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ISOMETRIC CLASSIFICATION OF SOBOLEV SPACES ON GRAPHS

BY

M. I. OSTROVSKII (Queens, NY)

Abstract. Isometric Sobolev spaces on finite graphs are characterized. The characterization implies that the following analogue of the Banach–Stone theorem is valid: if two Sobolev spaces on 3-connected graphs, with the exponent which is not an even integer, are isometric, then the corresponding graphs are isomorphic. As a corollary it is shown that for each finite group $\mathcal{G}$ and each p which is not an even integer, there exists $n \in \mathbb{N}$ and a subspace $L \subset \ell_p^n$ whose group of isometries is the direct product $\mathcal{G} \times \mathbb{Z}_2$.

1. Introduction. Let G be a finite simple graph. We denote by V_G and E_G its vertex set and edge set, respectively. Let d_v denote the degree of a vertex $v \in V_G$; we use the notation $d_{v,G}$ if v is a vertex of several graphs simultaneously. We omit the subscript G in E_G, V_G, etc., if G is clear from context. All undefined graph-theoretic terminology and notation follows [1] and/or [6].

Definition 1. Let $f : V_G \to \mathbb{R}$, and let $1 \le p < \infty$. The *Sobolev seminorm* of f corresponding to $E = E_G$ and p is defined by

$$\|f\| = \|f\|_{E,p} = \Big(\sum_{uv\in E} |f(u) - f(v)|^p\Big)^{1/p}.$$

If G is connected, then the only functions f satisfying $\|f\|_{E,p} = 0$ are constant functions, so $\|\cdot\|_{E,p}$ is a norm on each linear space of functions on $V = V_G$ which does not contain constants. Usually we shall consider the subspace in the space $\mathbb{R}^{V_G}$ of all functions on V_G given by $\sum_{v\in V} f(v)d_v = 0$. The resulting normed space will be called a *Sobolev space on* G and will be denoted by $S_p(G)$.

Sobolev seminorms have been used for work on spectral and isoperimetric problems of graph theory, problems on finite metric spaces and on the shapes of minimal-volume projections of cubes. We refer to [2], [3], [18], and [26] for

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more information on this matter. Isometries of classical Sobolev spaces were studied in [5].

In this paper by an *isometry* between two normed spaces X and Y we mean a linear bijection $T : X \to Y$ satisfying the condition $\|Tx\| = \|x\|$ for all $x \in X$. The main purpose of this paper is to answer an isometric version of the following general problem:

To what extent the geometry of the graph G is determined by the geometry of the space $S_p(G)$ (for $p \neq 2$)?

Recall the following well-known result (see [7, p. 442]).

Banach–Stone Theorem. *If the spaces $C(Q)$ and $C(R)$ of continuous functions on compact Hausdorff spaces are isometric, then Q and R are homeomorphic.*

The problem mentioned above can be considered as a problem about analogues of the Banach–Stone theorem for Sobolev spaces on graphs.

For graphs the assumption that $S_p(G)$ and $S_p(H)$ are isometric does not imply that G and H are isomorphic, even when $p \neq 2$. One of the easiest ways to show this is by observing that if G is a tree, then $S_p(G)$ is isometric to ℓ_p^n of the corresponding dimension. On the other hand, we prove (Theorem 2) that if p is not an even integer and the graphs G and H are 3-connected, then the isometric equivalence of $S_p(G)$ and $S_p(H)$ implies that G and H are isomorphic. It is also worth mentioning that Sobolev spaces of the same dimension can be "far" from each other. To state the corresponding result we recall that the *Banach—Mazur distance* $d(X,Y)$ between two finite-dimensional normed spaces of the same dimension is defined by

$$d(X,Y) = \inf\{\|T\| \cdot \|T^{-1}\| : \ T : X \to Y \text{ is an isomorphism}\}.$$

It was shown in [17] that there exist connected graphs G on n^2 vertices such that $d(S_1(G), \ell_1^{n^2-1}) \geq C\sqrt{\ln n}$, where $C > 0$ is an absolute constant.

Sobolev spaces on graphs (of non-trivial size), which are not 3-connected, can be isometric without the graphs being isomorphic. We describe (Theorem 1) the degree of similarity between graphs G and H which is equivalent to isometric equivalence of $S_p(G)$ and $S_p(H)$ for $p \notin \{2, 4, 6, 8, \dots, \infty\}$. (We shall write the last condition as $p \notin 2\mathbb{N}$. The restriction comes from the use of the extension theorem for L_p-isometries.)

