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STRONGLY GROUPOID GRADED RINGS AND COHOMOLOGY

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Abstract. We interpret the collection of invertible bimodules as a groupoid and call it the Picard groupoid. We use this groupoid to generalize the classical construction of crossed products to what we call groupoid crossed products, and show that these coincide with the class of strongly groupoid graded rings. We then use groupoid crossed products to obtain a generalization from the group graded situation to the groupoid graded case of the bijection from a second cohomology group, defined by the grading and the functor from the groupoid in question to the Picard groupoid, to the collection of equivalence classes of rings strongly graded by the groupoid.

1. Introduction. Many important mathematical objects are graded. By folklore this means that the object can be written as a direct sum subject to certain multiplicative relations that are coherent with the grading. A ring R (always assumed to be associative and equipped with a multiplicative identity) is *graded by a group* G if there is a set of additive subgroups, R_σ, $\sigma \in G$, of R such that $R = \bigoplus_{\sigma \in G} R_\sigma$ and $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Analogously, a left R-module M is called *graded by* G if there is a set of additive subgroups M_σ, $\sigma \in G$, of M such that $M = \bigoplus_{\sigma \in G} M_\alpha$ and $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. Group graded rings and modules have been studied extensively (see e.g. [3]–[6], [8], [10] and [14]).

Given two rings R and S graded by G, a ring homomorphism $R \to S$ is called *graded* if $f(R_\sigma) \subseteq S_\sigma$ for all $\sigma \in G$. An important question now is how to describe the isomorphism classes of rings graded by G. It turns out that for strongly graded rings, that is, rings R with the property that $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$, this question can be answered by using the language of cohomology in the following way. Each strongly graded ring can be presented (see [14] and [9]) as a so-called generalized crossed product (F, f), where F is a group homomorphism from G to the Picard group $\mathrm{Pic}(A)$ of isomorphism classes of invertible A-bimodules (see Section 2 for the precise definition), for a given ring A, and f is a factor set associated to F, that is, a collection of A-bimodule isomorphisms $f_{\sigma,\tau} : M_\sigma \otimes_A M_\tau \to M_{\sigma\tau}$ chosen so that the

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following diagram commutes:

$$\begin{array}{ccc} M_\sigma \otimes_A M_\tau \otimes_A M_\varrho & \xrightarrow{\mathrm{id}_{M_\sigma} \otimes f_{\tau,\varrho}} & M_\sigma \otimes_A M_{\tau\varrho} \\ {\scriptstyle f_{\sigma,\tau}\otimes \mathrm{id}_{M_\varrho}} \downarrow & & \downarrow {\scriptstyle f_{\sigma,\tau\varrho}} \\ M_{\sigma\tau} \otimes_A M_\varrho & \xrightarrow{f_{\sigma\tau,\varrho}} & M_{\sigma\tau\varrho} \end{array}$$

for all $\sigma, \tau, \varrho \in G$, where $F(\sigma) = [M_\sigma]$ (the isomorphism class of M_σ in $\mathrm{Pic}(A)$) for all $\sigma \in G$. The multiplication in $(F, f) = \bigoplus_{\sigma \in G} M_\sigma$ is defined by the bilinear extension of the relation $x \cdot y = f_{\sigma,\tau}(x \otimes y)$ for all $x \in M_\sigma$, $y \in M_\tau$, $\sigma, \tau \in G$. Since each M_σ is invertible it follows that we have an action of G on $C(A)$, the center of A, and hence also on $C(A)^*$, the corresponding group of units, defined by the relation $\sigma(a)x = xa$ for all $\sigma \in G$, $a \in C(A)$, $x \in M_\sigma$ (see [1]).

Năstăsescu and Van Oystaeyen [14] have shown that given a factor set f associated to F, the map from $Z^2(G, C(A)^*)$ to the collection of all factor sets associated to F, defined by $q \mapsto qf$, is a well defined bijection. Note that for a ring R strongly graded by the group G, we can set $A = R_e$ (e is the identity element of G) and define the group homomorphism $F : G \to \mathrm{Pic}(R_e)$ by $F(\sigma) = [R_\sigma]$ for all $\sigma \in G$. Furthermore, two rings R and S, strongly graded by G, are called *equivalent* if there is an isomorphism $f : R \to S$ of graded rings that is simultaneously an R_e-bimodule isomorphism. The R_e-bimodule structure on S is defined by $r \cdot s := f(r)s$ for all $r \in R_e$ and all $s \in S$. In the same paper Năstăsescu and Van Oystaeyen show that there is an induced bijection from $H^2(G, C(A)^*)$ to the collection of equivalence classes of rings strongly graded by G.

Many natural examples of rings, such as e.g. rings of matrices, crossed product algebras defined by separable extensions and groupoid rings (see [12], [13] and Example 4.3), are not in any natural way graded by groups, but instead by *groupoids*, i.e. categories with the property that all morphisms are isomorphisms. This inspired us in [11] to introduce and analyze the category of groupoid graded modules. The purpose of this article is to continue this line of work and give a homological answer to the question of when strongly groupoid graded rings are equivalent, analogous to the result in the group graded case described above. Since the components of groupoid graded rings are not in general bimodules where the left and right rings are equal, we first have to replace the Picard group from the group graded case by the collection of all invertible bimodules.

In Section 2, we equip this collection of modules with a groupoid structure (induced by the tensor product) and call it the Picard groupoid.

In Section 3, we prove some results concerning strongly groupoid graded rings that we need in later sections. After that, in Section 4, we introduce groupoid crossed products as a generalization of generalized crossed products and we show that each strongly groupoid graded ring can be presented in this way.

