\mathrm{s}}$ are isomorphic as translation quivers. Therefore, we may associate to A the numerical data $\Delta(A) = \Delta(A') = \mathbb{D}_{3m}$, $e(A) = e(A') = 3$, $t(A) = t(A') = 1$, $s(A) = s(A') = 1$, and $r(A) = r(A') = 2m - 1$.

2. Proof of the Theorem. Let A be a non-simple self-injective algebra of finite type. Assume first that A is non-standard. Then $\Delta(A) = \mathbb{D}_{3m}$, $s(A) = 1$, $t(A) = 1$ satisfy conditions (1)–(3) of the theorem. Moreover, A is a symmetric algebra (see [22], [27], [28]) with $\underline{\operatorname{mod}}\, A$ Calabi–Yau, because $\nu_A \cong 1_{\underline{\operatorname{mod}}\, A}$ and $\Omega_A^{-m-1} \cong 1_{\underline{\operatorname{mod}}\, A}$ for some $m \geq 0$ (see also [12, Theorem 4.3]).

From now on we assume that A is a non-simple standard self-injective algebra of finite type. Then A has a canonical presentation $A \cong \widehat{B}/(\sigma\varphi^s)$ given by Theorem 1.2. We abbreviate below the associated numerical data: $\Delta = \Delta(A)$, $e = e(A)$, $t = t(A)$, $s = s(A)$, and $r = r(A)$. We divide the proof into several steps.

PROPOSITION 2.1. *Assume $t(A) = 1$ and $\Delta(A) \neq \mathbb{A}_{4l-3}$ for $l \geq 2$. Then $\underline{\operatorname{mod}}\, A$ is Calabi–Yau if and only if* $\gcd(s(A), h^*_{\Delta(A)}) = 1$.

Proof. We know that $\underline{\operatorname{mod}}\, A$ is Calabi–Yau if and only if $\tau_A^{-1} \cong \Omega_A^d$ for some integer $d \geq -1$. Moreover, by Corollary 1.7, $r = stm_\Delta/e = sm_\Delta/e$ is the order of τ_A. According to Proposition 1.4, we have three cases to consider.

(a) Assume $\Delta = \mathbb{A}_1, \mathbb{D}_n$ (n even), $\mathbb{E}_7$, or $\mathbb{E}_8$. Then $h^*_\Delta = h_\Delta/2$. Applying Proposition 1.6, we conclude that $\Omega_A \cong \tau_A^{h^*_\Delta}$. Suppose $\underline{\operatorname{mod}}\, A$ is Calabi–Yau, and $\tau_A^{-1} \cong \Omega_A^d$ for some $d \geq -1$. Then we obtain the equivalences of functors

$$\tau_A^{dh^*_\Delta + 1} = (\tau_A^{h^*_\Delta})^d \tau_A \cong \Omega_A^d \tau_A \cong 1_{\underline{\operatorname{mod}}\, A},$$

and consequently, passing to the stable Auslander–Reiten quiver Γ_A^{s}, we obtain the equality $\tau_A^{dh^*_\Delta + 1} = 1_{\Gamma_A^{\mathrm{s}}}$. Hence

$$dh^*_\Delta + 1 \equiv 0 \pmod{r}.$$

Since $r = sm_\Delta/e$, this immediately implies the required condition $\gcd(s, h^*_\Delta) = 1$. Conversely, assume that $\gcd(s, h^*_\Delta) = 1$. Since $\gcd(m_\Delta, h_\Delta) = 1$ forces

$\gcd(m_\Delta/e, h^*_\Delta) = 1$, we then obtain $\gcd(r, h^*_\Delta) = 1$. But then $h^*_\Delta + r\mathbb{Z}$ generates the cyclic group $\mathbb{Z}/r\mathbb{Z}$, and consequently there exists $d \geq 1$ such that $d(h^*_\Delta + r\mathbb{Z}) = (r-1) + r\mathbb{Z}$, or equivalently $dh^*_\Delta + 1 \equiv 0 \pmod r$. Thus

$$\Omega^d_A \tau_A \cong (\tau_A^{h^*_\Delta})^d \tau_A = \tau_A^{dh^*_\Delta + 1} \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Hence, $\tau_A^{-1} \cong \Omega^d_A$, and so $\underline{\mathrm{mod}}\, A$ is Calabi–Yau.

