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ON COMPLETE SOLUTIONS AND COMPLETE SINGULAR SOLUTIONS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

BY

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Abstract. A complete solution of an implicit second order ordinary differential equation is defined by an immersive two-parameter family of geometric solutions on the equation hypersurface. We show that a completely integrable equation is either of Clairaut type or of first order type. Moreover, we define a complete singular solution, an immersive one-parameter family of singular solutions on the contact singular set. We give conditions for existence of a complete solution and a complete singular solution of implicit second order ordinary differential equations.

1. Introduction. An implicit second order ordinary differential equation is of the form

$$F(x, y, p, q) = 0,$$

where F is a smooth function of the independent variable x, of the function y, and of its first and second derivatives $p = dy/dx$ and $q = d^2y/dx^2$. It is natural to consider F as being defined on an open subset in the space $J^2(\mathbb{R}, \mathbb{R})$ of 2-jets of functions of one variable. Throughout this paper, we assume that 0 is a regular value of F. It follows that $F^{-1}(0)$ is a hypersurface in $J^2(\mathbb{R}, \mathbb{R})$. We call it the *equation hypersurface*. Let (x, y, p, q) be a local coordinate in $J^2(\mathbb{R}, \mathbb{R})$ and $\xi \subset TJ^2(\mathbb{R}, \mathbb{R})$ be the canonical contact system on $J^2(\mathbb{R}, \mathbb{R})$ described by the vanishing of the 1-forms $\alpha_1 = dy - pdx$ and $\alpha_2 = dp - qdx$. (It is worth noting that the field of planes ξ is an *Engel structure* on the 4-dimensional manifold $J^2(\mathbb{R}, \mathbb{R})$.)

We now define the notion of solutions. A *smooth solution* (or a *classical solution*) of $F = 0$ at z_0 is a smooth function germ $y = f(x)$ at a point t_0 such that $(t_0, f(t_0), f'(t_0), f''(t_0)) = z_0$ and $F(x, f(x), f'(x), f''(x)) = 0$. In other words, there exists a smooth function germ $f : (\mathbb{R}, t_0) \to \mathbb{R}$ such that the image of the 2-jet extension, $j^2f : (\mathbb{R}, t_0) \to (J^2(\mathbb{R}, \mathbb{R}), z_0)$, is contained in the equation hypersurface. It is easy to check that the map j^2f satisfies $(j^2f)^*\alpha_1 = (j^2f)^*\alpha_2 = 0$ (i.e. j^2f is an Engel immersion germ).

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More generally, a *geometric solution* of $F = 0$ at z_0 is an Engel immersion germ $\gamma : (\mathbb{R}, t_0) \to (J^2(\mathbb{R}, \mathbb{R}), z_0)$ such that the image of γ is contained in the equation hypersurface, that is, $\gamma' \neq 0$, $\gamma^*\alpha_1 = \gamma^*\alpha_2 = 0$ and $F(\gamma(t)) = 0$ for each $t \in (\mathbb{R}, t_0)$.

In this paper, the following notions are basic [1, 2, 4–6, 11]. Following the definition of parametrized version for smoothness of classical solutions, a *smooth complete solution* of $F = 0$ at z_0 is defined to be a two-parameter family of smooth function germs $y = f(t, r, s)$ such that

$$F\left(t, f(t, r, s), \frac{\partial f}{\partial t}(t, r, s), \frac{\partial^2 f}{\partial t^2}(t, r, s)\right) = 0$$

and the map germ $j_1^2 f : (\mathbb{R} \times \mathbb{R}^2, (t_0, r_0, s_0)) \to (F^{-1}(0), z_0)$ defined by

$$j_1^2 f(t, r, s) = \left(t, f(t, r, s), \frac{\partial f}{\partial t}(t, r, s), \frac{\partial^2 f}{\partial t^2}(t, r, s)\right)$$

is an immersion. The equation hypersurface is then foliated by a two-parameter family of classical solutions.

On the other hand, we consider the corresponding parametrized version of geometric solutions. Let $\Gamma : (\mathbb{R} \times \mathbb{R}^2, (t_0, r_0, s_0)) \to (F^{-1}(0), z_0)$ be a two-parameter family of geometric solutions of $F = 0$. We call Γ a *complete solution* at z_0 if

$$\operatorname{rank} \begin{pmatrix} \partial x/\partial t & \partial y/\partial t & \partial p/\partial t & \partial q/\partial t \\ \partial x/\partial r & \partial y/\partial r & \partial p/\partial r & \partial q/\partial r \\ \partial x/\partial s & \partial y/\partial s & \partial p/\partial s & \partial q/\partial s \end{pmatrix} (t_0, r_0, s_0) = 3,$$

where $\Gamma(t, r, s) = (x(t, r, s), y(t, r, s), p(t, r, s), q(t, r, s))$. It then follows that Γ is an immersion germ, that is, the equation hypersurface is foliated by a two-parameter family of geometric solutions. We say that the equation $F = 0$ is *smooth completely integrable* (respectively, *completely integrable*) at z_0 if there exists a smooth complete solution (respectively, a complete solution) of $F = 0$ at z_0.

Moreover, we say that a geometric solution $\gamma : (\mathbb{R}, 0) \to (F^{-1}(0), z_0)$ is a *singular solution* of $F = 0$ at z_0 if for any representative $\widetilde{\gamma} : I \to F^{-1}(0)$ of γ and any open subinterval $(a, b) \subset I$ at 0, $\widetilde{\gamma}|_{(a,b)}$ is never contained in a leaf of a complete solution (cf. [1, 5, 7]).

In order to consider a one-parameter family of singular solutions (i.e. a complete singular solution), we define subsets in $F^{-1}(0)$ as follows: Consider a point $z \in F^{-1}(0)$ such that the contact plane ξ_z intersects $T_z F^{-1}(0)$ transversally. Then it is easy to see that a complete solution exists around z by integrating the line field $\xi \cap TF^{-1}(0)$. We call points where transversality fails *contact singular points* and denote the set of such points by $\Sigma_{\mathrm{c}} = \Sigma_{\mathrm{c}}(F)$.