In Section 5, we recall the definitions concerning the cohomology theory of groupoids. We need this at the end of the article in Section 6 where we use the construction of groupoid crossed products to obtain a generalization from the group graded situation to the groupoid graded case of the bijection from the second cohomology group, defined by the grading and the functor from the groupoid in question to the Picard groupoid, to the collection of equivalence classes of rings strongly graded by the groupoid (see Theorem 6.3).

2. The Picard groupoid. In this section, we interpret the collection of invertible bimodules as a groupoid and call it the Picard groupoid. Next, we extend some well known sequences relating Picard groups and automorphism groups, to the Picard groupoid (see Propositions 2.1 and 2.2).

All rings are associative and are assumed to have multiplicative identities that act as identity maps on all modules and are preserved by ring homomorphisms. The center and the group of units of a ring A are denoted by $C(A)$ and A^* respectively.

Recall that a *groupoid* is a category with the property that all morphisms are isomorphisms. Equivalently, it can be defined as a nonempty collection Γ equipped with a unary operation $\Gamma \ni \sigma \mapsto \sigma^{-1} \in \Gamma$ and a partial binary operation $\Gamma \times \Gamma \ni (\sigma, \tau) \mapsto \sigma\tau \in \Gamma$ satisfying the following four axioms: (i) $d(\sigma) := \sigma^{-1}\sigma$ and $r(\sigma) := \sigma\sigma^{-1}$ are always defined ($d =$ "domain" and $r =$ "range"); (ii) $\sigma\tau$ is defined if and only if $d(\sigma) = r(\tau)$; (iii) if $\sigma\tau$ and $\tau\varrho$ are defined, then $(\sigma\tau)\varrho$ and $\sigma(\tau\varrho)$ are defined and equal; (iv) each of $d(\sigma)\tau$, $\tau d(\sigma)$, $r(\sigma)\tau$, and $\tau r(\sigma)$ is equal to τ if it is defined. Define $\Gamma_0 = \{d(\sigma) \mid \sigma \in \Gamma\} = \{r(\sigma) \mid \sigma \in \Gamma\}$ and for a positive integer n, let

$$\Gamma_n = \{(\sigma_0, \ldots, \sigma_{n-1}) \in \Gamma^n \mid d(\sigma_i) = r(\sigma_{i+1}),\ i = 0, \ldots, n-2\},$$

where Γ^n denotes the direct product of n copies of Γ. Let Γ' be another groupoid. If Γ' is a subcategory of Γ, closed under inverses, then we say that it is a *subgroupoid*. Let F be a homomorphism of groupoids, that is, a functor, from Γ to Γ'. Then the kernel and image of F, $\ker(F) = \{\sigma \in \Gamma \mid F(\sigma) \in \Gamma_0'\}$ and $\mathrm{im}(F) = \{F(\sigma) \mid \sigma \in \Gamma\}$, are subgroupoids of Γ and Γ' respectively. A sequence of homomorphisms of groupoids

$$\cdots \to \Gamma_{n-1} \xrightarrow{F_{n-1}} \Gamma_n \xrightarrow{F_n} \Gamma_{n+1} \to \cdots, \qquad n \in \mathbb{Z},$$

is called *exact* if $\mathrm{im}(F_{n-1}) = \ker(F_n)$ for all $n \in \mathbb{Z}$. For more details on groupoids, see e.g. [16].

Now we recall some definitions from [1] (see also [2] and [7]). Let A and B be rings. An A-B-bimodule M is called *invertible* if there is a B-A-bimodule N and isomorphisms $f : M \otimes_B N \to A$ (as A-bimodules) and $g : N \otimes_A M \to B$ (as B-bimodules) such that the following two diagrams commute:

$$\begin{array}{ccc} M \otimes_B N \otimes_A M & \xrightarrow{f \otimes \mathrm{id}_M} & A \otimes_A M \\ {\scriptstyle \mathrm{id}_M \otimes g}\downarrow & & \downarrow{\scriptstyle \mathrm{I}} \\ M \otimes_B B & \xrightarrow{\mathrm{II}} & M \end{array} \qquad \begin{array}{ccc} N \otimes_A M \otimes_B N & \xrightarrow{g \otimes \mathrm{id}_N} & B \otimes_B N \\ {\scriptstyle \mathrm{id}_N \otimes f}\downarrow & & \downarrow{\scriptstyle \mathrm{III}} \\ N \otimes_A A & \xrightarrow{\mathrm{IV}} & N \end{array}$$

where I–IV are the multiplication maps. We denote by $[M]$ the class of invertible A-B-bimodules that are isomorphic to M. We denote by PIC the collection of all such classes. Let C be another ring. If M' is an invertible B-C-bimodule, then we set $[M][M'] = [M \otimes_B M']$. It is easy to check that, with this partial binary operation, PIC is a groupoid, called the *Picard groupoid*. Note that if M is an invertible A-B-bimodule, then $r([M]) = [A]$ and $d([M]) = [B]$. We also remark that the Picard group $\mathrm{Pic}(A)$ of A (see [1]) is, in a natural way, a subgroupoid of PIC.

We denote by Ring the category of rings and ring homomorphisms. We denote by I, INN and ISO the subcategories of Ring with the same objects as Ring but with morphisms consisting of, respectively, the identities, the inner automorphisms and the isomorphisms. We denote by $i : \mathrm{I} \to \mathrm{INN}$ and $j : \mathrm{INN} \to \mathrm{ISO}$ the canonical functors. Note that I, INN and ISO are groupoids and that the functors i and j are homomorphisms of groupoids.

