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QUINTASYMPTOTIC PRIMES, LOCAL COHOMOLOGY AND IDEAL TOPOLOGIES

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Abstract. Let Φ be a system of ideals on a commutative Noetherian ring R, and let S be a multiplicatively closed subset of R. The first result shows that the topologies defined by $\{I_a\}_{I\in\Phi}$ and $\{S(I_a)\}_{I\in\Phi}$ are equivalent if and only if S is disjoint from the quintasymptotic primes of Φ. Also, by using the generalized Lichtenbaum–Hartshorne vanishing theorem we show that, if $(R,\mathfrak{m})$ is a d-dimensional local quasi-unmixed ring, then $H^d_\Phi(R)$, the dth local cohomology module of R with respect to Φ, vanishes if and only if there exists a multiplicatively closed subset S of R such that $S\cap\mathfrak{m}\neq\emptyset$ and the $S(\Phi)$-topology is finer than the Φ_a-topology.

1. Introduction. Throughout this paper, all rings considered will be commutative and Noetherian and will have non-zero identity elements. Such a ring will be denoted by R and a typical ideal of R will be denoted by I. Let $(\Lambda,\leq)$ be a (non-empty) directed partially ordered set. A *system of ideals of R over Λ* is an inverse family $\Phi=\{I_\alpha:\alpha\in\Lambda\}$ of ideals of R with the additional property that, for all $\alpha,\gamma\in\Lambda$, there exists $\delta\in\Lambda$ such that $I_\delta\subseteq I_\alpha I_\gamma$. Systems of ideals are a very useful generalization of the sets of powers of an ideal I in a ring R, and there are many important systems of ideals that are not powers. They have played an important role in many research papers, and there are numerous results concerning them in the literature (e.g., see [2], [3] and [7]).

Let Φ denote a system of ideals (of R) and S a multiplicatively closed subset of R. For an ideal I of R, the *S-component* of I, denoted by $S(I)$, is defined to be the union of $(I:_R s)$, where s varies in S. The *integral closure* of I in R is the ideal

$$I_a := \{x\in R : x \text{ satisfies an equation of the form } x^n+b_1x^{n-1}+\cdots+b_n=0,\ \text{where } b_i\in I^i \text{ for } i=1,\dots,n\}.$$

Also, the *radical* of I, denoted by $\operatorname{Rad}(I)$, is defined to be the ideal $\{x\in R : x^n\in I \text{ for some } n\in\mathbb{N}\}$. Furthermore, we denote by $V(\Phi)$ the subset

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$\bigcup_{I\in\Phi} V(I)$ of Spec R, where $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$. Finally, we define

$$\Phi_a := \{I_a : I \in \Phi\} \quad \text{and} \quad S(\Phi) := \{S(I) : I \in \Phi\}.$$

It is easily seen that the sets $S(\Phi)$, Φ_a and $S(\Phi_a)$ induce topologies on R which are called the $S(\Phi)$-*symbolic*, Φ_a-*integral closure* and $S(\Phi_a)$-*symbolic integral closure topologies*, respectively. The purpose of the present paper is to study the relationship between the vanishing of the general local cohomology modules $H^i_\Phi(R)$, and the comparison of the topologies induced by the sets $S(\Phi)$, Φ_a and $S(\Phi_a)$.

Let $(R, \mathfrak{m})$ be a local ring and let N be a non-zero finitely generated R-module. Then we denote by R^* (respectively N^*) the completion of R (respectively N) with respect to the $\mathfrak{m}$-adic topology. In particular, for any $\mathfrak{p} \in \operatorname{Spec} R$, $R^*_\mathfrak{p}$ and $N^*_\mathfrak{p}$ denote the $\mathfrak{p}R_\mathfrak{p}$-adic completions of $R_\mathfrak{p}$ and $N_\mathfrak{p}$, respectively. Also, we denote by $\operatorname{mAss}_R N$, the set of *minimal elements* of $\operatorname{Ass}_R N$. The ring R is said to be *quasi-unmixed* if $\dim R^*/\mathfrak{p} = \dim R$ for any $\mathfrak{p} \in \operatorname{mAss} R^*$. More generally, if R is not necessarily local, it is a *locally quasi-unmixed ring* if $R_\mathfrak{p}$ is quasi-unmixed for any $\mathfrak{p} \in \operatorname{Spec} R$. For any prime ideal $\mathfrak{p}$ of R, and $k \geq 0$ an integer, we define

$$\mathfrak{p}^{\langle k\rangle} = \bigcup_{s\in R\setminus\mathfrak{p}} ((\mathfrak{p}^k)_a : s).$$

For any ideal I of R, the set

$$\overline{Q}^*(I) := \{\mathfrak{p} \in \operatorname{Spec} R : \text{there is a } z \in \operatorname{mAss} R^*_\mathfrak{p} \text{ with} \operatorname{Rad}(IR^*_\mathfrak{p} + z) = \mathfrak{p}R^*_\mathfrak{p}\},$$

the *quintasymptotic prime ideals* of I, was systematically studied by S. McAdam in [15]. He proved that if I is an ideal of a Noetherian ring R and S is a multiplicatively closed subset of R, then S is disjoint from the quintasymptotic prime ideals of I if and only if the topologies defined by the filtrations $\{(I^n)_a\}_{n\geq1}$ and $\{S(I^n)_a\}_{n\geq1}$ are equivalent (cf. [15, Theorem 1.5]).

The main purpose of the second section is to introduce the concept of the quintasymptotic prime ideals with respect to a system of ideals and generalize McAdam's theorem to arbitrary systems of ideals Φ in a Noetherian ring R. More precisely, we will show that for any system of ideals Φ of a Noetherian ring R, and any multiplicatively closed subset S in R, the following conditions are equivalent:

(i) The $S(\Phi)$-symbolic topology is finer than the Φ_a-integral closure topology.
(ii) The $S(\Phi_a)$-symbolic integral closure topology is finer than the Φ_a-integral closure topology.
(iii) S is disjoint from the quintasymptotic primes of Φ.