We call Σ_{c} the *contact singular set* of $F^{-1}(0)$. It is easy to check that the contact singular set is given by

$$\Sigma_{\mathrm{c}}(F) = \{z \in J^2(\mathbb{R},\mathbb{R}) \mid F(z) = 0, F_x(z) + pF_y(z) + qF_p(z) = 0, F_q(z) = 0\}.$$

From the definition of singular solutions, it is easy to see that a geometric solution $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$ is a singular solution only if it is contained in $\Sigma_{\mathrm{c}}(F)$. We also consider the subset $\Delta = \Delta(F) \subset \Sigma_{\mathrm{c}}$ of points $z \in \Sigma_{\mathrm{c}}$ such that $T_zF^{-1}(0)$ coincides with the kernel of $\alpha_1(z)$. Explicitly, it is given by $\Delta = \{z \in \Sigma_{\mathrm{c}} \mid F_p(z) = 0\}$.

On the other hand, $J^2(\mathbb{R},\mathbb{R})$ has two natural projections. Let $\pi_1 : J^2(\mathbb{R},\mathbb{R}) \to J^1(\mathbb{R},\mathbb{R})$ and $\pi : J^2(\mathbb{R},\mathbb{R}) \to \mathbb{R}^2$ be the canonical projections given by $\pi_1(x, y, p, q) = (x, y, p)$ and $\pi(x, y, p, q) = (x, y)$. We call a point z_0 a π_1-*singular point* of $F = 0$ if $F = F_q = 0$ at z_0, and a π-*singular point* of $F = 0$ if $F = F_p = F_q = 0$ at z_0. We denote the sets of all π_1-singular points and of all π-singular points by $\Sigma_{\pi_1}(F)$ and $\Sigma_\pi(F)$ respectively. We define

$$\Sigma_* = \Sigma_*(F) = \{z \in J^2(\mathbb{R},\mathbb{R}) \mid F(z) = 0,\ F_x(z) + pF_y(z) + qF_p(z) = 0\}.$$

Observe that Σ_c is a subset of both Σ_{π_1} and Σ_*, and Δ is a subset of Σ_π.

We now assume that $F = 0$ is completely integrable at z_0 and Σ_{c} is a 2-dimensional submanifold around z_0. Then we say that an immersion germ $\Phi : (\mathbb{R} \times \mathbb{R}, (t_0, a_0)) \to (\Sigma_{\mathrm{c}}, z_0)$ such that for each $a \in (\mathbb{R}, a_0), \Phi(\cdot, a) : (\mathbb{R}, t_0) \to \Sigma_{\mathrm{c}}$ is a singular solution, is a *complete singular solution* at z_0.

Also, if ξ_z intersects $T_z\Sigma_{\mathrm{c}}$ transversally in $T_zF^{-1}(0)$ then integrating the line field $\xi \cap T\Sigma_{\mathrm{c}}$ yields a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}, (t_0, a_0)) \to (\Sigma_{\mathrm{c}}, z)$ such that $\Phi(\cdot, a)$ is a geometric solution for each $a \in (\mathbb{R}, a_0)$; if it is not contained in the complete solution, then it is a complete singular solution. Here we say that *Φ is not contained in the complete solution* if any representative of $\Phi(\cdot, a)$ is never contained in a leaf of the complete solution for each $a \in (\mathbb{R}, a_0)$. For a point where transversality does not hold, a complete singular solution need not exist. We call such points *second order contact singular points* and denote the set of such points by $\Sigma_{\mathrm{cc}} = \Sigma_{\mathrm{cc}}(F)$.

It is well-known that for a second order ordinary differential equation $F(x, y, p, q) = 0$, if the second derivative can be written as a single-valued function of x, y and p, so that $F = 0$ is given by an explicit equation $q = G(x, y, p)$ for some smooth function G, then at least locally there exists a (smooth) complete solution around a point on the equation hypersurface and there is no singular solution.

In [1] conditions for existence of a complete solution and a complete singular solution of implicit second order ordinary differential equations were given under a regularity condition.

THEOREM 1.1 ([1, Theorems 1.1, 1.2 and 1.3]). *Suppose that* 0 *is a regular value of* $F_q|_{F^{-1}(0)}$.

(1) $F = 0$ *is completely integrable at* z_0 *if and only if* $z_0 \notin \Sigma_{\rm c}$ *or* $\Sigma_{\rm c}$ *is a* 2*-dimensional manifold around* z_0.

(2) *Let* $F = 0$ *be completely integrable.*

  (i) *The leaves of the complete solution which meet* $\Sigma_{\rm c}$ *away from* Δ *intersect* $\Sigma_{\rm c}$ *transversally.*

  (ii) *The leaves of the complete solution which meet* Δ *meet* $\Sigma_{\rm c}$ *tangentially.*

(3) *Let* $F = 0$ *be completely integrable and* $\Sigma_{\rm c} \neq \emptyset$.

  (i) $F = 0$ *admits a complete singular solution around* $z_0 \in \Sigma_{\rm c}$ *if and only if either* $z_0 \notin \Sigma_{\rm cc}$, *or* $\Sigma_{\rm cc}$ *is a* 1*-dimensional manifold around* z_0.

  (ii) *Suppose that* $F = 0$ *admits a complete singular solution. Then each leaf of the complete singular solution intersects* $\Sigma_{\rm cc}$ *transversally.*

Here we give an example illustrating the notions of a complete solution and a complete singular solution. In [11], we discussed second order ordinary classical Clairaut equations given by

$$y = xp - \tfrac{1}{2}x^2 q + \varphi(q),$$

where $p = dy/dx$, $q = d^2y/dx^2$ and φ is a smooth function. This is an example of a (smooth) completely integrable second order ordinary differential equation (see [8–12] and Example 4.1 for details). We now consider the following case.

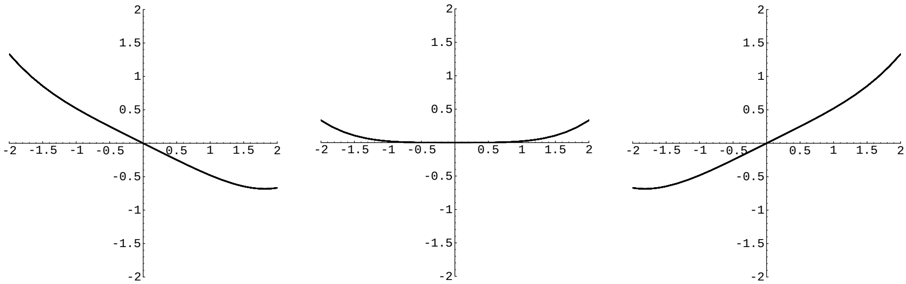
EXAMPLE 1.2 (Second order classical Clairaut equations with $\varphi(q) = q^2$). Let $F(x, y, p, q) = xp - \frac{1}{2}x^2 q + q^2 - y$. Since 0 is a regular value of $F_q|_{F^{-1}(0)}$, we can apply the results in Theorem 1.1 to this equation. In this case, $F_x + pF_y + qF_p \equiv 0$ and $F_q = -\frac{1}{2}x^2 + 2q$. Therefore the contact singular set is given by $\Sigma_{\rm c} = \big\{(x, y, p, q) \mid y = xp - \frac{1}{16}x^4,\ q = \frac{1}{4}x^2\big\}$. Since $\Sigma_{\rm c}$ is a 2-dimensional manifold and $\Sigma_{\rm cc} = \emptyset$, there exist a complete solution $\Gamma : \mathbb{R} \times \mathbb{R}^2 \to F^{-1}(0)$ and a complete singular solution $\Phi : \mathbb{R} \times \mathbb{R} \to \Sigma_{\rm c}$ which are given by