To state our first result, we need some more notation. Let A and A' be rings. If M is an A-A'-bimodule and $f : B \to A$ and $g : B' \to A'$ are ring homomorphisms, then let the B-B'-bimodule ${}_fM_g$ be defined as M as an additive group but with the multiplication of scalars defined by $b \cdot m \cdot b' = f(b)mg(b')$, $b \in S$, $m \in M$, $b' \in S'$. We define a homomorphism $k : \mathrm{ISO} \to \mathrm{PIC}$ of groupoids by $k(f) = {}_1A'_f$ for all ring isomorphisms f from A to A'. The map k respects the operations on ISO and PIC. In fact, if $f : A' \to A''$ and $g : A \to A'$ are ring isomorphisms, then the map ${}_1A''_f \otimes_{A'} {}_1A'_g \ni x \otimes y \mapsto xf(y) \in {}_1A''_{f \circ g}$ is an isomorphism of A''-A'-bimodules.

Our first result will not be needed in what follows, but we record it for its own interest.

2.1. Proposition. *With the above notations, the sequence of groupoids*

$$\mathrm{I} \xrightarrow{i} \mathrm{INN} \xrightarrow{j} \mathrm{ISO} \xrightarrow{k} \mathrm{PIC} \tag{1}$$

is exact.

Proof. Exactness at INN is trivial. Now we show exactness at ISO. Let A be a ring.

Assume first that we are given a ring automorphism f of A such that $[{}_1A_f] = [{}_1A_1]$, i.e., that there is an A-bimodule isomorphism g from ${}_1A_f$ to ${}_1A_1$. Set $u = g(1)$. If $a \in A$, then $au = ag(1) = g(a) = g(1) \cdot a = uf(a)$. Hence, since u is a unit, $f(a) = u^{-1}au$, $a \in A$. Thus f is inner.

On the other hand, if we are given an inner automorphism f of A, then $f(a) = u^{-1}au$, $a \in A$, for some unit u in A. Therefore we can define an A-bimodule isomorphism g from ${}_1A_f$ to ${}_1A_1$ by $g(x) = xu$, $x \in {}_1A_f$. Hence $[{}_1A_f] = [{}_1A_1]$. ∎

Note that (1) contains the exact sequence of groups $1 \to \mathrm{Inn}(A) \to \mathrm{Aut}(A) \to \mathrm{Pic}(A)$ from [7].

We denote by ISOC the groupoid of ring isomorphisms of all commutative rings, equipped with the same partial binary operation as the one we defined on ISO. Moreover, we define a map $l : \mathrm{PIC} \to \mathrm{ISOC}$ by the formula $l([M]) = f_M$ where $f_M : C(B) \to C(A)$ is the ring isomorphism defined by the relation $mb = f_M(b)m$, $m \in M$, $b \in C(B)$.

2.2. Proposition. *The map* $l : \mathrm{PIC} \to \mathrm{ISOC}$ *is a groupoid homomorphism.*

Proof. For the existence of the maps f_M, see e.g. [7]. It is easy to check that l is well defined. Next, if B' is another ring and N is an invertible B-B'-bimodule, then $m \otimes nb' = m \otimes f_N(b')n = mf_N(b') \otimes n = (f_M \circ f_N)(b')m \otimes n$ for all $m \in M$, $n \in N$, $b' \in C(B')$, and we get $l([M][N]) = l([M \otimes_B N]) = f_{M \otimes_B N} = f_M \circ f_N = l([M])l([N])$. ∎

3. Strongly graded modules. In this section we recall the definitions concerning groupoid graded rings and modules introduced in [11]. Our main goal in this section is to state and prove a result (see Proposition 3.3) that gives a functorial description of strongly graded rings and modules that will come in handy in later sections. To do that we need a few preliminary results (see Proposition 3.1 and Lemma 3.2).

For the rest of the article, we fix a groupoid Γ that is small considered as a category. We say that a ring R is *graded* if there is a family of additive subgroups, R_σ, $\sigma \in \Gamma$, of R such that $R = \bigoplus_{\sigma \in \Gamma} R_\sigma$, and for all $\sigma, \tau \in \Gamma$, we have $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ if $(\sigma, \tau) \in \Gamma_2$, and $R_\sigma R_\tau = \{0\}$ otherwise.

If R is a graded ring, then we say that a left R-module M is *graded* if there is a family of additive subgroups, M_σ, $\sigma \in \Gamma$, of M such that $M = \bigoplus_{\sigma \in \Gamma} M_\sigma$, and for all $\sigma, \tau \in \Gamma$, we have $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ if $(\sigma, \tau) \in \Gamma_2$, and $R_\sigma M_\tau = \{0\}$ otherwise. Let R-mod (R-gr) denote the category of (graded) left R-modules. The morphisms in the graded case are taken to be R-linear maps $f : M \to M'$ with the property $f(M_\sigma) \subseteq M'_\sigma$, $\sigma \in \Gamma$. Furthermore, we say that a graded ring or module is *strongly graded* if we have equality in the inclusions $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ (respectively, $R_\sigma M_\tau \subseteq M_{\sigma\tau}$) for any $(\sigma, \tau) \in \Gamma_2$. It is easy to see that R-gr is an abelian category. In fact, it is even a Grothendieck category.

For the rest of the article, we fix a graded ring R. Let M be a graded left R-module. Elements of $\bigcup_{\sigma\in\Gamma} M_\sigma$ are called *homogeneous elements* of M. Any nonzero $m \in M$ has a unique decomposition $m = \sum_{\sigma\in\Gamma} m_\sigma$, where $m_\sigma \in M_\sigma$, $\sigma \in \Gamma$, and all but a finite number of the m_σ are nonzero. The nonzero elements m_σ in the decomposition of m are called the *homogeneous components* of m. We set $M_0 = \bigoplus_{\sigma\in\Gamma_0} M_\sigma$.