(b) Assume $\Delta = \mathbb{A}_n$ ($n \geq 3$ odd), $\mathbb{D}_n$ (n odd), or $\mathbb{E}_6$. Then $h^*_\Delta = h_\Delta$, because $\Delta \neq \mathbb{A}_{4l-3}$ for $l \geq 1$. Applying Propositions 1.4 and 1.6, we conclude that $\Omega_A \cong \sigma\tau_A^{h_\Delta/2}$ for an automorphism σ of $\underline{\mathrm{mod}}\, A$ of order 2, commuting with τ_A. Suppose $\underline{\mathrm{mod}}\, A$ is Calabi–Yau, and $\tau_A^{-1} \cong \Omega^d_A$ for some $d \geq -1$. Then we obtain

$$\sigma^d \tau_A^{dh_\Delta/2+1} = (\sigma\tau_A^{h_\Delta/2})^d \tau_A \cong \Omega^d_A \tau_A \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Hence d is even and passing to Γ^{s}_A we conclude that $\sigma^d \tau_A^{dh_\Delta/2+1} = 1_{\Gamma^{\mathrm{s}}_A}$. Thus the order $r = sm_\Delta/e$ of τ_A divides $dh_\Delta/2+1$, or equivalently $dh_\Delta/2+1 \equiv 0 \pmod r$. Let $d = 2d_1$. Then $d_1 h_\Delta + 1 \equiv 0 \pmod r$, and hence $\gcd(s, h_\Delta) = 1$. Conversely, assume that $\gcd(s, h_\Delta) = 1$. Then, as in (a), we infer that there is an integer $d_1 \geq 1$ such that $d_1 h_\Delta + 1 \equiv 0 \pmod r$. Then, for $d = 2d_1$, we have $dh_\Delta/2 + 1 \equiv 0 \pmod r$. Hence,

$$\Omega^d_A \tau_A \cong (\sigma\tau_A^{h_\Delta/2})^d \tau_A = \sigma^{2d_1} \tau_A^{dh_\Delta/2+1} = \tau_A^{dh_\Delta/2+1} \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Thus, $\tau_A^{-1} \cong \Omega^d_A$, and consequently $\underline{\mathrm{mod}}\, A$ is Calabi–Yau.

(c) Assume $\Delta = \mathbb{A}_n$ (n even). Then $h^*_\Delta = h_\Delta$. Applying Propositions 1.4 and 1.6, we conclude that $\Omega_A \cong \varrho\tau_A^{m_\Delta/2}$ for an automorphism ϱ of $\underline{\mathrm{mod}}\, A$ with $\varrho^2 = \tau_A$. Suppose $\underline{\mathrm{mod}}\, A$ is Calabi–Yau, and $\tau_A^{-1} \cong \Omega^d_A$ for some $d \geq -1$. Then we obtain

$$\varrho^d \tau_A^{dm_\Delta/2+1} = (\varrho\tau_A^{m_\Delta/2})^d \tau_A \cong \Omega^d_A \tau_A \cong 1_{\underline{\mathrm{mod}}\, A},$$

and hence $d = 2d_1$ for some $d_1 \geq 0$. Then

$$\tau_A^{d_1 h_\Delta + 1} = \tau_A^{d_1} \tau_A^{d_1 m_\Delta + 1} = \varrho^d \tau_A^{dm_\Delta/2+1} \cong 1_{\underline{\mathrm{mod}}\, A},$$

and consequently passing to Γ^{s}_A we infer that $\tau_A^{d_1 h_\Delta + 1} = 1_{\Gamma^{\mathrm{s}}_A}$, and hence $r = sm_\Delta/e$ divides $d_1 h_\Delta + 1$. In particular, we obtain $\gcd(s, h^*_\Delta) = 1$, because $h^*_\Delta = h_\Delta$. Conversely, assume that $\gcd(s, h^*_\Delta) = 1$. Since $\gcd(m_\Delta/e, h_\Delta) = 1$, we have $\gcd(r, h_\Delta) = 1$, and so $h_\Delta + r\mathbb{Z}$ generates the group $\mathbb{Z}/r\mathbb{Z}$. Therefore, there exists an integer $d_1 \geq 1$ such that $d_1 h_\Delta + 1 \equiv 0 \pmod r$. Then, taking $d = 2d_1$, we obtain

$$\Omega^d_A \tau_A \cong (\varrho\tau_A^{m_\Delta/2})^d \tau_A = \varrho^{2d_1} \tau_A^{d_1 m_\Delta + 1} = \tau_A^{d_1 h_\Delta + 1} \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Thus, $\Omega^d_A \cong \tau_A^{-1}$, and hence $\underline{\mathrm{mod}}\, A$ is Calabi–Yau. ∎