$$\Gamma(t, r, s) = \big(t, rt + \tfrac{1}{2}st^2 + s^2, r + st, s\big),$$
$$\Phi(t, a) = \big(t, \tfrac{1}{48}t^4 + at, \tfrac{1}{12}t^3 + a, \tfrac{1}{4}t^2\big).$$

We have $\Delta = \{(x, y, p, q) \mid x = y = q = 0\}$. Hence the leaves of the complete solution intersect $\Sigma_{\rm c}$ transversally away from Δ and are tangent to $\Sigma_{\rm c}$ at points in Δ. Moreover, for each fixed $a \in (\mathbb{R}, 0)$, $\pi \circ \Phi(t, a) = \big(t, \frac{1}{48}t^4 + at\big)$ is the envelope of the two subfamilies of the complete solution which are given by

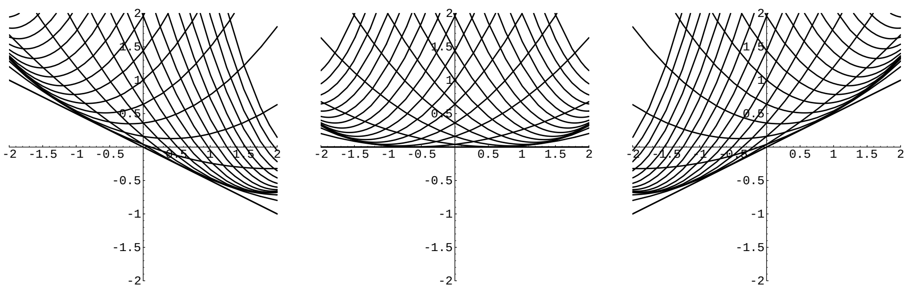
$$y = \left(\tfrac{4}{3}s\sqrt{s} + a\right)t + \tfrac{1}{2}st^2 + s^2, \quad y = \left(-\tfrac{4}{3}s\sqrt{s} + a\right)t + \tfrac{1}{2}st^2 + s^2.$$

See Figures 1 and 2.



$a < 0$ $a = 0$ $a > 0$

Fig. 1. Projection of the leaves of the complete singular solution to the (x, y)-plane



$a < 0$ $a = 0$ $a > 0$

Fig. 2. The two subfamilies of the complete solution

In this paper, we consider conditions for existence of a complete solution and a complete singular solution of implicit second order ordinary differential equations dropping the condition in Theorem 1.1. In §2, we give necessary and sufficient conditions for existence of complete solutions and smooth complete solutions. We show that $F = 0$ is completely integrable at z_0 if and only if $F = 0$ is either of Clairaut type or of first order type at z_0 (cf. Proposition 2.2). The Clairaut type has already appeared in [11] as a necessary and sufficient condition for existence of a smooth complete solution. Moreover, we consider properties of completely integrable equations. In §3, we give conditions for existence of a complete singular solution for Clairaut type and first order type equations. In §4, we give some examples.

All map germs and manifolds considered are differentiable of class C^∞.

2. Complete solutions and smooth complete solutions. In this section, we consider conditions for existence of a complete solution of implicit second order ordinary differential equations. We apply the following lemma.

LEMMA 2.1 ([1, Lemma 3.1]). *Let $F = 0$ be a second order ordinary differential equation. The equation $F = 0$ is completely integrable at $z_0 \in F^{-1}(0)$ if and only if there exist function germs $\alpha, \beta : (F^{-1}(0), z_0) \to \mathbb{R}$, which do not vanish simultaneously, such that*

$$\alpha \cdot (F_x + pF_y + qF_p)|_{F^{-1}(0)} + \beta \cdot F_q|_{F^{-1}(0)} \equiv 0.$$

We say that an equation $F = 0$ is of *second order Clairaut type* (for short, *Clairaut type*) at z_0 if there exist smooth function germs $A, B : (J^2(\mathbb{R}, \mathbb{R}), z_0) \to \mathbb{R}$ such that

$$F_x + p \cdot F_y + q \cdot F_p = A \cdot F + B \cdot F_q,$$

and of *first order type* at z_0 if there exist smooth function germs $A', B' : (J^2(\mathbb{R}, \mathbb{R}), z_0) \to \mathbb{R}$ such that

$$F_q = A' \cdot F + B' \cdot (F_x + p \cdot F_y + q \cdot F_p).$$

We showed in [11, Theorem 3.1] that an equation $F = 0$ has a smooth complete solution at z_0 if and only if it is of Clairaut type at z_0. We remark that if $F = 0$ is of Clairaut type then $\Sigma_{\mathrm{c}} = \Sigma_{\pi_1}$, $\Delta = \Sigma_\pi$, and if it is of first order type then $\Sigma_{\mathrm{c}} = \Sigma_*$.

The following is a consequence of Lemma 2.1 and the fact that $F = 0$ is regular.

PROPOSITION 2.2. *$F = 0$ is completely integrable at z_0 if and only if $F = 0$ is either of Clairaut type or of first order type at z_0.*

The following results correspond to Theorem 1.1(1),(2) and [11, Proposition 3.2].