By the next proposition, we can always assume that Γ_0 is finite.

3.1. Proposition. *With the above notations, we get:*

(a) $1 \in \bigoplus_{\sigma\in\Gamma_0} R_\sigma$.

If we set $\Gamma' = \{\sigma \in \Gamma \mid 1_{d(\sigma)}, 1_{r(\sigma)} \neq 0\}$, *then*

(b) *The set* Γ', *with the operations induced from* Γ, *is a groupoid.*
(c) Γ'_0 *is finite.*
(d) $R = \bigoplus_{\sigma\in\Gamma'} R_\sigma$.

Proof. Straightforward (see [11] for the details). ∎

By the last proposition, we can write $1 = \sum_{e\in\Gamma_0} 1_e$, where $1_e \in R_e \setminus \{0\}$, $e \in \Gamma_0$. Note that if M is a graded left R-module, then this implies that

$$1_{r(\sigma)} m = m \tag{2}$$

for all $m \in M_\sigma$, $\sigma \in \Gamma$.

3.2. Lemma. *The ring R is strongly graded if and only if*

$$R_{r(\sigma)} = R_\sigma R_{\sigma^{-1}} \tag{3}$$

for all $\sigma \in \Gamma$.

Proof. If R is strongly graded, then, trivially, (3) holds. On the other hand, if (3) holds, then, by (2), we get $R_\sigma R_\tau \subseteq R_{\sigma\tau} = R_{r(\sigma)} R_{\sigma\tau} \subseteq R_\sigma R_{\sigma^{-1}} R_{\sigma\tau} \subseteq R_\sigma R_\tau$ for all $(\sigma, \tau) \in \Gamma_2$. Hence R is strongly graded. ∎

Let i denote the inclusion $R_0 \subseteq R$. Since R_0 is a ring, i is a ring homomorphism. Now we introduce the graded restriction and induction functors ${}_{\mathrm{gr}}i_*$ from R-gr to R_0-mod and ${}_{\mathrm{gr}}i^*$ from R_0-mod to R-gr, respectively. By mimicking the ungraded case we define these functors by ${}_{\mathrm{gr}}i_*(M) = M_0$, with the induced left R_0-module structure, for all graded left R-modules M, and ${}_{\mathrm{gr}}i^*(N) = R \otimes_{R_0} N$, with the induced left R-module structure, and a grading defined by $(R \otimes_{R_0} N)_\sigma = R_\sigma \otimes_{R_0} N$ for all $\sigma \in \Gamma$ and all left R_0-modules N. It is easy to check that ${}_{\mathrm{gr}}i_*$ is a right adjoint of ${}_{\mathrm{gr}}i^*$, and that the corresponding unit $\alpha : \mathrm{id}_{R_0\text{-mod}} \to {}_{\mathrm{gr}}i_* {}_{\mathrm{gr}}i^*$ and counit $\beta : {}_{\mathrm{gr}}i^* {}_{\mathrm{gr}}i_* \to \mathrm{id}_{R\text{-gr}}$ are the natural maps $\alpha_N : N \to R_0 \otimes_{R_0} N$, $N \in R_0$-mod, and $\beta_M : R \otimes_{R_0} M_0 \to M$, $M \in R$-gr.

3.3. Proposition. *With the above notations, the following three conditions are equivalent:*

(i) *The ring R is strongly graded.*
(ii) *Every graded left R-module is strongly graded.*
(iii) *The natural transformations α and β are natural equivalences.*

Proof. Suppose that (i) holds. We show (ii). Take a graded left R-module M. If $(\sigma,\tau) \in \Gamma_2$, then, by (3), we get $R_\sigma M_\tau \subseteq M_{\sigma\tau} = R_{r(\sigma)} M_{\sigma\tau} = R_\sigma R_{\sigma^{-1}} M_{\sigma\tau} \subseteq R_\sigma M_\tau$. Hence M is strongly graded.

Now suppose that (ii) holds. We show (iii). Take a graded left R-module M. By the assumptions, β_M is surjective. Set $K = \ker(\beta_M)$. Take $\sigma \in \Gamma$. Then, since $\beta_M|_{R_0 \otimes_{R_0} M} : R_0 \otimes_{R_0} M \to M$ is an isomorphism, we get

$$K_\sigma = R_\sigma K_0 = R_\sigma((R_0 \otimes_{R_0} M) \cap K) = R_\sigma \ker(\beta_M|_{R_0 \otimes_{R_0} M}) = R_\sigma 0 = 0.$$

Therefore $K = 0$. Also, α_N is an isomorphism for all left R_0-modules N. In fact, the inverse of α_N is given by the map $R_0 \otimes_{R_0} N \ni r \otimes n \mapsto rn \in N$, $r \in R_0$, $n \in N$.

If (iii) holds, then trivially (ii) and hence (i) holds. ∎

To state the next result, we need some more notation. For a graded left R-module M and $\sigma \in \Gamma$, let $M(\sigma)$, the *σ-suspension* of M, be the graded submodule of M defined by the new grading $M(\sigma)_\tau = M_{\tau\sigma}$ if $(\tau,\sigma) \in \Gamma_2$ and $M(\sigma)_\tau = \{0\}$ otherwise, for all $\tau \in \Gamma$.

3.4. Corollary. *Suppose that R is a strongly graded ring and M a strongly graded left R-module. With the above notations, we get:*

(a) *The multiplication map $R \otimes_{R_0} M_\tau \to M(\tau)$ is an isomorphism of graded left R-modules.*
(b) *If $(\sigma,\tau) \in \Gamma_2$, then the multiplication map $R_\sigma \otimes_{R_{r(\tau)}} R_\tau \to R_{\sigma\tau}$ is an isomorphism of $R_{r(\sigma)}$-$R_{d(\tau)}$-bimodules.*

Proof. (a) By (the proof of) Proposition 3.3, the multiplication map $\beta_M : R \otimes_{R_0} M \to M$ is an isomorphism of graded left R-modules. This map restricts to an isomorphism $R \otimes_{R_0} M_\tau \to M(\tau)$ of graded left R-modules.