LEMMA 2.2. *Assume* $t(A) = 1$ *and* $\Delta(A) = \mathbb{A}_{4l-3}$ *for some* $l \geq 2$. *Then* $\underline{\mathrm{mod}}\, A$ *is Calabi–Yau if and only if* $\gcd(s(A), h^*_{\Delta(A)}) = 1$ *and* $s(A)$ *is odd.*

Proof. We have $h^*_\Delta = h_\Delta/2 = 2l - 1$. Further, since $4l - 3$ is odd and greater than 1, applying Propositions 1.4 and 1.6, we have $\Omega_A \cong \sigma\tau_A^{h^*_\Delta}$ for an automorphism σ of $\underline{\mathrm{mod}}\, A$ of order 2, commuting with τ_A. We also note that $\gcd(s, h_\Delta) = 1$ if and only if s is odd and $\gcd(s, h^*_\Delta) = 1$. Then the lemma follows by the argument applied in part (b) of the proof of Proposition 2.1. ■

LEMMA 2.3. *Assume* $t(A) = 3$. *Then* $\underline{\mathrm{mod}}\, A$ *is not Calabi–Yau.*

Proof. Since $t = t(A) = 3$, it follows from Proposition 1.3 that $\Delta = \mathbb{D}_4$, and so $h_\Delta = 6$, $h^*_\Delta = 3$, $m_\Delta = 5$. Applying Proposition 1.4, we conclude that $\Omega_A \cong \tau_A^3$. Suppose $\underline{\mathrm{mod}}\, A$ is Calabi–Yau, and $\tau_A^{-1} \cong \Omega_A^d$ for some $d \geq -1$. Then we obtain

$$\tau_A^{3d+1} = (\tau_A^3)^d\tau_A \cong \Omega_A^d\tau_A \cong 1_{\underline{\mathrm{mod}}\, A}.$$

But then passing to Γ_A^{s} we conclude that $\tau_A^{3d+1} = 1_{\Gamma_A^{\mathrm{s}}}$, and so the order $r = stm_\Delta = 15s$ of τ_A divides $3d + 1$, a contradiction. Therefore, $\underline{\mathrm{mod}}\, A$ is not Calabi–Yau. ■

We will now analyse the case when $t = t(A) = 2$. Observe that, by Proposition 1.3, $t(A) = 2$ forces $\Delta = \Delta(A)$ to be one of the graphs: $\mathbb{A}_n$ ($n \geq 3$ odd), $\mathbb{D}_n$, or $\mathbb{E}_6$.

PROPOSITION 2.4. *Assume* $t(A) = 2$ *and* $\Delta(A) = \mathbb{D}_n$ *with* n *even. Then* $\underline{\mathrm{mod}}\, A$ *is Calabi–Yau if and only if* $\gcd(s(A), h^*_{\Delta(A)}) = 1$.

Proof. Observe first that $h^*_\Delta = h_\Delta/2 = n-1$ is odd, and hence $\gcd(s, h^*_\Delta) = 1$ is equivalent to $\gcd(st, h^*_\Delta) = 1$. Since $\gcd(m_\Delta/e, h^*_\Delta) = 1$, the latter is also equivalent to $\gcd(r, h^*_\Delta) = \gcd(stm_\Delta/e, h^*_\Delta) = 1$. Further, it follows from Propositions 1.4 and 1.6 that $\Omega_A \cong \tau_A^{h^*_\Delta} = \tau_A^{n-1}$. Assume $\underline{\mathrm{mod}}\, A$ is Calabi–Yau, and $\tau_A^{-1} \cong \Omega_A^d$ for some $d \geq -1$. Then we have

$$\tau_A^{dh^*_\Delta+1} = (\tau_A^{h^*_\Delta})^d\tau_A \cong \Omega_A^d\tau_A \cong 1_{\underline{\mathrm{mod}}\, A},$$

and consequently passing to Γ_A^{s} we infer that $\tau_A^{dh^*_\Delta+1} = 1_{\Gamma_A^{\mathrm{s}}}$, and hence $r = stm_\Delta/e$ divides $dh^*_\Delta+1$. Then $\gcd(r, h^*_\Delta) = 1$, and hence $\gcd(s, h^*_\Delta) = 1$. Conversely, if $\gcd(s, h^*_\Delta) = 1$ then $\gcd(r, h^*_\Delta) = 1$ and there is $d \geq 1$ such that $dh^*_\Delta + 1 \equiv 0 \pmod r$. Hence,

$$\Omega_A^d\tau_A \cong (\tau_A^{h^*_\Delta})^d\tau_A = \tau_A^{dh^*_\Delta+1} \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Thus, $\tau_A^{-1} \cong \Omega_A^d$ and $\underline{\mathrm{mod}}\, A$ is Calabi–Yau. ■

PROPOSITION 2.5. *Assume $t(A) = 2$ and $\Delta(A)$ is one of the graphs $\mathbb{D}_n$ (n odd), $\mathbb{E}_6$, or $\mathbb{A}_n$ ($n \geq 3$ odd but not of the form $4l-3$ for some $l \geq 2$). Then $\underline{\mathrm{mod}}\, A$ is Calabi–Yau if and only if* $\gcd(s(A), h^*_{\Delta(A)}) = 1$.