2. Surgeries preserving the isometric class of Sobolev spaces

Definition 2. A connected induced subgraph O in a graph G is called *2-joined* if $3 \leq |V_O| < |V_G|$ and there exist $u, v \in V_O$, $u \neq v$, such that

- Each path from a vertex of O to a vertex which is not in O has either u or v among its vertices.
- Both u and v are adjacent to vertices which are not in O.

The vertices u and v are called *junction vertices* of O.

REMARK. The following is an immediate consequence of the definitions: A connected graph G with $|V_G| \geq 4$ contains a 2-joined subgraph if and only if G is not 3-connected.

It can be easily verified in a straightforward way that all results of this paper are valid in the case $|V_G| \leq 3$. We assume $|V_G| \geq 4$ without mentioning this explicitly.

THEOREM 1. *Let G and H be connected graphs. Let $1 \leq p < \infty$, $p \notin 2\mathbb{N}$. The spaces $S_p(G)$ and $S_p(H)$ are isometric if and only if the graph G is isomorphic to a graph obtained from H by using finitely many surgeries of the following two types.*

TYPE 1. *Let v be a cutvertex of G, and let O be one of the components of $G - v$. We choose a vertex u ($\neq v$) in $G - O$. For each vertex w in O which is adjacent to v we delete the edge wv and introduce a new edge wu.*

TYPE 2. *Let O be a 2-joined subgraph of G with junction vertices u and v. Suppose that O has at least one vertex, distinct from u and v, which is not adjacent to both u and v. We "twist" O in G. More formally, we do simultaneously the following two procedures*: (1) *for each vertex $w \in V_O \backslash \{u, v\}$ which is adjacent to u, but not to v, we delete the edge wu and introduce a new edge wv*; (2) *for each vertex $w \in V_O \backslash \{u, v\}$ which is adjacent to v, but not to u, we delete the edge wv and introduce a new edge wu.*

Proof. The "if" part of the theorem is true for each $1 \leq p < \infty$. It is an immediate consequence of the following result.

PROPOSITION 1. *Let $1 \leq p < \infty$. Let H be a graph obtained from G by using one of the surgeries described in Theorem 1. Then $S_p(H)$ is isometric to $S_p(G)$.*

Proof. Let $\mathcal{A}_G$ be the linear operator $\mathcal{A}_G : \mathbb{R}^{V_G} \to \mathbb{R}^{V_G}$ given by

$$(\mathcal{A}_G f)(u) = f(u) - \frac{\sum_v f(v) d_{v,G}}{\sum_v d_{v,G}}.$$

It is easy to see that $\mathcal{A}_G$ maps each function from $\mathbb{R}^{V_G}$ into $S_p(G)$, and that $\|\mathcal{A}_G f\|_{E,p} = \|f\|_{E,p}$.

First we show that for each surgery there exists a natural bijection S from E_G onto E_H.

Type 1 surgeries: The bijection coincides with the identity mapping on all edges from E_G which are also in E_H. On the remaining edges, S is defined as follows: $S(wv) = wu$ for $w \in V_O$ with $wv \in E_G$.

Type 2 surgeries: The bijection S coincides with the identity mapping on all edges from $E_G \backslash E_O$, and on all edges of E_O which are not incident to v or u. On the remaining edges it is defined as follows:

- $S(wu) = wv$ for each $w \in V_O \setminus \{v\}$ with $wu \in E_O$.
- $S(wv) = wu$ for each $w \in V_O \setminus \{u\}$ with $wv \in E_O$.
- $S(uv) = uv$ if $uv \in E_O$.

Observe that to prove the proposition it is enough to find a linear mapping $L : \mathbb{R}^{V_G} \to \mathbb{R}^{V_H}$ such that for $yz = S(wx)$ we have

$$|(Lg)(y) - (Lg)(z)| = |g(w) - g(x)|. \tag{1}$$

In fact, if there is an L satisfying (1), then $\mathcal{A}_H L : S_p(G) \to S_p(H)$ is an isometry.

Straightforward verification shows that the following mappings satisfy (1).