We denote by $\mathcal{C}(R)$ the category of all R-modules and R-homomorphisms between them. The system of ideals Φ determines the Φ-torsion functor $\Gamma_\Phi : \mathcal{C}(R) \to \mathcal{C}(R)$. This is a subfunctor of the identity functor on $\mathcal{C}(R)$, for which $\Gamma_\Phi(G) = \{g \in G : \mathfrak{a}g = 0 \text{ for some } \mathfrak{a} \in \Phi\}$ for each R-module G. Note that in [2], Γ_Φ is denoted by L_Φ and called the "general local cohomology functor with respect to Φ". For each $i \geq 0$, the ith right derived functor of Γ_Φ is denoted by H^i_Φ. See [5] and [8] for the basic results on local cohomology.

Recently, Marti-Farre generalized Schenzel's theorem (cf. [17, Corollary 4.3]) to Noetherian quasi-unmixed local rings. Namely, he showed that if I is an ideal in a d-dimensional Noetherian quasi-unmixed local ring $(R, \mathfrak{m})$, then $H^d_I(R) = 0$ if and only if there exists a multiplicatively closed subset S of R such that $\mathfrak{m} \cap S \neq \emptyset$, and that the topologies defined by $\{(I^n)_a\}_{n\geq 1}$ and $\{S(I^n)_a\}_{n\geq 1}$ are equivalent (cf. [12, Proposition 2.1]). In the third section we establish the relationship between the vanishing of the general local cohomology modules $H^i_\Phi(R)$ and the comparison of the $S(\Phi)$-symbolic and the Φ_a-integral closure topologies. Then we obtain the following result which generalizes the characterization given by Marti-Farre.

Let $(R, \mathfrak{m})$ be a local quasi-unmixed ring with $\dim R = d$. Then for any system of ideals Φ of R, the following conditions are equivalent:

(i) $H^d_\Phi(R) = 0$.

(ii) There is a multiplicatively closed subset S of R such that $\mathfrak{m} \cap S \neq \emptyset$ and the $S(\Phi)$-symbolic topology is finer than the Φ_a-integral closure topology.

At the end of Section 3 we give some applications to generalized local cohomology.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, Φ will be an arbitrary system of ideals of R, and N will be a finitely generated R-module.

2. Quintasymptotic primes and ideal topologies. The purpose of this section is to introduce the concept of quintasymptotic primes of Φ with respect to a module N over R. The main point of our investigations is to establish a relationship between the topologies induced by $\{I_a\}_{I\in\Phi}$, $\{S(I)\}_{I\in\Phi}$ and $\{S(I_a)\}_{I\in\Phi}$ by using the quintasymptotic primes of Φ with respect to R. The main results are Theorems 2.5 and 2.8. Before stating them, let us give a definition.

Definition. A prime ideal $\mathfrak{p}$ of R is called a *quintasymptotic prime ideal* of Φ with respect to N if there exists $z \in \operatorname{mAss}_{R^*_\mathfrak{p}} N^*_\mathfrak{p}$ such that $\operatorname{Rad}(IR^*_\mathfrak{p} + z) = \mathfrak{p}R^*_\mathfrak{p}$ for all proper ideals $I \in \Phi$. The set of quintasymptotic primes of Φ with respect to N is denoted by $\overline{Q}^*(\Phi, N)$. Note that $\overline{Q}^*(\Phi, N) \subseteq \bigcap_{I\in\Phi}\overline{Q}^*(I, N)$, and if J is a proper ideal of R and $\Phi = \{J^n\}_{n\geq 0}$, then $\overline{Q}^*(\Phi, N) = \overline{Q}^*(J, N)$, where $\overline{Q}^*(J, N)$ is the set of quintasymptotic prime ideals of J with respect to N (see [1, 3.1]).

LEMMA 2.1. *Let S be a multiplicatively closed subset of R. Then*:

(i) *For any prime ideal $\mathfrak{p}$ of R disjoint from S, $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$ if and only if $S^{-1}\mathfrak{p} \in \overline{Q}^*(S^{-1}\Phi, S^{-1}N)$. Here $S^{-1}\Phi := \{S^{-1}I : I \in \Phi\}$ is a system of ideals of $S^{-1}R$.*

(ii) *If $\mathfrak{p} \in \operatorname{mAss}_R N/IN$ for all proper ideals $I \in \Phi$, then $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$.*

(iii) *Let Ψ be a system of ideals of R such that for any $\mathfrak{p} \in \operatorname{Spec} R$, the ideals $IR_{\mathfrak{p}}$ and $JR_{\mathfrak{p}}$ are proper for all $I \in \Phi$ and $J \in \Psi$. If Ψ is comparable to Φ, then $\overline{Q}^*(\Phi, N) = \overline{Q}^*(\Psi, N)$.*

(iv) *If $z \in \operatorname{mAss}_R N$, and $\mathfrak{p} \in \operatorname{mAss}_R R/I + z$ for all proper ideals $I \in \Phi$, then $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$.*

Proof. Statement (i) follows from the isomorphisms $(S^{-1}N)_{S^{-1}\mathfrak{p}} \cong N_{\mathfrak{p}}$ and $(S^{-1}R)_{S^{-1}\mathfrak{p}} \cong R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset$. For (ii), let $\mathfrak{p} \in \operatorname{mAss}_R N/IN$ for all proper ideals $I \in \Phi$. Then it is easy to see that $\mathfrak{p}R^*_{\mathfrak{p}} = \operatorname{Rad}(IR^*_{\mathfrak{p}} + \operatorname{Ann}_{R^*_{\mathfrak{p}}} N^*_{\mathfrak{p}})$ for all proper ideals $I \in \Phi$. Now it is easy to show that $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$.