LEMMA 2.3. *Suppose that 0 is a regular value of $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$. Then $F = 0$ is completely integrable at z_0 if and only if $z_0 \notin \Sigma_{\mathrm{c}}$ or Σ_{c} is a 2-dimensional manifold around z_0. Moreover, if $z_0 \in \Sigma_{\mathrm{c}}$ then $F = 0$ is of first order type at z_0.*

Proof. Let $F = 0$ be completely integrable at z_0 and $z_0 \in \Sigma_{\mathrm{c}}$. By Lemma 2.1, there exist function germs $\alpha, \beta : (F^{-1}(0), z_0) \to \mathbb{R}$, which do not vanish simultaneously, such that

$$\alpha \cdot (F_x + pF_y + qF_p)|_{F^{-1}(0)} + \beta \cdot F_q|_{F^{-1}(0)} \equiv 0.$$

If $\beta(z_0) = 0$, then $\alpha(z_0) \neq 0$ and $\nabla(F_x + pF_y + qF_p)|_{F^{-1}(0)} = 0$ at z_0. This contradicts the fact that 0 is a regular value of $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$. Hence $\beta(z_0) \neq 0$. It follows that Σ_{c} is a 2-dimensional manifold around z_0 and $F = 0$ is of first order type at z_0.

Conversely, suppose that Σ_{c} is a 2-dimensional manifold around z_0. By the assumption, the set Σ_* is also a 2-dimensional manifold around z_0. Hence $\Sigma_{\mathrm{c}} = \Sigma_*$ around z_0, as set germs. Since 0 is a regular value of $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$, $F = 0$ is of first order type at z_0. ∎

PROPOSITION 2.4. *Suppose that* 0 *is a regular value of the restriction* $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$ *and* $F = 0$ *is completely integrable. Then*:

(i) *The leaves of the complete solution which meet* Σ_{c} *away from* Δ *intersect* Σ_{c} *transversally.*
(ii) *The leaves of the complete solution which meet* Δ *are tangent to* Σ_{c}.

The proof is analogous to that of Theorem 1.2 in [1], so it is omitted.

By Theorem 1.1, Lemma 2.3 and [1, Proposition 3.5], we have the following corollary.

COROLLARY 2.5. *Suppose that* $z_0 \in \Sigma_{\mathrm{c}} \setminus \Delta$. *Then* $F = 0$ *is completely integrable at* z_0 *if and only if* Σ_{c} *is a* 2*-dimensional manifold around* z_0 *and* z_0 *is a regular point of either* $F_q|_{F^{-1}(0)}$ *or* $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$.

Now suppose that Σ_{c} is a 2-dimensional manifold around z_0. Then we can consider the second order contact singular set Σ_{cc}. If $z_0 \notin \Sigma_{\mathrm{cc}}$, then we have a condition for existence of a complete solution at z_0.

PROPOSITION 2.6. *Suppose that* Σ_{c} *is a* 2*-dimensional manifold around* z_0 *and* $z_0 \notin \Sigma_{\mathrm{cc}}$. *Then* Σ_{c} *coincides with either* Σ_{π_1} *or* Σ_* *around* z_0 *if and only if* $F = 0$ *is completely integrable at* z_0. *More precisely*,

(1) $\Sigma_{\mathrm{c}} = \Sigma_{\pi_1}$ *around* z_0 *if and only if* $F = 0$ *is of Clairaut type at* z_0.
(2) $\Sigma_{\mathrm{c}} = \Sigma_*$ *around* z_0 *if and only if* $F = 0$ *is of first order type at* z_0.

Proof. (1) Suppose that $\Sigma_{\mathrm{c}} = \Sigma_{\pi_1}$ around z_0. Since $\nabla F(z_0) \neq 0$ and $z_0 \in \Sigma_{\mathrm{c}}$, $F_y(z_0) \neq 0$ or $F_p(z_0) \neq 0$. If $F_y(z_0) \neq 0$, by the implicit function theorem, there exists a smooth function $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^3$, such that in a neighborhood of z_0, $(x, y, p, q) \in F^{-1}(0)$ if and only if $f(x, p, q) - y = 0$. Thus we may assume without loss of generality that

$$F(x, y, p, q) = f(x, p, q) - y.$$

Define

$$\phi : U \to F^{-1}(0), \quad (x, p, q) \mapsto (x, f(x, p, q), p, q), \quad \text{and} \quad u_0 = \phi^{-1}(z_0).$$

It follows that $\phi^{-1}(\Sigma_{\mathrm{c}}) = {f_q}^{-1}(0)$. From the definition of Σ_{cc} and $z_0 \notin \Sigma_{\mathrm{cc}}$, either $(f_{qx} + q \cdot f_{qp})(u_0) \neq 0$ or $f_{qq}(u_0) \neq 0$. Therefore we have $\nabla f_q \neq 0$ at u_0.

On the other hand, if $F_p(z_0) \neq 0$, again by the implicit function theorem, there exists a smooth function $g : V \to \mathbb{R}$, where V is an open set in $\mathbb{R}^3$, such that in a neighborhood of z_0, $(x, y, p, q) \in F^{-1}(0)$ if and only if

$g(x,y,q)-p=0$. Thus we may assume without loss of generality that

$$F(x,y,p,q)=g(x,y,q)-p.$$

Define

$$\psi : V \to F^{-1}(0), \quad (x,y,q)\mapsto (x,y,g(x,y,q),q), \quad \text{and} \quad v_0=\psi^{-1}(z_0).$$

Then $\psi^{-1}(\Sigma_{\mathrm{c}})={g_q}^{-1}(0)$. By definition of Σ_{cc} and $z_0\notin\Sigma_{\mathrm{cc}}$, we have either $(g_{qx}+g\cdot g_{qp})(v_0)\neq 0$ or $g_{qq}(v_0)\neq 0$. Hence also $\nabla g_q\neq 0$ at v_0. In both cases, z_0 is a regular point of $F_q|_{F^{-1}(0)}$. It follows that $F=0$ is completely integrable at z_0 by Theorem 1.1(1). We conclude by Proposition 3.2 in [11] that $F=0$ is of Clairaut type at z_0.

(2) The argument is similar to that in case (1). The condition $z_0\notin\Sigma_{\mathrm{cc}}$ guarantees that z_0 is a regular point of $(F_x+pF_y+qF_p)|_{F^{-1}(0)}$. By Lemma 2.3, $F=0$ is of first order type at z_0. ∎

Below, if $F_y(z_0)\neq 0$ or $F_p(z_0)\neq 0$, we keep the respective assumptions and notations of the above proof.

For a completely integrable equation, we have the following property.