(b) This follows from (a) and the canonical isomorphism $R_\sigma \otimes_{R_0} R_\tau \to R_\sigma \otimes_{R_{r(\tau)}} R_\tau$ of $R_{r(\sigma)}$-$R_{d(\tau)}$-bimodules. ∎

4. Groupoid crossed products. In this section, we begin by defining groupoid crossed products. Then we show that every groupoid crossed product is a strongly graded ring and that, conversely, every strongly graded ring can be presented as a groupoid crossed product (see Proposition 4.1). We end this section by showing that there are many natural examples of groupoid crossed products (see Example 4.3).

Let $F : \mathit{\Gamma} \to \mathrm{PIC}$ be a groupoid homomorphism. Set $F(\sigma) = [M_\sigma]$, $\sigma \in \mathit{\Gamma}$, where each M_σ is an invertible $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule, for some rings $A_{r(\sigma)}$ and $A_{d(\sigma)}$. Then a *factor set associated to* F is a family $f = \{f_{\sigma,\tau} \mid (\sigma,\tau) \in \mathit{\Gamma}_2\}$, where each $f_{\sigma,\tau} : M_\sigma \otimes_{A_{d(\sigma)}} M_\tau \to M_{\sigma\tau}$ is an isomorphism of $A_{r(\sigma)}$-$A_{d(\tau)}$-bimodules, making the diagram

$$\text{(4)} \qquad \begin{array}{ccc} M_\sigma \otimes_{A_{d(\sigma)}} M_\tau \otimes_{A_{d(\tau)}} M_\varrho & \xrightarrow{\mathrm{id}_{M_\sigma} \otimes f_{\tau,\varrho}} & M_\sigma \otimes_{A_{d(\sigma)}} M_{\tau\varrho} \\ {\scriptstyle f_{\sigma,\tau}\otimes \mathrm{id}_{M_\varrho}} \downarrow & & \downarrow {\scriptstyle f_{\sigma,\tau\varrho}} \\ M_{\sigma\tau} \otimes_{A_{d(\tau)}} M_\varrho & \xrightarrow{f_{\sigma\tau,\varrho}} & M_{\sigma\tau\varrho} \end{array}$$

commute for all $(\sigma,\tau,\varrho) \in \mathit{\Gamma}_3$. If f is a factor set associated to F, then we define the groupoid crossed product

$$(F,f) = \bigoplus_{\sigma\in\mathit{\Gamma}} M_\sigma,$$

the additive group with multiplication defined by the bilinear extension of the rule $x \cdot y = f_{\sigma,\tau}(x \otimes y)$ if $(\sigma,\tau) \in \mathit{\Gamma}_2$ and $x \cdot y = 0$ otherwise, for all $x \in M_\sigma$ and $y \in M_\tau$ and $\sigma,\tau \in \mathit{\Gamma}$.

4.1. Proposition. *Let* $F : \mathit{\Gamma} \to \mathrm{PIC}$ *be a groupoid homomorphism and* f *a factor set associated to* F*.*

(a) (F,f) *is a strongly graded ring.*
(b) *If* R *is a strongly graded ring, then there is a groupoid homomorphism* $F : \mathit{\Gamma} \to \mathrm{PIC}$ *and a factor set* f *associated to* F *such that* R *is isomorphic to* (F,f)*.*

Proof. (a) Set $S = (F,f)$. By (4) multiplication in S is associative and by the definition of this multiplication, $S_\sigma S_\tau = S_{\sigma\tau}$ for any $(\sigma,\tau) \in \mathit{\Gamma}_2$. All that is left to show is that S has a multiplicative identity. Since $F(e) = [M_e] = [A_e]$, $e \in \mathit{\Gamma}_0$, we see that $M_e \cong A_e$ as A_e-bimodules. Thus, there exist $m_e \in M_e$, $e \in \mathit{\Gamma}_0$, such that $M_e = A_e m_e = m_e A_e$ and $a m_e = m_e a$, $a \in A_e$. Since each $f_{e,e} : M_e \otimes_{A_e} M_e \to M_e$, $e \in \mathit{\Gamma}_0$, is an isomorphism of A_e-bimodules, we can write $f_{e,e}(m_e \otimes m_e) = c_e m_e$ for some $c_e \in C(A_e)^*$ (note that $m \otimes m$ is a generator of a left and right R-module $M \otimes M$). Now set $n_e = c_e^{-1} m_e$, $e \in \mathit{\Gamma}_0$. Then $f_{e,e}(n_e \otimes n_e) = n_e$, $e \in \mathit{\Gamma}_0$. Hence $n := \sum_{e\in\mathit{\Gamma}_0} n_e$ is a multiplicative identity of S. Notice that if $x \in M_\sigma$ for some $\sigma \in \mathit{\Gamma}$, then, since $M_{r(\sigma)} = n_{r(\sigma)} A_{d(\sigma)}$, there is $y \in M_\sigma$ such that $x = f_{r(\sigma),\sigma}(n_{r(\sigma)} \otimes y)$. Thus, by (4), we get

$$\begin{aligned} n \cdot x = n_{r(\sigma)} \cdot x = f_{r(\sigma),\sigma}(n_{r(\sigma)} \otimes x) &= f_{r(\sigma),\sigma}(n_{r(\sigma)} \otimes f_{r(\sigma),\sigma}(n_{r(\sigma)} \otimes y)) \\ &= f_{r(\sigma),\sigma}(f_{r(\sigma),r(\sigma)}(n_{r(\sigma)} \otimes n_{r(\sigma)}) \otimes y) = f_{r(\sigma),\sigma}(n_{r(\sigma)} \otimes y) = x. \end{aligned}$$

Analogously, $x \cdot n = x$.