In order to show (iii) assume $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$. Then there exists $z \in \operatorname{mAss}_{R^*_{\mathfrak{p}}} N^*_{\mathfrak{p}}$ such that $\mathfrak{p}R^*_{\mathfrak{p}} = \operatorname{Rad}(IR^*_{\mathfrak{p}} + z)$ for all proper ideals $I \in \Phi$. Now, let J be an arbitrary proper ideal in Ψ. Then there is an ideal K in Φ such that $K \subseteq J$. Therefore $\mathfrak{p}R^*_{\mathfrak{p}} = \operatorname{Rad}(JR^*_{\mathfrak{p}} + z)$ for some $z \in \operatorname{mAss}_{R^*_{\mathfrak{p}}} N^*_{\mathfrak{p}}$ and for all proper ideals $J \in \Psi$, and so $\mathfrak{p} \in \overline{Q}^*(\Psi, N)$. The opposite inclusion is proved in a similar way.

Finally, to prove (iv) let $z \in \operatorname{mAss}_R N$ and assume that $\mathfrak{p} \in \operatorname{mAss}_R R/I + z$ for all proper ideals $I \in \Phi$. In view of (i), we may assume that $(R, \mathfrak{p})$ is local. Let $\mathfrak{q} \in \operatorname{Spec} R$ be a minimal ideal of zR^*. Then $\mathfrak{q} \in \operatorname{Ass}_{R^*} R^*/zR^*$ and $\mathfrak{p}R^* \in \operatorname{mAss}_{R^*} R^*/IR^* + \mathfrak{q}$ for all proper ideals $I \in \Phi$. Now by [4, Corollary 1, p. 280], $\mathfrak{q} \in \operatorname{mAss}_{R^*} N^*$. Accordingly, $\mathfrak{p}R^* = \operatorname{Rad}(IR^* + \mathfrak{q})$ for some $\mathfrak{q} \in \operatorname{mAss}_{R^*} N^*$ and for all proper ideals $I \in \Phi$. Consequently, $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$, as desired. ∎

Before stating the next result we fix some notation. For an ideal J of R, we use $\Phi + J$ to denote $\{I + J : I \in \Phi\}$. It is easy to see that $\Phi + J$ is a system of ideals of R.

PROPOSITION 2.2. *Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$ if and only if there exists $\mathfrak{q} \in \operatorname{mAss}_R N$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p}/\mathfrak{q} \in \overline{Q}^*(\Phi + \mathfrak{q}/\mathfrak{q}, R/\mathfrak{q})$.*

Proof. In view of Lemma 2.1(i), we may assume that R is local at $\mathfrak{p}$. Let $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$. Then there exists $z \in \operatorname{mAss}_{R^*} N^*$ such that $\operatorname{Rad}(IR^* + z) = \mathfrak{p}R^*$ for all proper ideals $I \in \Phi$. In view of [4, Corollary 1, p. 280], we have $w := z \cap R \in \operatorname{mAss}_R N$ and $z \in \operatorname{mAss}_{R^*}(R^*/wR^*)$. Hence $z/wR^* \in$

$\mathrm{mAss}_{R^*/wR^*}(R^*/wR^*)$ and $\mathrm{Rad}((I+\mathfrak{q}/\mathfrak{q})R^*/wR^* + z/wR^*) = \mathfrak{p}R^*/wR^*$ for all proper ideals $I \in \Phi$, and so $\mathfrak{p}/w \in \overline{Q}^*(\Phi + w/w, R/w)$, as desired.

Conversely, let $\mathfrak{q} \in \mathrm{mAss}_R N$ with $\mathfrak{p} \supseteq \mathfrak{q}$ and $\mathfrak{p}/\mathfrak{q} \in \overline{Q}^*(\Phi + \mathfrak{q}/\mathfrak{q}, R/\mathfrak{q})$. Then there exists $z \in \operatorname{Spec} R^*$ such that $z/\mathfrak{q}R^* \in \mathrm{mAss}_{R^*/\mathfrak{q}R^*} R^*/\mathfrak{q}R^*$ and $\mathrm{Rad}((IR^* + \mathfrak{q}R^*)/\mathfrak{q}R^* + z/\mathfrak{q}R^*) = \mathfrak{p}R^*/\mathfrak{q}R^*$ for all proper ideals $I \in \Phi$. Consequently, $z \in \mathrm{mAss}_{R^*} R^*/\mathfrak{q}R^*$ and $\mathrm{Rad}(IR^* + z) = \mathfrak{p}R^*$ for all proper ideals $I \in \Phi$. It is easy to see that $z \in \mathrm{mAss}_R N^*$, and so $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$. This completes the proof. ∎

The following lemma, which is a generalization of Chevalley's theorem (see [16, Theorem 30.1] and [4, Ch. IV, Section 2.5, Corollary 4]), plays a key role in this section.

LEMMA 2.3 (see [11, Lemma 3.3] and [7, Proposition 2.9]). *Let $(R, \mathfrak{m})$ be a complete local ring and let M be a submodule of a finitely generated R-module N. Let $\{N_i : i \in \Lambda\}$, where Λ is some index set, be a set of submodules of N such that for all $j, k \in \Lambda$, there exists $i \in \Lambda$ for which $N_i \subseteq N_j \cap N_k$. Suppose that the family $\{M + N_i : i \in \Lambda\}$ has a minimal element. Then there exists $j \in \Lambda$ such that $N_j \subseteq M + \bigcap_{i \in \Lambda} N_i$.*

Before we state Theorem 2.5, which is one of our main tools, we prove the following proposition that will be used in the proof of that theorem. Proposition 2.4 gives a characterization of $\overline{Q}^*(\Phi)$.