PROPOSITION 2.7. *Let $F=0$ be completely integrable at z_0 and Σ_{c} be a 2-dimensional manifold around z_0. Then $\Sigma_{\mathrm{cc}}\subset\Delta$.*

Proof. By Proposition 2.2, we may assume that $F=0$ is either of Clairaut type or of first order type at z_0. If $F=0$ is of Clairaut type at z_0, then we have already shown that $\Sigma_{\mathrm{cc}}\subset\Delta$ in [11, Proposition 3.3]. Therefore we may assume that $F=0$ is of first order type at z_0. By definition, there exist smooth function germs A and B at z_0 such that $F_q=A\cdot F+B\cdot(F_x+pF_y+qF_p)$. We assume that $\Sigma_{\mathrm{cc}}\neq\emptyset$ and $z_0\in\Sigma_{\mathrm{cc}}$.

Suppose that $F_y(z_0)\neq 0$. By the definition of Σ_{cc}, we have

$$(f_x-p+qf_p)_x(u)+q(f_x-p+qf_p)_p(u)=0, \qquad (f_x-p+qf_p)_q(u)=0,$$

where $u=(x,p,q)\in\phi^{-1}(\Sigma_{\mathrm{cc}})$. Differentiating $f_q=\alpha\cdot(f_x-p+qf_p)$ with respect to x and p, we have

$$f_{qx}=\alpha_x\cdot(f_x-p+qf_p)+\alpha\cdot(f_x-p+qf_p)_x,$$
$$f_{qp}=\alpha_p\cdot(f_x-p+qf_p)+\alpha\cdot(f_x-p+qf_p)_p,$$

where $\alpha=B\circ\phi$. For any $z=(x,y,p,q)\in\Sigma_{\mathrm{cc}}$, if we compare the above equalities, then $f_p(u)=0$ and hence $z\in\Delta$.

Suppose that $F_p(z_0)\neq 0$. By the definition of Σ_{cc}, we have

$$(g_x+g\cdot g_p-q)_x(v)+g(v)\cdot(g_x+g\cdot g_p-q)_p(v)=0,\ (g_x+g\cdot g_p-q)_q(v)=0,$$

where $v=(x,y,q)\in\psi^{-1}(\Sigma_{\mathrm{cc}})$. Differentiating $g_q=\beta\cdot(g_x+g\cdot g_p-q)$ with respect to x and p, we have

$$g_{qx}=\beta_x\cdot(g_x+g\cdot g_p-q)+\beta\cdot(g_x+g\cdot g_p-q)_x,$$
$$g_{qp}=\beta_p\cdot(g_x+g\cdot g_p-q)+\beta\cdot(g_x+g\cdot g_p-q)_p,$$

where $\beta = B \circ \psi$. However, this case does not occur for any $z = (x, y, p, q) \in \Sigma_{cc}$. ■

3. Complete singular solutions. In this section, we consider the uniqueness of complete singular solutions, analogously to the uniqueness results in [6, 11], and give conditions for existence of a complete singular solution. Throughout this section, we assume that $F = 0$ is completely integrable at z_0 and Σ_c is a 2-dimensional manifold around z_0. Moreover, we use the notation of the previous section.

The uniqueness of the complete singular solution is dealt with in the following result.

PROPOSITION 3.1. *Let* $\Phi_1 : (\mathbb{R} \times \mathbb{R}, (t_1, a_1)) \to (\Sigma_c, z_0)$ *and* $\Phi_2 : (\mathbb{R} \times \mathbb{R}, (t_2, a_2)) \to (\Sigma_c, z_0)$ *be complete singular solutions around* z_0. *Then there exists a diffeomorphism germ* $\Psi : (\mathbb{R} \times \mathbb{R}, (t_1, a_1)) \to (\mathbb{R} \times \mathbb{R}, (t_2, a_2))$ *of the form* $\Psi(t, a) = (\psi_1(t, a), \psi_2(a))$ *such that* $\Phi_2 \circ \Psi = \Phi_1$.

Proof. Suppose that the assertion does not hold. Since the complete singular solution is a one-parameter family of geometric solutions in Σ_c, there exists a point $z_1 \in (\Sigma_c, z_0)$ such that $\Phi_{1,c_1} = \Phi_1(\cdot, c_1)$ and $\Phi_{2,c_2} = \Phi_2(\cdot, c_2)$ are transversal at z_1. Then we can construct an immersion germ $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \to (\Sigma_c, z_1)$ which satisfies the conditions

$$\frac{\partial y}{\partial a}(t,a) = p(t,a)\,\frac{\partial x}{\partial a}(t,a), \qquad \frac{\partial p}{\partial a}(t,a) = q(t,a)\,\frac{\partial x}{\partial a}(t,a),$$
$$\frac{\partial y}{\partial t}(t,a) = p(t,a)\,\frac{\partial x}{\partial t}(t,a), \qquad \frac{\partial p}{\partial t}(t,a) = q(t,a)\,\frac{\partial x}{\partial t}(t,a),$$

where $\Phi(t, a) = (x(t, a), y(t, a), p(t, a), q(t, a))$. If we calculate the second order partial derivatives of the above equalities, we get

$$\frac{\partial^2 p}{\partial t \partial a} = \frac{\partial q}{\partial t} \cdot \frac{\partial x}{\partial a} + q \cdot \frac{\partial^2 x}{\partial t \partial a}, \qquad \frac{\partial^2 p}{\partial a \partial t} = \frac{\partial q}{\partial a} \cdot \frac{\partial x}{\partial t} + q \cdot \frac{\partial^2 x}{\partial a \partial t}.$$

Therefore we obtain the equality $(\partial q/\partial t) \cdot (\partial x/\partial a) = (\partial q/\partial a) \cdot (\partial x/\partial t)$. This contradicts the fact that Φ is an immersion germ. ■

3.1. *Clairaut type equations.* By Proposition 2.2, we may assume that $F = 0$ is either of Clairaut type or of first order type at z_0. First we consider the case when $F = 0$ is of Clairaut type at $z_0 \in \Sigma_c$. Then $F = 0$ satisfies either $F_y(z_0) \neq 0$ or $F_p(z_0) \neq 0$. If $F_p(z_0) \neq 0$, then $z_0 \notin \Delta$ and hence $z_0 \notin \Sigma_{cc}$ by Proposition 2.7. Thus there exists a complete singular solution around z_0 by Corollary 2.5 and Theorem 1.1(2). On the other hand, if $F_y(z_0) \neq 0$, we have the following results.