(b) Define $F : \Gamma \to \mathrm{PIC}$ by $F(\sigma) = [R_\sigma]$, $\sigma \in \Gamma$, and a factor set f associated to F by the multiplication maps $f_{\sigma,\tau} : R_\sigma \otimes_{R_{d(\sigma)}} R_\tau \to R_{\sigma\tau}$, $(\sigma, \tau) \in \Gamma_2$. The claim now follows immediately from Corollary 3.4(b). ∎

Now we show that the isomorphism class of (F, f) does not depend on the choice of the bimodules M_σ. To do that we need some more notation and a lemma.

Let F and F' be homomorphisms of groupoids from Γ to PIC that coincide on Γ_0. Take factor sets f and f' associated to F and F' respectively and set $F(\sigma) = [M_\sigma]$, $F'(\sigma) = [M'_\sigma]$, $\sigma \in \Gamma$. A *morphism* from f to f' is defined to be a family $\alpha = (\alpha_\sigma)_{\sigma\in\Gamma}$, where each $\alpha_\sigma : M_\sigma \to M'_\sigma$ is an $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule homomorphism such that the diagram

$$\begin{array}{ccc} M_\sigma \otimes_{A_{d(\sigma)}} M_\tau & \xrightarrow{f_{\sigma,\tau}} & M_{\sigma\tau} \\ {\scriptstyle \alpha_\sigma\otimes\alpha_\tau}\downarrow & & \downarrow{\scriptstyle \alpha_{\sigma\tau}} \\ M'_\sigma \otimes_{A_{d(\sigma)}} M'_\tau & \xrightarrow{f'_{\sigma\tau}} & M'_{\sigma\tau} \end{array} \tag{5}$$

is commutative for all $(\sigma, \tau) \in \Gamma_2$.

4.2. Lemma. *With the above notations, a morphism α from f to f' induces a homomorphism of graded rings α from (F, f) to (F', f'). If each α_e, $e \in \Gamma_0$, is surjective, then $\alpha(1) = 1$. Moreover, α is an isomorphism if and only if each α_e, $e \in \Gamma_0$, is bijective.*

Proof. By (5), α is multiplicative. Let n denote the multiplicative identity of (F, f) and set $n' = \alpha(n)$. Fix $e \in \Gamma_0$ and take $y \in M'_e$. If α_e is surjective, then there is $x \in M_e$ such that $\alpha_e(x) = y$. Hence $n' \cdot y = \alpha(n) \cdot \alpha(x) = \alpha(n \cdot x) = \alpha(x) = y$. Analogously, $y \cdot n' = y$. Since $e \in \Gamma_0$ was arbitrarily chosen, n' is a multiplicative identity of (F', f'). The last statement of the lemma is obvious. ∎

By the above lemma, the isomorphism class of (F, f) does not depend on the choice of the bimodules M_σ, $\sigma \in \Gamma$. In fact, if $F(\sigma) = [M_\sigma] = [M'_\sigma]$, $\sigma \in \Gamma$, then there exist $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule isomorphisms $\alpha_\sigma : M'_\sigma \to M_\sigma$, $\sigma \in \Gamma$. If we now set $f'_{\sigma,\tau} = \alpha_{\sigma\tau}^{-1} \circ f_{\sigma,\tau} \circ (\alpha_\sigma \otimes \alpha_\tau)$ for $(\sigma, \tau) \in \Gamma_2$, then f' is a factor set associated to F and (5) commutes.

We end this section by showing that there are lots of natural examples of groupoid crossed products.

4.3. Example. (a) If T is an associative ring, then the groupoid ring $T[\Gamma]$ of T over Γ is defined to be the set of all formal sums $\sum_{\sigma\in\Gamma} t_\sigma\sigma$ with $t_\sigma \in T$, $\sigma \in \Gamma$, and $t_\sigma = 0$ for almost all $\sigma \in \Gamma$. Addition is defined pointwise and multiplication is defined by the T-linear extension of the rule $\sigma \cdot \tau = \sigma\tau$ if $(\sigma, \tau) \in \Gamma_2$ and $\sigma \cdot \tau = 0$ otherwise. The grading is, of course, defined by

$T[\Gamma]_\sigma = T\sigma$, $\sigma \in \Gamma$. By defining the maps $f_{\sigma,\tau}$ via the multiplication in $T[\Gamma]$, it is easy to see that $T[\Gamma]$ is a groupoid crossed product. Note also that if Γ is a group, then $T[\Gamma]$ is the usual group ring of T over Γ. On the other hand, if $\Gamma = I \times I$, where I is a finite set of cardinality n, and Γ is equipped with the partial binary operation defined by letting $(i,j)(k,l)$ be defined and equal to (i,l) precisely when $j = k$, then $T[\Gamma]$ is the ring of $n \times n$ matrices over T. For results concerning the separability and semisimplicity of groupoid rings, see [12].