PROPOSITION 2.4. *Let $\mathfrak{p} \in V(\Phi)$. Then the following conditions are equivalent*:

(i) $\mathfrak{p} \in \overline{Q}^*(\Phi)$.
(ii) *There is an integer $k \geq 0$ such that $I :_R \langle \mathfrak{p} \rangle \not\subseteq \mathfrak{p}^{\langle k \rangle}$ for all $I \in \Phi$.*
(iii) *There is an integer $k \geq 0$ such that $I_a :_R \langle \mathfrak{p} \rangle \not\subseteq \mathfrak{p}^{\langle k \rangle}$ for all $I \in \Phi$.*

Proof. (i)⇒(ii). Let $\mathfrak{p} \in \overline{Q}^*(\Phi)$. Then in view of Lemma 2.1(i) and [13, Theorem 7.4(iii)] and the fact that $(\mathfrak{p}^k R_{\mathfrak{p}})_a \cap R = \mathfrak{p}^{\langle k \rangle}$, without loss of generality, we may assume that $(R, \mathfrak{p})$ is local. Then there exists $z \in \mathrm{mAss}\, R^*$ such that for all proper ideals $I \in \Phi$ we have $\mathrm{Rad}(IR^* + z) = \mathfrak{p}R^*$. By [15, Lemma 3.1] there exists a non-zero $x \in R^* \setminus z$ such that for every ideal J of R^* with $\mathrm{Rad}(J + z) = \mathfrak{p}R^*$, either $x \in J$ or $\mathfrak{p}R^* \in \mathrm{Ass}_R R^*/J$. Then, by [15, Lemma 1.4(i)], $x \notin \bigcap_{n \geq 1} (\mathfrak{p}^n R^*)_a$. Hence for sufficiently large k, we have $x \notin (\mathfrak{p}^k R^*)_a$.

Now suppose, to the contrary, that (ii) is not true. Then, for such k, there is an ideal $I_k \in \Phi$ with $I_k :_R \langle \mathfrak{p} \rangle \subseteq (\mathfrak{p}^k)_a$. Then by [13, Theorem 7.4(iii)] and [14, Lemma 3.15] we deduce that $x \notin I_k R^* :_{R^*} \langle \mathfrak{p}R^* \rangle$. Because of $\mathrm{Rad}(I_k R^* :_{R^*} \langle \mathfrak{p}R^* \rangle + z) = \mathfrak{p}R^*$ by [15, Lemma 3.1] we have $\mathfrak{p}R^* \in \mathrm{Ass}_{R^*} R^*/I_k R^* :_{R^*} \langle \mathfrak{p}R^* \rangle$. (Note that $\mathrm{Rad}(I_k R^* + z) = \mathfrak{p}R^*$.) Consequently, $\mathfrak{p} \in \operatorname{Ass}_R R/I_k :_R \langle \mathfrak{p} \rangle$, which provides the required contradiction. The implication (ii)⇒(iii) is obviously true.

In order to prove that (iii)⇒(i), suppose the contrary, that is, $\mathfrak{p} \notin \overline{Q}^*(\Phi)$. Since $\overline{Q}^*(\Phi)$ and the integral closure behave well under localization (see Lemma 2.1(i) and [18, Lemma 2.3]), without loss of generality, we may assume that $(R, \mathfrak{p})$ is local. Now, let $\operatorname{mAss} R^* = \{z_1, \dots, z_n\}$. Then, for each $i = 1, \dots, n$, there exists $I_i \in \Phi$ such that $\operatorname{Rad}(I_i R^* + z_i) \neq \mathfrak{p} R^*$. Since Φ is a system of ideals, it follows that there is $I \in \Phi$ such that $I \subseteq \bigcap_{i=1}^n I_i$. Then $\operatorname{Rad}(IR^* + z_i) \neq \mathfrak{p}R^*$ for every $i = 1, \dots, n$. Again, because Φ is a system of ideals, for each $n, k \in \mathbb{N}$, one easily sees that there is $J \in \Phi$ such that $J \subseteq I^n \cap \mathfrak{p}^k$. Hence

$$\bigcap_{J \in \Phi} ((JR^*)_a :_{R^*} \langle \mathfrak{p}R^* \rangle) \subseteq \bigcap_{n \geq 1} ((I^n R^*)_a :_{R^*} \langle \mathfrak{p}R^* \rangle).$$

So, [15, Lemma 3.2(c)] implies that $\bigcap_{J \in \Phi} ((JR^*)_a :_{R^*} \langle \mathfrak{p}R^* \rangle) = \bigcap_{i=1}^n z_i = \operatorname{nil}(R^*)$. Using Lemma 2.3, we see that for all $k \geq 0$ there exists $J \in \Phi$ such that $(JR^*)_a :_{R^*} \langle \mathfrak{p}R^* \rangle \subseteq \operatorname{nil}(R^*) + (\mathfrak{p}^k R^*)_a$. Note that the module $((\mathfrak{p}^k R^*)_a :_{R^*} \langle \mathfrak{p}R^* \rangle + (\mathfrak{p}^k R^*)_a)/(\mathfrak{p}^k R^*)_a$ has finite length. Therefore, for each integer $k \geq 0$ there is $J \in \Phi$ such that $(JR^*)_a :_{R^*} \langle \mathfrak{p}R^* \rangle \subseteq (\mathfrak{p}^k R^*)_a = (\mathfrak{p}R^*)^{\langle k \rangle}$; note that $\mathfrak{p}R^*$ is the unique maximal ideal of R^*. Now, it is easy to see that [14, Lemma 3.15] provides a contradiction. ∎

We are now in a position to state and prove the first main theorem of this section. Theorem 2.5 shows that $\overline{Q}^*(\Phi)$ behaves nicely with respect to faithfully flat extensions.

THEOREM 2.5. *Let $R \subseteq T$ be a faithfully flat extension of Noetherian rings.*

(i) *If $\mathfrak{q} \in \overline{Q}^*(\Phi T)$, then $\mathfrak{q} \cap R \in \overline{Q}^*(\Phi)$.*
(ii) *If $\mathfrak{p} \in \overline{Q}^*(\Phi)$, then for each minimal prime ideal $\mathfrak{q}$ of $\mathfrak{p}T$, $\mathfrak{q} \in \overline{Q}^*(\Phi T)$ and $\mathfrak{q} \cap R = \mathfrak{p}$.*

Proof. (i) Let $\mathfrak{q} \in \overline{Q}^*(\Phi T)$. It follows from Proposition 2.4 that there is an integer $k \geq 0$ such that $(IT :_T \langle \mathfrak{q} \rangle) \nsubseteq \mathfrak{q}^{\langle k \rangle}$ for all $I \in \Phi$. Let $\mathfrak{q} \cap R = \mathfrak{p}$. Then in view of Proposition 2.4 it is sufficient to show that $(I :_R \langle \mathfrak{p} \rangle) \nsubseteq \mathfrak{p}^{\langle k \rangle}$. Suppose this is not the case. Then, by [13, Theorem 7.4] and [14, Lemma 3.15], it is easy to see that $(IT :_T \langle \mathfrak{q} \rangle) \subseteq \mathfrak{q}^{\langle k \rangle}$, which is a contradiction. So $\mathfrak{q} \cap R \in \overline{Q}^*(\Phi)$ and (i) follows.