LEMMA 3.2. *Suppose that* $\Sigma_{cc} \neq \Sigma_c$. *If there exists a complete singular solution* $\Phi : (\mathbb{R} \times \mathbb{R}, (t_0, a_0)) \to (\Sigma_c, z_0)$ *then there exist function germs*

$\lambda, \mu : (\phi^{-1}(\Sigma_{\mathrm{c}}), u_0) \to \mathbb{R}$, *which do not vanish simultaneously*, *such that*

$$(*) \qquad \lambda \cdot (f_{qx} + q f_{qp})|_{\phi^{-1}(\Sigma_{\mathrm{c}})} + \mu \cdot f_{qq}|_{\phi^{-1}(\Sigma_{\mathrm{c}})} \equiv 0.$$

Conversely, *if there exist* λ, μ *as above*, *then they yield a diffeomorphism germ* Φ *such that* $\Phi(\cdot, a)$ *is a geometric solution for each* $a \in (\mathbb{R}, a_0)$; *if* Φ *is not contained in the complete solution*, *then it is a complete singular solution.*

Proof. Suppose that $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \to (\Sigma_{\mathrm{c}}, z_0)$ is a complete singular solution around z_0. Differentiating $\phi^{-1} \circ \Phi$ with respect to t yields a vector field $X : \phi^{-1}(\Sigma_{\mathrm{c}}) \to T\phi^{-1}(\Sigma_{\mathrm{c}})$ given by $X(\phi^{-1} \circ \Phi(t, a)) = (\phi^{-1} \circ \Phi)_t(t, a)$. By definition of a complete singular solution and of the smooth map ϕ, X has the form $X = (\lambda, q\lambda, \mu)$ for function germs $\lambda, \mu : (\phi^{-1}(\Sigma_{\mathrm{c}}), u_0) \to \mathbb{R}$ which do not vanish simultaneously. Since $F = 0$ is of Clairaut type at z_0, $\phi^{-1}(\Sigma_{\mathrm{c}}) = f_q^{-1}(0)$. Then $X(u)$ lies in $T_u f_q^{-1}(0)$. It follows that the identity $(*)$ holds.

Reversing the above argument yields a diffeomorphism germ Φ : $(\mathbb{R} \times \mathbb{R}, (t_0, a_0)) \to (\Sigma_{\mathrm{c}}, z_0)$ such that $\Phi(\cdot, a)$ is a geometric solution for each $a \in (\mathbb{R}, a_0)$. If Φ is not contained in the complete solution, that is, any representative of $\Phi(\cdot, a)$ is never contained in a leaf of the complete solution for each $a \in (\mathbb{R}, a_0)$, then Φ is a complete singular solution around z_0. ∎

COROLLARY 3.3. *Under the condition* $(*)$, *we have the following*:

(1) *If* $\Sigma_{\mathrm{cc}} \neq \Sigma_{\mathrm{c}}$, $\lambda(u_0) = 0$ *and* $\mu(u_0) \neq 0$, *then there exists a complete singular solution around* z_0.
(2) *If* Δ *is a* 1*-dimensional manifold* (*generally*, *if* $\Delta \neq \Sigma_{\mathrm{c}}$) *around* z_0, *then there exists a complete singular solution around* z_0.

Proof. (1) Since $F = 0$ is of Clairaut type, we conclude that the diffeomorphism Φ is not contained in the complete solution by the form of the vector field of X in Lemma 3.2.

(2) Suppose that $z \in \Sigma_{\mathrm{c}} \setminus \Delta$. By Corollary 2.5 and Theorem 1.1(2), the leaves of the complete solution which meet Σ_{c} away from Δ intersect Σ_{c} transversally. Hence the diffeomorphism Φ is not contained in the complete solution. ∎

THEOREM 3.4. *Suppose that* Σ_{cc} *is a* 1*-dimensional manifold around* z_0 *and that* z_0 *is a regular point of* $F_p|_{\Sigma_{\mathrm{c}}}$. *Then there exists a complete singular solution around* z_0.

Proof. Since $F = 0$ is of Clairaut type at z_0, there exists a function germ $\beta : (F^{-1}(0), z_0) \to \mathbb{R}$ such that $f_x - p + q f_p = \beta \cdot f_q$. By definition of Σ_{c}, Δ and Σ_{cc}, we have

$$\phi^{-1}(\Sigma_{\mathrm{c}}) = f_q^{-1}(0), \quad \phi^{-1}(\Delta) = \{(x, p, q) \in \phi^{-1}(\Sigma_{\mathrm{c}}) \mid f_p(x, p, q) = 0\}$$

and

$$\phi^{-1}(\Sigma_{\rm cc}) = \{(x,p,q) \in \phi^{-1}(\Sigma_{\rm c}) \mid f_{xq}(x,p,q) + q f_{pq}(x,p,q) = f_{qq}(x,p,q) = 0\}.$$

Since z_0 is a regular point of $F_p|_{\Sigma_{\rm c}}$, $\phi^{-1}(\Delta)$ is also a 1-dimensional manifold around u_0. By Proposition 2.7, $\phi^{-1}(\Sigma_{\rm cc}) \subset \phi^{-1}(\Delta)$. It follows that $\phi^{-1}(\Sigma_{\rm cc}) = \phi^{-1}(\Delta)$ around u_0 and hence there exist function germs $k, \ell : (\phi^{-1}(\Sigma_{\rm c}), u_0) \to \mathbb{R}$ such that

$$(f_{xq} + q f_{pq})|_{f_q{}^{-1}(0)} = k \cdot f_p|_{f_q{}^{-1}(0)}, \qquad f_{qq}|_{f_q{}^{-1}(0)} = \ell \cdot f_p|_{f_q{}^{-1}(0)}.$$

On the other hand, differentiating $f_x - p + q f_p = \beta \cdot f_q$ with respect to q, we have

$$f_{xq} + f_p + q f_{pq} = \beta_q \cdot f_q + \beta \cdot f_{qq}.$$

Restricting the equality to $f_q^{-1}(0)$, we have $(k - \beta \cdot \ell + 1) \cdot f_p|_{f_q^{-1}(0)} = 0$. Since z_0 is a regular point of $f_p|_{f_q^{-1}(0)}$, we have $k - \beta \cdot \ell + 1 = 0$ at u_0. It follows that either $k \neq 0$ or $\ell \neq 0$ at u_0. By Lemma 3.2 and Corollary 3.3, there exists a complete singular solution around z_0. ∎

3.2. *First order type equations.* Let $F = 0$ be of first order type at $z_0 \in \Sigma_{\rm c}$, so either $F_y(z_0) \neq 0$ or $F_p(z_0) \neq 0$. By Proposition 2.4 and Corollary 2.5, if $F_p(z_0) \neq 0$, then there exists a complete singular solution around z_0. If $F_y(z_0) \neq 0$, we can prove the following results using the same arguments in the proofs of Lemma 3.2, Corollary 3.3 and Theorem 3.4.