Type 1 surgeries:

$$(Lg)(z) = \begin{cases} g(z) & \text{if } z \in G - O, \\ g(z) - g(v) + g(u) & \text{if } z \in V_O. \end{cases}$$

Type 2 surgeries:

$$(Lg)(z) = \begin{cases} g(z) & \text{if } z \in G - O,\ z = u, \text{ or } z = v, \\ g(u) + g(v) - g(z) & \text{if } z \in V_O \setminus \{u, v\}. \ \blacksquare \end{cases}$$

To prove the "only if" part of the theorem we need the so-called extension theorem for isometries of subspaces of L_p. The theorem in the form used by us is due to C. Hardin [11]. Results of the same spirit were proved earlier by W. Lusky [16] and A. Plotkin (see [20]–[22]). See [4], [8, Section 3.3], [12], [13, Section 2], [23], and [24] for related information and historical comments.

Let F be a set of functions on a measure space $(\Omega_1, \Sigma_1, \mu_1)$. We assume, for simplicity, that F contains a function whose support is Ω_1. Let $\varrho(F)$ denote the least σ-algebra in which all quotients f/g $(f, g \in F)$ are measurable; here the quotients are allowed to have ∞ as one of their values (and $0/0$ is defined to be ∞). We denote by $\mathcal{R}(F)$ the set of all $\varrho(F)$-measurable functions on Ω_1, and by $\mathcal{R}(F) \cdot F$ the set of all functions of the form rf, where $r \in \mathcal{R}(F)$, $f \in F$.

Extension Theorem. *Let $p \in (0, \infty)$, $p \notin 2\mathbb{N}$, H be a closed subspace of $L_p(\Omega_1, \Sigma_1, \mu_1)$, and $T : H \to L_p(\Omega_2, \Sigma_2, \mu_2)$ be a linear isometric embedding. Then T can be extended to a linear isometric embedding of $\mathcal{R}(H) \cdot H \cap L_p(\Omega_1, \Sigma_1, \mu_1)$ into $L_p(\Omega_2, \Sigma_2, \mu_2)$.*

There is a natural isometric embedding of $S_p(G)$ into $\ell_p(E_G)$. To define it we choose a direction for each edge $uv \in E_G$ and let

$$(\mathcal{C}_G g)(uv) = g(u) - g(v)$$

for $g \in S_p(G)$, where uv is directed from u to v. We identify $S_p(G)$ with $\mathcal{C}_G(S_p(G))$. The embedding $\mathcal{C}_G$ makes the extension theorem applicable to Sobolev spaces on graphs. Using the extension theorem we prove

PROPOSITION 2. *Let $p \notin 2\mathbb{N}$, let $T : S_p(G) \to S_p(H)$ be an isometry, and let an orientation of edges of G and H be given. Then there exist a function $\theta : E_H \to \{-1, 1\}$ and a bijection $B : E_H \to E_G$ such that:*

1. *If $f \in \ell_p(E_G)$ is in $\mathcal{C}_G(S_p(G))$, then $g \in \ell_p(E_H)$ given by $g(uv) = \theta(uv)f(B(uv))$ is in $\mathcal{C}_H(S_p(H))$ and $Tf = g$.*
2. *The bijection B is cycle-preserving (a set of edges forming a cycle in H is mapped onto a similar set in G).*

Proof. Let $T : S_p(G) \to S_p(H)$ be an isometry. Without loss of generality we assume that the numbers of edges of G and H satisfy $|E_G| \geq |E_H|$. We consider $S_p(G)$ and $S_p(H)$ as subspaces of $\ell_p(E_G)$ and $\ell_p(E_H)$, respectively, by means of the natural embedding defined above. In order to use the terminology and notation of the extension theorem we identify $\ell_p(E_G)$ with $L_p(E_G, \Sigma_1, \mu_1)$ and $\ell_p(E_H)$ with $L_p(E_H, \Sigma_2, \mu_2)$, where Σ_1 and Σ_2 are the σ-algebras of all subsets, and μ_1 and μ_2 are the counting measures.

LEMMA 1. *If $S_p(G)$ is embedded into $L_p(E_G, \Sigma_1, \mu_1)$ using $\mathcal{C}_G$, then $\varrho(S_p(G)) = \Sigma_1$.*

Proof. The image of $S_p(G)$ in $L_p(E_G, \Sigma_1, \mu_1)$ contains functions of full support: indeed, consider $\mathcal{C}_G(\mathcal{A}_G s)$ for any function $s : V_G \to \mathbb{R}$ with $s(u) \neq s(v)$ for $u \neq v$. Hence for each cut $C \subset E_G$ there is a function of the form f/g, with $f, g \in \mathcal{C}_G(S_p(G))$, supported on C. Hence $C \subset \varrho(S_p(G))$. On the other hand, the σ-algebra generated by all cuts of G is Σ_1. In fact, for each edge $uv \in E_G$ consider $C(u) \cap C(v)$, where $C(u)$ (resp. $C(v)$) is the cut containing all edges incident to u (resp. v). Since G is assumed to be without multiple edges, it follows that $C(u) \cap C(v) = \{uv\}$. ∎

By Lemma 1, the extension theorem implies that there exists an isometric embedding $T' : \ell_p(E_G) \to \ell_p(E_H)$ which extends the isometry $T : S_p(G) \to S_p(H)$. The assumption $|E_G| \geq |E_H|$ implies that T' is surjective.