In order to prove (ii), let $\mathfrak{p} \in \overline{Q}^*(\Phi)$ and let $\mathfrak{q} \in \operatorname{Spec} T$ be a minimal prime of $\mathfrak{p}T$. Then $\mathfrak{p} \subseteq \mathfrak{q} \cap R$, so that using the going-down theorem we get $\mathfrak{p} = \mathfrak{q} \cap R$. Since $R_\mathfrak{p} \subseteq T_\mathfrak{q}$ is a faithfully flat extension (see [13, Theorem 9.5]), it is easy to see that so is $R^*_\mathfrak{p} \subseteq T^*_\mathfrak{q}$; and $\operatorname{Rad}(\mathfrak{p}T_\mathfrak{q}) = \mathfrak{q}T_\mathfrak{q}$. Moreover, because $\overline{Q}^*(\Phi)$ and $\overline{Q}^*(\Phi T)$ behave well under localization, we may assume that $(R, \mathfrak{p})$ and $(T, \mathfrak{q})$ are local rings, and $\operatorname{Rad}(\mathfrak{p}T) = \mathfrak{q}$. Then, since $\mathfrak{p} \in \overline{Q}^*(\Phi)$

by definition, there exists $z \in \operatorname{mAss} R^*$ such that $\operatorname{Rad}(IR^* + z) = \mathfrak{p}R^*$ for all proper ideals $I \in \Phi$. By [13, Theorem 23.2], there is $w \in \operatorname{mAss} T^*$ such that $w \cap R^* = z$. Then $zT^* \subseteq w$, and so $\operatorname{Rad}(IT^* + w) = \mathfrak{q}T^*$ for all proper ideals $I \in \Phi$. Therefore by definition $\mathfrak{q} \in \overline{Q}^*(IT)$ as required. ∎

COROLLARY 2.6. *Let $R \subseteq T$ be a faithfully flat extension of Noetherian rings and N a non-zero finitely generated R-module.*

(i) *If $\mathfrak{q} \in \overline{Q}^*(\Phi T, N \otimes_R T)$ then $\mathfrak{q} \cap R \in \overline{Q}^*(\Phi, N)$.*
(ii) *If $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$ then, for each minimal prime ideal $\mathfrak{q}$ of $\mathfrak{p}T$, $\mathfrak{q} \in \overline{Q}^*(IT, N \otimes_R T)$ and $\mathfrak{q} \cap R = \mathfrak{p}$.*

Proof. By the definition, ΦT is a system of ideals of T. In order to prove (i) let $\mathfrak{q} \in \overline{Q}^*(\Phi T, N \otimes_R T)$. Then, by Proposition 2.2, there exists $z \in \operatorname{mAss}_T(N \otimes_R T)$ such that $z \subseteq \mathfrak{q}$ and $\mathfrak{q}/z \in \overline{Q}^*(\Phi T + z/z, T/z)$. By [13, Theorem 23.2], there exists $w \in \operatorname{mAss}_R N$ such that $z \cap R = w$, and $\mathfrak{q}/wT \in \operatorname{mAss}_{T/wT}(T/wT)$. Hence $\mathfrak{q}/wT \in \overline{Q}^*(\Phi T + wT/wT, T/wT)$. Since T/wT is a faithfully flat extension of R/w, in view of Theorem 2.5, $\mathfrak{q}/wT \cap R/w \in \overline{Q}^*(\Phi + w/w, R/w)$. That is, $\mathfrak{q} \cap R/w \in \overline{Q}^*(\Phi + w/w, R/w)$. Consequently, by Proposition 2.2, we have $\mathfrak{q} \cap R \in \overline{Q}^*(\Phi, N)$ as required.

For the proof of (ii), let $\mathfrak{p} \in \overline{Q}^*(\Phi, N)$ and $\mathfrak{q} \in \operatorname{Spec} T$ be a minimal prime ideal of $\mathfrak{p}T$. Then, by Proposition 2.2, there is $z \in \operatorname{mAss}_R N$ such that $\mathfrak{p}/z \in \overline{Q}^*(\Phi + z/z, R/z)$. Since $\mathfrak{q}/zT$ is a minimal prime ideal of $\mathfrak{p}T/zT$, and T/zT is a faithfully flat extension of R/z, by Theorem 2.5 we have $\mathfrak{q}/zT \in \overline{Q}^*(\Phi T + zT/zT, T/zT)$. Hence by Proposition 2.2, there exists a minimal prime ideal w/zT of T/zT such that $\mathfrak{q}/w \in \overline{Q}^*(\Phi T + w/w, T/w)$. But, since $w \in \operatorname{mAss}_T(N \otimes_R T)$ by [4, Corollary 1, p. 280], we deduce that $\mathfrak{q} \in \overline{Q}^*(\Phi T, N \otimes_R T)$. This completes the proof. ∎

The following lemma, which is a consequence of [15, Lemma 1.4] and of the definition of a system of ideals, is of assistance in the proof of the second main theorem of this section.

LEMMA 2.7. *Let $(R, \mathfrak{m})$ be a local ring and let S be a multiplicatively closed subset of R such that $S \cap \mathfrak{m} = \emptyset$. Then $\bigcap_{I \in \Phi} S(I_a)$ is the intersection of all minimal primes of R, i.e., the nilradical* $\operatorname{nil}(R)$.