LEMMA 3.5. *Suppose that $\Sigma_{\rm cc} \neq \Sigma_{\rm c}$. If there exists a complete singular solution $\Phi : (\mathbb{R} \times \mathbb{R}, (t_0, a_0)) \to (\Sigma_{\rm c}, z_0)$ then there exist function germs $\lambda, \mu : (\phi^{-1}(\Sigma_{\rm c}), u_0) \to \mathbb{R}$, which do not vanish simultaneously, such that*

$$(**) \qquad \lambda \cdot ((f_x - p + q f_p)_x + q(f_x - p + q f_p)_p)|_{\phi^{-1}(\Sigma_{\rm c})} + \mu \cdot (f_x - p + q f_p)_q|_{\phi^{-1}(\Sigma_{\rm c})} \equiv 0.$$

Conversely, if there exist λ, μ as above, then they yield a diffeomorphism germ Φ such that $\Phi(\cdot, a)$ is a geometric solution for each $a \in (\mathbb{R}, a_0)$; if Φ is not contained in the complete solution, then it is a complete singular solution.

COROLLARY 3.6. *Under the condition $(**)$, we have the following:*

(1) *If $\Sigma_{\rm cc} \neq \Sigma_{\rm c}$, $\lambda(u_0) \neq 0$ and $\mu(u_0) = 0$, then there exists a complete singular solution around z_0.*
(2) *If Δ is a 1-dimensional manifold (generally, if $\Delta \neq \Sigma_{\rm c}$) around z_0, then there exists a complete singular solution around z_0.*

THEOREM 3.7. *Suppose that $\Sigma_{\rm cc}$ is a 1-dimensional manifold around z_0 and that z_0 is a regular point of $F_p|_{\Sigma_{\rm c}}$. Then there exists a complete singular solution around z_0.*

The following result corresponds to Theorem 1.1(3). Its proof is similar to that of Theorem 1.3 in [1].

PROPOSITION 3.8. *Suppose that* 0 *is a regular value of the restriction* $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$ *and* $F = 0$ *is completely integrable at* $z_0 \in \Sigma_{\mathrm{c}}$.

(i) *If* $F = 0$ *admits a complete singular solution* $\Phi : (\mathbb{R} \times \mathbb{R}, (t_0, a_0)) \to (\Sigma_{\mathrm{c}}, z_0)$ *then* $z_0 \notin \Sigma_{\mathrm{cc}}$ *or* Σ_{cc} *is a* 1*-dimensional manifold around* z_0. *Conversely, if* $z_0 \notin \Sigma_{\mathrm{cc}}$ *or* Σ_{cc} *is a* 1*-dimensional manifold around* z_0, *then they yield a diffeomorphism germ* Φ *such that* $\Phi(\cdot, a)$ *is a geometric solution for each* $a \in (\mathbb{R}, a_0)$; *if* Φ *is not contained in the complete solution, then it is a complete singular solution.*

(ii) *Suppose that* $F = 0$ *admits a complete singular solution. Then each leaf of the complete singular solution intersects* Σ_{cc} *transversally.*

We remark that there is an important difference between the case where 0 is a regular value of $F_q|_{F^{-1}(0)}$ and the case where it is a regular value of $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$. Namely, if 0 is a regular value of $F_q|_{F^{-1}(0)}$ and $z_0 \in \Delta$, then Δ is a 1-dimensional manifold around z_0 by Proposition 3.6 in [1]. However, Δ is not necessarily a 1-dimensional manifold even if 0 is a regular value of $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$ (cf. Example 4.3 in §4).

4. Examples. In this section we give examples of second order Clairaut type and first order type equations.

EXAMPLE 4.1 (Second order classical Clairaut equations). Consider the second order classical Clairaut equations $F(x, y, p, q) = xp - \frac{1}{2}x^2 q + \varphi(q) - y$, where φ is a smooth function. Suppose that $\varphi'(q) \geq 0$ and $\varphi''(0) = 0$ (cf. Example 1.2). In this case, $F_x + pF_y + qF_p \equiv 0$ and $F_q = -\frac{1}{2}x^2 + \varphi'(q)$. Therefore $F = 0$ is of Clairaut type and hence there exists a (smooth) complete solution $\Gamma : \mathbb{R} \times \mathbb{R}^2 \to F^{-1}(0)$ which is given by $\Gamma(t, r, s) = \big(t, \frac{1}{2}st^2 + rt + \varphi(s), st + r, s\big)$. The contact singular set is

$$\Sigma_{\mathrm{c}}^{\pm} = \{(x, y, p, q) \mid x = \pm\sqrt{2}\,\varphi'(q)^{1/2},\ y = \pm\sqrt{2}\,\varphi'(q)^{1/2} p - \varphi'(q) q + \varphi(q)\}.$$

We have $\Delta = \{(x, y, p, q) \mid x = 0,\ \varphi'(q) = 0,\ y = \varphi(q)\}$ and $\Sigma_{\mathrm{cc}} = \{(x, y, p, q) \mid x = 0,\ \varphi'(q) = 0,\ \varphi''(q) = 0,\ y = \varphi(q)\}$. By Lemma 3.2, if there exists a complete singular solution, then there exist functions $\lambda, \mu : \phi^{-1}(\Sigma_{\mathrm{c}}) \to \mathbb{R}$, not vanishing simultaneously, such that $\lambda \cdot (\mp\sqrt{2}\,\varphi'(q)^{1/2}) + \mu \cdot \varphi''(q) \equiv 0$. We may assume that λ and μ only depend on q, and $\mu \neq 0$. Then by direct calculations, we have $dx = (\lambda/\mu) \cdot dq$, and a complete singular solution $\Phi^{\pm} : \mathbb{R} \times \mathbb{R} \to \Sigma_{\mathrm{c}}^{\pm}$ is given by

$$\Phi^{\pm}(t, a) = \Big(\pm\sqrt{2}\,\varphi'(t)^{1/2}, t\varphi'(t) - 2\varphi'(t)^{1/2}\int \varphi'(t)^{1/2}\,dt \pm \sqrt{2}\,\varphi'(t)^{1/2} a + \varphi(t),$$
$$\pm\sqrt{2}\Big(t\varphi'(t)^{1/2} - \int \varphi'(t)^{1/2}\,dt\Big) + t, a\Big).$$

In particular, if $\varphi(q)$ is a constant c, then $F(x,y,p,q) = -y + xp - \frac{1}{2}x^2q + c$ and $\Sigma_{\mathrm{c}} = \Delta = \Sigma_{\mathrm{cc}} = \{(x,y,p,q) \mid x = 0,\ y = c\}$. Nevertheless, there exists a complete singular solution $\Phi(t,a) = (0,c,t,a)$.