We recall the description of isometries of ℓ_p^n, $p \neq 2$ (see, e.g., [14, p. 112]): each of them is formed by permutations of the unit vectors and multiplication of them by ± 1. Therefore, for each isometry $T' : \ell_p(E_G) \to \ell_p(E_N)$ there exists a bijection $B : E_H \to E_G$, and a function $\theta : E_H \to \{-1, 1\}$, such that

$$T'f(uv) = \theta(uv)f(B(uv)), \qquad uv \in E_H. \tag{2}$$

It remains to show that B is cycle-preserving. Denote by e_v^* $(v \in V_H)$ the functional on $\mathbb{R}^{V_H}$ given by $e_v^*(f) = f(v)$. Denote by e_{uv}^* $(uv \in E_H)$ the functional on $\ell_p(E_H)$ given by $e_{uv}^*(h) = h(uv)$. It is clear that the restriction of e_{uv}^* to $S_p(H)$ is equal to $e_u^* - e_v^*$ or $e_v^* - e_u^*$, depending on the choice of the direction of the edge uv, which was used to define the natural embedding. The formula (2) can be rewritten as

$$(T')^*(e_{uv}^*) = \theta(uv)e_{B(uv)}^*.$$

Let $u_1v_1, u_2v_2, \dots, u_nv_n \in E_H$ be a set of edges forming a cycle. We know that $e_{u_iv_i}^*|_{S_p(H)} = \theta_i(e_{u_i}^* - e_{v_i}^*)$ for some $\theta_i \in \{-1, 1\}$. Since $\{u_iv_i\}_{i=1}^n$ form a cycle, there exist $\tau_i \in \{-1, 1\}$ such that

$$\sum_{i=1}^n \tau_i\theta_i(e_{u_i}^* - e_{v_i}^*) = 0 \quad \text{or} \quad \Big(\sum_{i=1}^n \tau_i e_{u_iv_i}^*\Big)\Big|_{S_p(H)} = 0.$$

Since T' maps $S_p(G)$ into $S_p(H)$, this implies

$$\Big(\sum_{i=1}^n \tau_i\theta(u_iv_i)e_{B(u_iv_i)}^*\Big)\Big|_{S_p(G)} = \Big((T')^*\Big(\sum_{i=1}^n \tau_i e_{u_iv_i}^*\Big)\Big)\Big|_{S_p(G)} = 0.$$

Let $B(u_iv_i) = w_iy_i$. The discussion above implies that $e_{B(u_iv_i)}^*|_{S_p(G)} = \gamma_i(e_{w_i}^* - e_{y_i}^*)$ for some $\gamma_i \in \{-1, 1\}$. We get

$$\sum_{i=1}^n \tau_i\theta(u_iv_i)\gamma_i(e_{w_i}^* - e_{y_i}^*) = 0.$$

This can happen only if each of the e_v^* is repeated in this sum an even number of times (half of them with negative sign). The well-known argument of the Euler's theorem (see, e.g., [1, p. 17]) implies that $\{w_iy_i\}_{i=1}^n$ is a union of cycles. Since we can interchange the roles of G and H in this argument, it is a single cycle. ∎

Remark. It can also be shown that each direction-preserving bijection B satisfying condition 2 of Proposition 2 can be used to define an isometry as described in condition 1 of Proposition 2. This observation explains why in the rest of the proof it is enough to use the cycle-preserving property of B only.

The fact that the existence of a bijection B satisfying the conditions of Proposition 2 implies that the graph G can be obtained from H by using finitely many surgeries of types 1 and 2 can be considered as part of Whitney's 2-isomorphism theorem [29]. Usually this theorem is stated in terms of matroids and for general, not necessarily connected graphs (see [27] or [19, p. 148]). Stated for connected graphs and without matroid terminology, the theorem is:

Whitney's 2-isomorphism Theorem. *If G and H are connected graphs such that there exists a bijection between E_G and E_H which is also a bijection between the sets of cycles, then G can be obtained from H by using finitely many surgeries of types* 1 *and* 2.