Proof. Let J be a proper ideal of R such that $J \in \Phi$. Then it is easy to see that $\bigcap_{I \in \Phi} S^{-1}(I_a) \subseteq \bigcap_{n \geq 0}(S^{-1}J^n)_a$. Hence $\bigcap_{I \in \Phi} S^{-1}(I_a) \subseteq \bigcap_{n \geq 0}(S^{-1}\mathfrak{m}^n)_a$. Since $S \cap \mathfrak{m} = \emptyset$, from [15, Lemma 1.4(i)] we deduce that $\bigcap_{n \geq 0}(S^{-1}\mathfrak{m}^n)_a = \operatorname{nil}(S^{-1}R)$. Hence we have $\bigcap_{I \in \Phi} S^{-1}(I_a) = \operatorname{nil}(S^{-1}R)$. Pulling back to R, and noting that S is disjoint from every minimal prime of R, we get $\bigcap_{I \in \Phi} S(I_a) = \operatorname{nil}(R)$, as desired. ∎

We are now ready to state and prove the second main theorem of this section, which gives a characterization of quintasymptotic primes in terms of the equivalence between the topologies induced by Φ_a, $S(\Phi)$ and $S(\Phi_a)$.

THEOREM 2.8. *Let S be a multiplicatively closed subset of R and assume that each element of $V(\Phi)$ contains an element of $\overline{Q}^*(\Phi)$. Then the following conditions are equivalent*:

(i) $S \subseteq R \setminus \bigcup\{\mathfrak{p} \in \overline{Q}^*(\Phi)\}$.
(ii) *The $S(\Phi_a)$-topology is finer than the topology induced by $\mathfrak{p}^{\langle n\rangle}, n \geq 1$, for all $\mathfrak{p} \in \overline{Q}^*(\Phi)$.*
(iii) *The $S(\Phi)$-topology is finer than the topology induced by $\mathfrak{p}^{\langle n\rangle}, n \geq 1$, for all $\mathfrak{p} \in \overline{Q}^*(\Phi)$.*
(iv) *The $S(\Phi_a)$-topology is finer than the Φ_a-topology.*
(v) *The $S(\Phi)$-topology is finer than the Φ_a-topology.*

Proof. (i)⇒(ii). Let $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \in \overline{Q}^*(\Phi)$ and let $k \geq 1$. We need to show that there exists an ideal $I \in \Phi$ such that $S(I_a) \subseteq \mathfrak{p}^{\langle k\rangle}$. To this end, let S' be the natural image of S in $R_\mathfrak{p}$. Then, in view of assumption (i) and Lemma 2.1, we have $S' \subseteq R_\mathfrak{p} \setminus \bigcup\{\mathfrak{q} \in \overline{Q}^*(\Phi R_\mathfrak{p})\}$. Here $\Phi R_\mathfrak{p} = \{IR_\mathfrak{p} : I \in \Phi\}$ is a system of ideals of $R_\mathfrak{p}$. Also, one easily sees that $S'((IR_\mathfrak{p})_a) \subseteq (\mathfrak{p}^k R_\mathfrak{p})_a$ implies that $S(I_a) \subseteq \mathfrak{p}^{\langle k\rangle}$ (see [18, Lemma 2.3]). Therefore we may assume that R is local at $\mathfrak{p}$. Furthermore, by [14, Lemma 3.15], we may assume in addition in view of [13, Theorem 7.4] and Theorem 2.5 that R is complete. Now, since $\mathfrak{p} \in \overline{Q}^*(\Phi)$, we see that $\mathfrak{p} \cap S = \emptyset$. Putting this together with Lemma 2.7, we find that $\bigcap_{I\in\Phi} S(I_a) = \operatorname{nil}(R)$. Therefore by Lemma 2.3, for each k there exists $I \in \Phi$ such that $S(I_a) \subseteq \operatorname{nil}(R) + (\mathfrak{p}^k)_a$. Because $\operatorname{nil}(R) \subseteq (\mathfrak{p}^k)_a$, it follows that $S(I_a) \subseteq (\mathfrak{p}^k)_a$, as required.

The implication (ii)⇒(iii) is obviously true. To prove (iii)⇒(i), let $\mathfrak{p} \in \overline{Q}^*(\Phi)$. We have to show that $\mathfrak{p} \cap S = \emptyset$. Suppose not and let $s \in S \cap \mathfrak{p}$. Then by Proposition 2.4 there is an integer $k \geq 0$ such that $(I :_R \langle\mathfrak{p}\rangle) \nsubseteq \mathfrak{p}^{\langle k\rangle}$ for all $I \in \Phi$. On the other hand, (iii) says that $S(J) \subseteq (\mathfrak{p}^k)_a$ for some $J \in \Phi$. Consequently, for such J, we have $(J :_R \langle\mathfrak{p}\rangle) \nsubseteq S(J)$. Now, let $x \in (J :_R \langle\mathfrak{p}\rangle) \setminus S(J)$. Then $\mathfrak{p}^l x \subseteq J$ for sufficiently large l. Hence $s^l x \in J$, and so $x \in S(J)$, which is a contradiction. Consequently, $\mathfrak{p} \cap S = \emptyset$, as desired.

In order to prove (ii)⇒(iv), let $I \in \Phi$. We need to show that there is an ideal $J \in \Phi$ such that $S(J_a) \subseteq I_a$. To achieve this, consider a normal primary decomposition $I_a = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ of I_a in which each component $\mathfrak{q}_i$ is integrally closed (see [19, Lemma 5.2]). Suppose that $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary for all $1 \leq i \leq r$. Then, for sufficiently large k_i, we have $(\mathfrak{p}_i^{k_i})_a \subseteq (\mathfrak{q}_i)_a = \mathfrak{q}_i$ for all $1 \leq i \leq r$. Moreover, by Lemma 2.1 and the assumption there exists an ideal $J_i \in \Phi$ such that $S((J_i)_a) \subseteq (\mathfrak{p}_i^{k_i})_a$. Hence $S((J_i)_a) \subseteq \mathfrak{q}_i$ for all $i = 1, \ldots, r$.