(b) Let K/F be a separable finite field extension and $\overline{F}$ a fixed algebraic closure of F containing K. Let L denote the normal closure of K/F in $\overline{F}$ and let Gal be the Galois group of L/F. Let $K_1, \ldots, K_n$ be the different conjugate fields of K under the action of Gal. Furthermore, if $1 \le i, j \le n$, then let Γ_{ij} denote the set of $\sigma \in \mathrm{Gal}$ such that $\sigma(K_j) = K_i$. If we let $\Gamma = \bigcup_{i,j} \Gamma_{i,j}$, then Γ is, in a natural way, a groupoid. We will use the notation $r(\sigma) = i$ and $d(\sigma) = j$ for $\sigma \in \Gamma_{ij}$. Let $(K/F, f)$ denote the direct sum $\bigoplus_{\sigma\in\Gamma} K_{r(\sigma)} u_\sigma$ equipped with the multiplication $au_\sigma bu_\tau = a\sigma(b) f_{\sigma,\tau} u_{\sigma\tau}$ if $(\sigma,\tau) \in \Gamma_2$ and $au_\sigma bu_\tau = 0$ otherwise, for all $a \in K_{r(\sigma)}$, $b \in K_{r(\tau)}$, $\sigma, \tau \in \Gamma$, where $f_{\sigma,\tau} \in K_{r(\sigma)}$, $(\sigma,\tau) \in \Gamma_2$, and $f_{\sigma,\tau\varrho}\sigma(f_{\tau,\varrho}) = f_{\sigma\tau,\varrho} f_{\sigma,\tau}$ for all $(\sigma,\tau,\varrho) \in \Gamma_3$. By interpreting this equation as a commutative diagram of the type (4), it is easy to see that the F-algebra $(K/F, f)$ is a groupoid crossed product. Note also that if K/F is actually Galois, then the construction of $(K/F, f)$ coincides with the classical construction of a crossed product relative the extension K/F (see e.g. [15]). For results concerning separability of crossed product algebras (and heredity of crossed product orders) defined by separable field extensions, see [13].

5. Cohomology of groupoids. For use in the last section, we now recall the cohomology theory for groupoids. For details see e.g. [16]. Let $X = (X_e)_{e\in\Gamma_0}$ be a collection of abelian groups. Let $I(X)$ denote the set of isomorphisms $\varphi_{e,e'} : X_{e'} \to X_e$, $e, e' \in \Gamma_0$. Note that $I(X)$ is in a natural way a groupoid with respect to composition and that there is a canonical bijection $\Gamma_0 \to I(X)_0$, where $e \mapsto \mathrm{id}_{X_e}$, $e \in \Gamma_0$. Furthermore, X is called a *left Γ-module* if there is a homomorphism of groupoids $L : \Gamma \to I(X)$ such that $L(e) = \mathrm{id}_{X_e}$, $e \in \Gamma_0$. Let X' denote the disjoint union of the X_e, $e \in \Gamma_0$. Let n be a nonnegative integer. A function $c : \Gamma_n \to X'$ is called an *n-cochain* if for all $(\sigma_0, \ldots, \sigma_{n-1}) \in \Gamma_n$, the following two conditions hold:

(i) $c(\sigma_0, \ldots, \sigma_{n-1}) \in X_{r(\sigma_0)}$;
(ii) if $0 \le i \le n-1$ and $\sigma_i \in \Gamma_0$, then $c(\sigma_0, \ldots, \sigma_{n-1}) = 0$.

The collection of n-cochains forms an abelian group which is denoted by $C^n(\Gamma, X)$. Define a map δ^n from $C^n(\Gamma, X)$ to $C^{n+1}(\Gamma, X)$ by $\delta^0(\sigma_0) =$

$L(\sigma_0)(c(r(\sigma_0))) - c(d(\sigma_0))$ for $n = 0$, $c \in C^0(\Gamma, X)$ and $\sigma_0 \in \Gamma_0$, and

$$\delta^n(c)(\sigma_0, \ldots, \sigma_n) = L(\sigma_0)(c(\sigma_1, \ldots, \sigma_n)) + \sum_{i=1}^{n}(-1)^i c(\sigma_0, \ldots, \sigma_{i-2}, \sigma_{i-1}\sigma_i, \sigma_{i+1}, \ldots, \sigma_n) + (-1)^{n+1}c(\sigma_0, \ldots, \sigma_{n-1})$$

for $n > 0$, $c \in C^n(\Gamma, X)$ and $(\sigma_0, \ldots, \sigma_n) \in \Gamma_{n+1}$. Then $(C^n(\Gamma, X), \delta^n)$ is a cochain complex. Now set $Z^n(\Gamma, X) = \ker(\delta^n)$ and for positive n, set $B^n(\Gamma, X) = \mathrm{im}(\delta^{n-1})$. Then $H^n(\Gamma, X) := Z^n(\Gamma, X)/B^n(\Gamma, X)$ is called the nth *cohomology group* of the Γ-module X.

6. Cohomology and strongly graded rings. In this section, we obtain a bijection from the second cohomology group to the collection of equivalence classes of rings strongly graded by the groupoid (see Theorem 6.3). This bijection is a generalization from the group graded situation to the groupoid graded case. To accomplish this, we need some more notations and a lemma.

Let $F : \Gamma \to \mathrm{PIC}$ be a groupoid homomorphism. We set $F(\sigma) = [M_\sigma]$, $\sigma \in \Gamma$, where M_σ is an invertible $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule. By Proposition 2.2 there is a homomorphism of groupoids l from PIC to ISOC. If we set $A = (C(A_e)^*)_{e \in \Gamma_0}$ and $L = l \circ F$, then, in the language of Section 5, A is a left Γ-module.