It is clear that application of this theorem completes the proof of Theorem 1. ∎

3. Analogue of the Banach–Stone theorem for Sobolev spaces on 3-connected graphs and groups of isometries of subspaces of ℓ_p^n. The next result is an immediate corollary of Theorem 1, because (as already observed) 3-connected graphs do not have 2-joined subgraphs, and hence, in this case, the conclusion of Theorem 1 implies that G and H are isomorphic.

Theorem 2. *Let G and H be 3-connected graphs and let $1 \le p < \infty$, $p \notin 2\mathbb{N}$. If $S_p(G)$ and $S_p(H)$ are isometric Banach spaces, then G and H are isomorphic.*

Remark. For 3-connected graphs, each mapping $B : E_H \to E_G$ satisfying condition 2 of Proposition 2 corresponds to an isomorphism of H and G (see [28, p. 156] and [19, Lemma 5.3.2, p. 148]).

An interesting corollary of this remark and Proposition 2 is:

Theorem 3. *For each $1 \le p < \infty$, $p \notin 2\mathbb{N}$, and each finite group $\mathcal{G}$ there exists $n \in \mathbb{N}$ and a subspace $X \subset \ell_p^n$ such that the direct product $\mathcal{G} \times \mathbb{Z}_2$ is isomorphic to the group of all isometries of X.*

Proof. First we prove that each isometry $T : S_p(H) \to S_p(H)$ for a 3-connected graph H and $1 \le p < \infty$, $p \notin 2\mathbb{N}$, corresponds to a pair (φ, θ), where φ is an automorphism of H and $\theta = \pm 1$. In fact, let $B : E_H \to E_H$ be the cycle-preserving bijection whose existence is proved in Proposition 2. By the remark after Theorem 2, B corresponds to an automorphism of H, say φ. Also, according to Proposition 2, the extension T' of T to $\ell_p(E_H)$ is given by $(T'f)(uv) = \theta_T(uv) f(B(uv))$. It remains to show that if two isometries of $S_p(H)$, say T and S, correspond to the same automorphism φ of H, then either $\theta_T(uv) = \theta_S(uv)$ for each $uv \in E_H$, or $\theta_T(uv) = -\theta_S(uv)$ for each $uv \in E_H$.

Assume the contrary, that is, there exist edges uv and wz such that $T^* e^*_{uv} = S^* e^*_{uv}$ and $T^* e^*_{wz} = -S^* e^*_{wz}$. It is well known that in a 2-connected graph any two edges are contained in a cycle. Let C be a cycle containing both uv and wz. We infer (see Proposition 2) that for some collection $\tau_{xy} \in \{-1, 1\}$,

$$\Big(\sum_{xy \in C} \tau_{xy} e^*_{xy}\Big)\Big|_{S_p(H)} = 0.$$

Hence

$$\left(\sum_{xy\in C}\tau_{xy}T^*e^*_{xy}\right)\Big|_{S_p(H)}=0 \quad\text{and}\quad \left(\sum_{xy\in C}\tau_{xy}S^*e^*_{xy}\right)\Big|_{S_p(H)}=0. \tag{3}$$

Subtracting the equations from (3) and using the assumptions, we deduce that the values of functions from $\mathcal{C}_H(S_p(H))$ on a proper subset of the cycle $B(C)$ satisfy a non-trivial linear equation. It is easy to see that this leads to a contradiction.

Therefore it suffices to construct a 3-connected graph H whose group of automorphisms is isomorphic to $\mathcal{G}$. To do this we use the result of R. Frucht [9] (see also [15, §12.8]) stating that for each finite group $\mathcal{G}$ there is a finite 3-regular graph F whose group of automorphisms is isomorphic to $\mathcal{G}$. To finish the proof we use the following observations (the first comes from Frucht's construction, the other two are immediate consequences of the definitions):

- Graphs in Frucht's construction can be required to have ≥ 10 vertices.
- The group of automorphisms of the complement H of a graph F is the same as the group of automorphisms of F.
- If F is 3-regular and has ≥ 10 vertices, then its complement H is 3-connected.

Hence H has the required properties. ∎

REMARK. Y. Gordon–R. Loewy [10] and J. Stern [25] proved similar "universality" results with X being Hilbert spaces with an equivalent norm, obtained by a slight perturbation of the original norm.

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Department of Mathematics and Computer Science
St. John's University
8000 Utopia Parkway
Queens, NY 11439, U.S.A.
E-mail: ostrovsm@stjohns.edu

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