Since Φ is a system of ideals of R, there is an ideal $J \in \Phi$ such that $J \subseteq J_i$ for all $i = 1, \ldots, r$. Consequently, $S(J_a) \subseteq S((J_i)_a)$ for every $i = 1, \ldots, r$, and so $S(J_a) \subseteq \bigcap_{i=1}^r \mathfrak{q}_i$. Therefore $S(J_a) \subseteq I_a$, as desired.

Next, we prove (iv)⇒(ii). To do this, let $\mathfrak{p} \in \overline{Q}^*(\Phi)$ and $k \geq 1$. Then there exists $I \in \Phi$ such that $I \subseteq \mathfrak{p}$. Hence $I^k \subseteq \mathfrak{p}^k$. Since Φ is a system of ideals, there is an ideal $J \in \Phi$ with $J \subseteq I^k$. Condition (iv) says that $S(K_a) \subseteq J_a$ for some $K \in \Phi$. Hence $S(K_a) \subseteq (I^k)_a$. Consequently, $S(K_a) \subseteq (\mathfrak{p}^k)_a$, as required.

In the final step we have to show the equivalence between (iii) and (v). The proof of (iii)⇒(v) is similar to the proof of (ii)⇒(iv). In order to prove that (v)⇒(iii), let $\mathfrak{p} \in \overline{Q}^*(\Phi)$ and $k \geq 1$. Then $\mathfrak{p} \supseteq I$ for some $I \in \Phi$. Since Φ is a system of ideals, there exists $J \in \Phi$ such that $J \subseteq I^k \subseteq \mathfrak{p}^k$. But, by assumption (v), there is $K \in \Phi$ with $S(K) \subseteq J_a$. Consequently, $S(K) \subseteq (\mathfrak{p}^k)_a$, and the result follows. ∎

3. Local cohomology and ideal topologies. The main purpose of this section is to establish a connection between the vanishing of the local cohomology modules and the comparison of topologies. The main result of this section is Theorem 3.3. Before stating it we recall the definition and we provide a short introduction about local cohomology.

The important concept of local cohomology was first introduced by Grothendieck in the early 1960s, partly to answer a conjecture of Pierre Samuel [8]. More details about the definition and basic results on local cohomology can be found in the book [5] by M. P. Brodmann and R. Y. Sharp; we just briefly summarize some of the main aspects.

Let I be an ideal of a Noetherian ring R and let N be an R-module. The ith local cohomology module of N with respect to I is by definition

$$H_I^i(N) := \varinjlim_n \operatorname{Ext}_R^i(R/I^n, N).$$

There are some fundamental vanishing and non-vanishing results for local cohomology. A necessary and sufficient condition is given by the Lichtenbaum–Hartshorne vanishing theorem. It states that if $(R, \mathfrak{m})$ is a local (Noetherian) ring with $\dim R = d$, then $H_I^d(R) = 0$ if and only if $\dim R^*/(IR^* + \mathfrak{p}) > 0$ for all $\mathfrak{p} \in \operatorname{Ass} R^*$ with $\dim R^*/\mathfrak{p} = d$. The proofs of this theorem use the fact that, under certain circumstances, the I-adic topology on R is equivalent to the topology defined by a certain filtration. Recently, Schenzel [17] has shown that if $(R, \mathfrak{m})$ is a local (Noetherian) complete quasi-Gorenstein ring with $\dim R = d$, then $H_I^d(R) = 0$ if and only if the topology defined by the filtration $\{I^n :_R \langle \mathfrak{m} \rangle\}_{n \geq 1}$ is equivalent to the I-adic topology on R. Here, for any ideal J of R, $J :_R \langle \mathfrak{m} \rangle = \bigcup_{n \geq 0} (J :_R \mathfrak{m}^n)$. Later, Marti-Farre generalized the result of Schenzel to Noetherian quasi-unmixed local rings.

Let $Z = V(I)$ be a closed subscheme of an affine scheme $X = \operatorname{Spec} R$. Then, for any R-module N, the local cohomology groups $H^*_Z(X, M)$ are isomorphic to the limit Ext-groups in the category of R-modules:

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^*_R(R/I^n, N).$$

Thus it is quite natural to consider the groups

$$H^*_\Phi(N) := \varinjlim_{I\in\Phi} \operatorname{Ext}^*_R(R/I, N)$$

for more general inverse systems of ideals. These groups (introduced in [2]), called generalized local cohomology groups, are considered below.

The following lemma, which is a generalization of the Lichtenbaum–Hartshorne vanishing theorem, will be used in the proof of Theorem 3.2.

Lemma 3.1 (see [7, Theorem 2.8]). *Let* $(R, \mathfrak{m})$ *be a local* (*Noetherian*) *ring such that* $\dim R = d$. *Then the following statements are equivalent*:

(i) $H^d_\Phi(R) = 0$.
(ii) *For all prime ideals* $\mathfrak{p}$ *of* R^* *with* $\dim R^*/\mathfrak{p} = d$, *there is an ideal* I *in* Φ *such that* $\dim R^*/(IR^* + \mathfrak{p}) > 0$.

We are now ready to state and prove the main theorem of this section, which gives a generalization of [12, Proposition 2.1] in the context of general local cohomology modules.

Theorem 3.2. *Let* $(R, \mathfrak{m})$ *be local of dimension* d *and assume that each element of* $V(\Phi)$ *contains an element of* $\overline{Q}^*(\Phi)$. *Consider the following conditions*:

(i) *There exists a multiplicatively closed subset* S *of* R *such that* $\mathfrak{m} \cap S \neq \emptyset$ *and the* $S(\Phi)$*-topology is finer than the* Φ_a*-topology.*
(ii) $H^d_\Phi(R) = 0$.