Proof. Observe that, in the cases considered, $h^*_\Delta = h_\Delta$ and $h_\Delta/2$ are even. Since $A = \widehat{B}/(\sigma\nu^s_{\widehat{B}})$ for a rigid automorphism σ of $\widehat{B}$ of order 2, invoking Propositions 1.5 and 1.6 we conclude that $\tau_A^{sm_\Delta} = \sigma$ for the induced automorphism σ of $\underline{\mathrm{mod}}\, A$ of order 2. Hence, $\tau_A^{2sm_\Delta} = 1_{\underline{\mathrm{mod}}\, A}$ implies that $\tau_A = \sigma\tau_A^{sm_\Delta+1}$. Further, applying Propositions 1.4 and 1.6, we obtain $\Omega_A \cong \sigma\tau_A^{h_\Delta/2}$ for the same automorphism σ of $\underline{\mathrm{mod}}\, A$. Assume $\underline{\mathrm{mod}}\, A$ is Calabi–Yau, and let $\tau_A^{-1} \cong \Omega_A^d$ for some $d \geq -1$. Then we have

$$\sigma^{d+1}\tau_A^{dh_\Delta/2+sm_\Delta+1} = (\sigma\tau_A^{h_\Delta/2})^d(\sigma\tau_A^{sm_\Delta+1}) \cong \Omega_A^d\tau_A \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Then

$$\tau_A^{2(dh_\Delta/2+sm_\Delta+1)} = (\sigma^{d+1}\tau_A^{dh_\Delta/2+sm_\Delta+1})^2 \cong 1_{\underline{\mathrm{mod}}\, A},$$

and consequently passing to Γ^{s}_A we conclude that $\tau_A^{2(dh_\Delta/2+sm_\Delta+1)} = 1_{\Gamma^{\mathrm{s}}_A}$, and so the order $r = 2sm_\Delta$ of τ_A divides $2(dh_\Delta/2 + sm_\Delta + 1)$. Hence sm_Δ divides $dh_\Delta/2 + 1$ and we have $\gcd(s, h_\Delta/2) = 1$. Since $h_\Delta/2$ is even, s is odd, and therefore we conclude that $\gcd(s, h^*_\Delta) = \gcd(s, h_\Delta) = 1$.

Conversely, assume that $\gcd(s, h_\Delta) = 1$. Then s is odd and $\gcd(m_\Delta, h_\Delta) = 1$ forces $\gcd(sm_\Delta, h_\Delta) = 1$. Hence there is an integer $d_1 \geq 1$ such that

$$d_1h_\Delta + (h_\Delta/2 + 1) \equiv 0 \ (\mathrm{mod}\, sm_\Delta),$$

or equivalently

$$d_1h_\Delta + h_\Delta/2 + 1 + sm_\Delta \equiv 0 \ (\mathrm{mod}\, sm_\Delta).$$

On the other hand, sm_Δ is odd, and so $1 + sm_\Delta$ is even. Since h_Δ and $h_\Delta/2$ are also even, we then conclude that

$$d_1h_\Delta + h_\Delta/2 + sm_\Delta + 1 \equiv 0 \ (\mathrm{mod}\, 2sm_\Delta).$$

Taking $d = 2d_1 + 1$, we obtain

$$\Omega_A^d\tau_A \cong \sigma^{d+1}\tau_A^{dh_\Delta/2+sm_\Delta+1} = \tau_A^{d_1h_\Delta+h_\Delta/2+sm_\Delta+1} \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Hence, $\Omega_A^d \cong \tau_A^{-1}$, and consequently $\underline{\mathrm{mod}}\, A$ is Calabi–Yau. ∎

The following proposition completes the proof of the theorem.

PROPOSITION 2.6. *Assume $t(A) = 2$ and $\Delta(A) = \mathbb{A}_{4l-3}$ for some $l \geq 2$. Then $\underline{\mathrm{mod}}\, A$ is Calabi–Yau if and only if $s(A)$ is even and* $\gcd(s(A), h^*_{\Delta(A)}) = 1$.