6.1. Lemma. *Let f and g be factor sets associated to F.*

(a) *If $q \in Z^2(\Gamma, A)$, then fq is a factor set associated to F.*
(b) *There is $q \in Z^2(\Gamma, A)$ such that $g = qf$.*
(c) *A cocycle $q \in Z^2(\Gamma, A)$ belongs to $B^2(\Gamma, A)$ if and only if there is a graded ring isomorphism α from (F, f) to (F, qf) such that each graded restriction α_σ to M_σ, $\sigma \in \Gamma$, is an $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule isomorphism.*

Proof. (a) Set $h = fq$. We have to verify that (4) commutes for h. Take $(\sigma, \tau, \varrho) \in \Gamma_3$, $x \in M_\sigma$, $y \in M_\tau$ and $z \in M_\varrho$. Then

$$\begin{aligned}
&(h_{\sigma\tau,\varrho} \circ (h_{\sigma,\tau} \otimes \mathrm{id}_{M_\sigma}))(x \otimes y \otimes z) \\
&\quad = q_{\sigma\tau,\varrho} q_{\sigma,\tau}(f_{\sigma\tau,\varrho} \circ (f_{\sigma,\tau} \otimes \mathrm{id}_{P_\sigma}))(x \otimes y \otimes z) \\
&\quad = (q_{\sigma,\tau\varrho}\sigma(q_{\tau,\varrho}))(f_{\sigma,\tau\varrho} \circ (\mathrm{id}_{M_\sigma} \otimes f_{\tau,\varrho}))(x \otimes y \otimes z) \\
&\quad = (q_{\sigma,\tau\varrho}\sigma(q_{\tau,\varrho}))f_{\sigma,\tau\varrho}(x \otimes f_{\tau,\varrho}(y \otimes z)) = h_{\sigma,\tau\varrho}(\sigma(q_{\tau,\varrho}x \otimes f_{\tau,\varrho}(y \otimes z))) \\
&\quad = h_{\sigma,\tau\varrho}(xq_{\tau,\varrho} \otimes f_{\tau,\varrho}(y \otimes z)) = h_{\sigma,\tau\varrho}(x \otimes h_{\tau,\varrho}(y \otimes z)) \\
&\quad = (h_{\sigma,\tau\varrho} \circ (\mathrm{id}_{M_\sigma} \otimes h_{\tau,\varrho}))(x \otimes y \otimes x).
\end{aligned}$$

(b) Set $h_{\sigma,\tau} = g_{\sigma,\tau} \circ f_{\sigma,\tau}^{-1}$ for all $(\sigma, \tau) \in \Gamma_2$. Then $h_{\sigma,\tau}$ is an $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule automorphism of $M_{\sigma\tau}$. Hence, by Proposition 2.2, there is

$q_{\sigma,\tau} \in C(A_{r(\sigma)})^*$ such that $h_{\sigma,\tau}(x) = q_{\sigma,\tau}x$, $x \in M_{\sigma\tau}$. By (4) it follows that $q \in Z^2(\Gamma, A)$.

(c) Suppose now that $q \in B^2(\Gamma, A)$. Then there is $c \in C^1(\Gamma, A)$ such that $q_{\sigma,\tau} = \sigma(c_\tau)c_\sigma c_{\sigma\tau}^{-1}$, $(\sigma,\tau) \in \Gamma_2$. Define a map α from (F, qf) to (F, f) by $\alpha(x) = c_\sigma x$, $x \in M_\sigma$. If $x \in M_\sigma$, $y \in M_\tau$ and $(\sigma,\tau) \in \Gamma_2$, then $\alpha(xy) = c_{\sigma\tau}q_{\sigma,\tau}f_{\sigma,\tau}(x \otimes y) = q_{\sigma,\tau}^{-1}\sigma(c_\tau)c_\sigma q_{\sigma,\tau}f_{\sigma,\tau}(x \otimes y) = f_{\sigma,\tau}(c_\sigma x \otimes c_\tau y) = \alpha(x)\alpha(y)$. Clearly, each α_σ, $\sigma \in \Gamma$, is an $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule isomorphism.

On the other hand, suppose that there is an isomorphism β of graded rings from (F, qf) to (F, f) such that each β_σ, $\sigma \in \Gamma$, is an $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule isomorphism. Hence (as above) there are $d_\sigma \in C(A_{r(\sigma)})^*$ such that $\beta_\sigma(x) = d_\sigma x$, $x \in M_\sigma$. Therefore, for all $x \in M_\sigma$, $y \in M_\tau$, $(\sigma,\tau) \in \Gamma_2$, we obtain $\beta(xy) = \beta(x)\beta(y) \Leftrightarrow d_{\sigma\tau}q_{\sigma,\tau}f_{\sigma,\tau}(x \otimes y) = f_{\sigma,\tau}(d_\sigma x \otimes d_\tau y) \Leftrightarrow d_{\sigma\tau}q_{\sigma,\tau}f_{\sigma,\tau}(x \otimes y) = d_\sigma \sigma(d_\tau) f_{\sigma,\tau}(x \otimes y)$. Thus $q \in B^2(\Gamma, A)$. ■

6.2. Corollary. *If f is a factor set associated to F, then the map from $Z^2(\Gamma, A)$ to the collection of factor sets associated to F, defined by $q \mapsto qf$, is bijective.*

Finally, to state the main result of this article, we need some more notations. If f and f' are factor sets associated to F, then we write $(F, f) \approx (F, f')$ if there is an isomorphism of graded rings from (F, f) to (F, f') such that each graded restriction to M_σ, $\sigma \in \Gamma$, is an $A_{r(\sigma)}$-$A_{d(\sigma)}$-bimodule isomorphism. Let (F) denote the collection of equivalence classes of groupoid crossed products (F, f) modulo $\approx$, where f runs over all factor sets associated to F.

6.3. Theorem. *If f is a factor set associated to F, then the map from $H^2(\Gamma, A)$ to (F), defined by $q \mapsto qf$, is bijective.*

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