Example 4.2. Let $F(x,y,p,q) = -y + pq^2 - \frac{2}{5}q^5$. In this case, $F_x + pF_y + qF_p = -p + q^3$ and $F_q = -2q(-p+q^3)$. Hence $F = 0$ is of first order type and 0 is a regular value of $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$. Moreover,

$$\Sigma_{\mathrm{c}} = \big\{(x,y,p,q) \bigm| p = q^3,\ y = \tfrac{3}{5}q^5\big\},$$
$$\Delta = \Sigma_{\mathrm{cc}} = \big\{(x,y,p,q) \mid y = p = q = 0\big\}.$$

Thus Σ_{c} is a 2-dimensional manifold, and Δ and Σ_{cc} are 1-dimensional manifolds. By Lemma 2.3, Proposition 3.8 and Corollary 3.6, there exist a complete solution and a complete singular solution. Indeed, the complete solution $\Gamma : \mathbb{R} \times \mathbb{R}^2 \to F^{-1}(0)$ and the complete singular solution $\Phi : \mathbb{R} \times \mathbb{R} \to \Sigma_{\mathrm{c}}$ are given by

$$\Gamma(t,r,s) = \big(t^2 + r, \tfrac{4}{15}t^5 + st^2, \tfrac{2}{3}t^3 + s, t\big), \qquad \Phi(t,a) = \big(\tfrac{3}{2}t^2 + a, \tfrac{3}{5}t^5, t^3, t\big).$$

Moreover, by Propositions 2.4 and 3.8, the leaves of the complete solution intersect Σ_{c} transversally away from Δ and are tangent to Σ_{c} at points in Δ, and each leaf of the complete singular solution intersects Σ_{cc} transversally.

Example 4.3. Let $F(x,y,p,q) = -y - \frac{1}{2}p^2q$. In this case, $F_x + pF_y + qF_p = -p(1+q^2)$ and $F_q = -\frac{1}{2}p^2$. Hence $F = 0$ is of first order type and 0 is a regular value of $(F_x + pF_y + qF_p)|_{F^{-1}(0)}$. Moreover,

$$\Sigma_{\mathrm{c}} = \Delta = \{(x,y,p,q) \mid y = p = 0\},\ \Sigma_{\mathrm{cc}} = \{(x,y,p,q) \mid y = p = q = 0\}.$$

Thus Σ_{c} and Δ are 2-dimensional manifolds, and Σ_{cc} is a 1-dimensional manifold. By Lemma 2.3, $F = 0$ is completely integrable and the complete solution is given by

$$\Gamma(t,r,s) = \Big(-\tfrac{1}{2}r\int(1+t^2)^{-5/4}\,dt + s, -\tfrac{1}{2}r^2t(1+t^2)^{-1/2}, r(1+t^2)^{-1/4}, t\Big).$$

Notice that the contact singular set Σ_{c} is foliated by a 1-parameter family of geometric solutions $\Phi(t,a) = (a,0,0,t)$. However, the family Φ is contained in the complete solution and thus its members are not singular solutions even if Σ_{cc} is a 1-dimensional manifold.

Example 4.4 (First order classical Clairaut equations viewed as second order equations). Let $F(x,y,p,q) = -y + px + \chi(p)$, where χ is a smooth function. This equation is called the *first order classical Clairaut equation* (cf. [3, 6]). In this case, $F_x + pF_y + qF_p = q(x + \chi'(p))$ and $F_q \equiv 0$. Therefore $F = 0$ is of first order type and the complete solution is given by $\Gamma(t,r,s) = (s, rs + \chi(r), r, t)$. The contact singular set Σ_{c} decomposes into the union $\Sigma_{\mathrm{c}}^1 \cup \Sigma_{\mathrm{c}}^2$ of two 2-dimensional manifolds intersecting transversally in $F^{-1}(0)$, where

$$\Sigma_{\rm c}^1 = \{(x,y,p,q) \mid y = px + \chi(p),\, q = 0\},$$
$$\Sigma_{\rm c}^2 = \{(x,y,p,q) \mid x = -\chi'(p),\, y = -p\chi'(p) + \chi(p)\}.$$

Notice that $\Sigma_{\rm c}^1$ and $\Sigma_{\rm c}^2$ are foliated by 1-parameter families of geometric solutions

$$\Phi_1(t,a) = (t, at + \chi(a), a, 0), \qquad \Phi_2(t,a) = (-\chi'(a), -a\chi'(a) + \chi(a), a, t)$$

respectively. The family Φ_1 is not contained in the complete solution and thus constitutes a complete singular solution. However, Φ_2 is contained in the complete solution and thus its members are not singular solutions. We can show that the second order contact singular set $\Sigma_{\rm cc}$ decomposes into the union $\Sigma_{\rm cc}^1 \cup \Sigma_{\rm cc}^2$ of two 1-dimensional manifolds, where

$$\Sigma_{\rm cc}^1 = \{(x,y,p,q) \mid x = -\chi'(p),\, y = -p\chi'(p) + \chi(p),\, q = 0\},$$
$$\Sigma_{\rm cc}^2 = \{(x,y,p,q) \mid x = -\chi'(p),\, y = -p\chi'(p) + \chi(p),\, q = -1/\chi''(p)\}.$$

In this case, we have $\Delta = \Sigma_{\rm c}^2$, $\Sigma_{\rm cc}^1 = \Sigma_{\rm c}^1 \cap \Sigma_{\rm c}^2$ and $\Sigma_{\rm cc}^2 \subset \Sigma_{\rm c}^2$. If $z_0 \in \Sigma_{\rm cc}^1$, then z_0 is a regular point of $F_p|_{\Sigma_{\rm c}^1}$. By Theorem 3.7, there exists a complete singular solution on $\Sigma_{\rm c}^1$ around z_0. On the other hand, since $F_p|_{\Sigma_{\rm c}^2} = 0$, it is not regular. Thus we cannot establish the existence of a complete singular solution on $\Sigma_{\rm c}^2$.

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