Proof. Observe first that $h^*_\Delta = h_\Delta/2 = 2l - 1$ is odd. As in Proposition 2.5, we have $\tau_A^{sm_\Delta} = \sigma$, $\tau_A = \sigma\tau_A^{sm_\Delta+1}$ and $\Omega_A \cong \sigma\tau_A^{h_\Delta/2}$ for an automorphism σ of $\underline{\mathrm{mod}}\, A$ of order 2. Assume that $\underline{\mathrm{mod}}\, A$ is Calabi–Yau, and let $\tau_A^{-1} \cong \Omega_A^d$ for some $d \geq -1$. Then we conclude as above that

$$\sigma^{d+1}\tau_A^{dh^*_\Delta + sm_\Delta + 1} \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Hence, passing to Γ_A^{s} we conclude that $\sigma^{d+1}\tau_A^{dh^*_\Delta + sm_\Delta + 1} = 1_{\Gamma_A^{\mathrm{s}}}$. Then $r = 2sm_\Delta$ divides $2(dh^*_\Delta + sm_\Delta + 1)$, and so sm_Δ divides $dh^*_\Delta + 1$. In particular, we get $\gcd(s, h^*_\Delta) = 1$. We claim that s is even. Suppose s is odd. Since $h^*_\Delta = h_\Delta/2$ is odd, we obtain

$$\Omega_A^d \cong (\sigma\tau_A^{h^*_\Delta})^d = (\sigma^{h^*_\Delta}\tau_A^{h^*_\Delta})^d = \sigma^{dh^*_\Delta}\tau_A^{dh^*_\Delta}.$$

Observe that $dh^*_\Delta = p + asm_\Delta$ for some $p \in \{1, \ldots, sm_\Delta - 1\}$ and an integer $a \geq 0$. Further, since $sm_\Delta - 1$ is even, we have

$$\sigma^{sm_\Delta}\tau_A^{sm_\Delta} = \sigma^{sm_\Delta - 1}\tau_A^{-1}(\sigma\tau_A^{sm_\Delta + 1}) \cong \tau_A^{-1}\tau_A = 1_{\underline{\mathrm{mod}}\, A}.$$

Therefore we obtain

$$\Omega_A^d \cong \sigma^{dh^*_\Delta}\tau_A^{dh^*_\Delta} = \sigma^{p + asm_\Delta}\tau_A^{p + asm_\Delta} = \sigma^p\tau_A^p(\sigma^{sm_\Delta}\tau_A^{sm_\Delta})^a \cong \sigma^p\tau_A^p.$$

Hence

$$\sigma^p\tau_A^{p+1} \cong \Omega_A^d\tau_A \cong 1_{\underline{\mathrm{mod}}\, A} \quad \text{and} \quad \tau_A^{2(p+1)} \cong (\sigma^p\tau_A^{p+1})^2 \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Then, passing to Γ_A^{s} we deduce that $\tau_A^{2(p+1)} = 1_{\Gamma_A^{\mathrm{s}}}$, and hence the order $r = 2sm_\Delta$ of τ_A divides $2(p + 1)$, and so sm_Δ divides $p + 1$. Therefore, by our assumption $0 < p < sm_\Delta$, we conclude that $p = sm_\Delta - 1$, and hence p is even. Finally, we obtain

$$\sigma \cong \tau_A^{sm_\Delta} \cong \sigma^p\tau_A^{p+1} \cong 1_{\underline{\mathrm{mod}}\, A},$$

a contradiction. Therefore, s is even.

Conversely, assume that s is even and $\gcd(s, h^*_\Delta) = 1$. Since $\gcd(m_\Delta, h^*_\Delta) = 1$ and h^*_Δ is odd, we then have $\gcd(2sm_\Delta, h^*_\Delta) = 1$. Therefore, there exists an integer $d \geq 1$ such that

$$dh^*_\Delta + (sm_\Delta + 1) \equiv 0 \pmod{2sm_\Delta}.$$

Observe that d is odd, because h^*_Δ is odd and s is even. Then we conclude that

$$\Omega_A^d\tau_A \cong \sigma^{d+1}\tau_A^{dh_\Delta/2 + sm_\Delta + 1} = \tau_A^{dh^*_\Delta + sm_\Delta + 1} \cong 1_{\underline{\mathrm{mod}}\, A}.$$

Hence, $\Omega_A^d \cong \tau_A^{-1}$, and consequently $\underline{\mathrm{mod}}\, A$ is Calabi–Yau. ∎

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