Then (i)$\Rightarrow$(ii); *and these conditions are equivalent whenever* R *is quasi-unmixed.*

Proof. In order to prove (i)$\Rightarrow$(ii), suppose that there is a multiplicatively closed subset S of R with $\mathfrak{m} \cap S \neq \emptyset$ and such that the $S(\Phi)$-topology is finer than the Φ_a-topology. Then it follows from Theorem 2.8 that $S \subseteq R \setminus \bigcup\{\mathfrak{p} \in \overline{Q}^*(\Phi)\}$. Since $\mathfrak{m} \cap S \neq \emptyset$, we deduce that $\mathfrak{m} \notin \overline{Q}^*(\Phi)$. Hence, for all $z \in \operatorname{mAss} R^*$ there exists $I \in \Phi$ such that $\operatorname{Rad}(IR^* + z) \neq \mathfrak{m}R^*$. Now, using the generalized Lichtenbaum–Hartshorne vanishing theorem (see Lemma 3.1) it follows that $H^d_\Phi(R) = 0$, as desired.

Next, we show the converse when R is quasi-unmixed. It then follows that $\dim R^*/z = \dim R = d$ for every $z \in \operatorname{mAss} R^*$. Hence, in view of Lemma 3.1, condition (ii) says that $\dim R^*/(IR^* + z) > 0$ for all $z \in \operatorname{mAss} R^*$ and for some $I \in \Phi$. Now, let S be the multiplicatively closed subset of

R defined by $S := R \setminus \bigcup\{\mathfrak{p} \in \overline{Q}^*(\Phi)\}$. Since $\mathfrak{m} \notin \overline{Q}^*(\Phi)$ it follows that $\mathfrak{m} \cap S \neq \emptyset$. Moreover, in view of Theorem 2.8, the $S(\Phi)$-topology is finer than the Φ_a-topology. This completes the proof. ∎

Theorem 3.3. *Under the assumptions of Theorem* 3.2, *consider the following conditions*:

(i) *The $S(\Phi)$-topology is finer than the Φ_a-topology.*
(ii) $H^{\operatorname{ht}\mathfrak{p}}_{\Phi R_\mathfrak{p}}(R_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in V(\Phi)$ *with* $\mathfrak{p} \cap S \neq \emptyset$, *where* $\operatorname{ht}\mathfrak{p} = \dim R_\mathfrak{p}$.

Then (i)$\Rightarrow$(ii); *and these conditions are equivalent whenever R is locally quasi-unmixed.*

Proof. For the first part, suppose the $S(\Phi)$-topology is finer than the Φ_a-topology. Then, from Theorem 2.8, we have $S \subseteq R \setminus \bigcup\{\mathfrak{q} \in \overline{Q}^*(\Phi)\}$. Now, let $\mathfrak{p} \in V(\Phi)$ with $\mathfrak{p} \cap S \neq \emptyset$. Then we see that $\mathfrak{p} \notin \overline{Q}^*(\Phi)$. Hence by Lemma 2.1, $\mathfrak{p}R_\mathfrak{p} \notin \overline{Q}^*(\Phi R_\mathfrak{p})$. Therefore, in view of the generalized Lichtenbaum–Hartshorne vanishing theorem (see Lemma 3.1), we have $H^{\operatorname{ht}\mathfrak{p}}_{\Phi R_\mathfrak{p}}(R_\mathfrak{p}) = 0$. Hence (i) implies (ii).

Now, let R be a locally quasi-unmixed ring, and assume that for all $\mathfrak{p} \in V(\Phi)$ with $\mathfrak{p} \cap S \neq \emptyset$, we have $H^{\operatorname{ht}\mathfrak{p}}_{\Phi R_\mathfrak{p}}(R_\mathfrak{p}) = 0$. Then, by the generalized Lichtenbaum–Hartshorne vanishing theorem, for all $z \in \operatorname{mAss} R^*_\mathfrak{p}$ there exists an ideal $I \in \Phi$ such that $\dim R^*_\mathfrak{p}/(IR^*_\mathfrak{p} + z) > 0$. (Note that R is locally quasi-unmixed.) Hence $\mathfrak{p}R_\mathfrak{p} \notin \overline{Q}^*(\Phi R_\mathfrak{p})$, and so by Lemma 2.1, $\mathfrak{p} \notin \overline{Q}^*(\Phi)$. Consequently, we have $S \subseteq R \setminus \bigcup\{\mathfrak{p} \in \overline{Q}^*(\Phi)\}$. Now, the result follows from Theorem 2.8. ∎

Before giving an application of Theorem 3.3 we need the following definition.

Definition. The *cohomological dimension* of Φ is defined as

$$\operatorname{cd}_\Phi(R) = \sup\{\operatorname{cd}_I(R) : I \in \Phi\},$$

where $\operatorname{cd}_I(R)$ is the cohomological dimension of I (see [6], [9], [10]).

Corollary 3.4. *Suppose $(R, \mathfrak{m})$ is quasi-unmixed local with* $\dim R = d$. *Let S be a multiplicatively closed subset of R defined by $S = R \setminus \bigcup\{\mathfrak{p} \in \operatorname{mAss} R/I$ for all $I \in \Phi\}$ and suppose each element of $V(\Phi)$ contains an element of $\overline{Q}^*(\Phi)$. Suppose that every ideal in Φ is unmixed. Then the following statements are equivalent*:

(i) *The $S(\Phi)$-topology is finer than the Φ_a-topology.*
(ii) $\operatorname{Supp} H^i_\Phi(R) \subseteq \{\mathfrak{p} \in V(\Phi) : \operatorname{ht}\mathfrak{p} \geq i+1\}$ *for all $I \in \Phi$ and i with* $\operatorname{ht} I < i \leq d$.

Proof. First we show (i)$\Rightarrow$(ii). Let $\mathfrak{p} \in \operatorname{Supp} H^i_\Phi(R)$. Then, in view of [2, Lemma 2.1] and the fact that the local cohomology functor commutes with direct limits and by the flat base change theorem (see [5, Theorem 4.2.1]), we have $\mathfrak{p} \in V(\Phi)$. Hence $\operatorname{Supp} H^i_\Phi(R) \subseteq V(\Phi)$. Furthermore, if $\mathfrak{p} \in V(\Phi)$ is such that $\operatorname{ht}\mathfrak{p} < i$ then, by [5, Theorems 4.3.2 and 6.1.2] and [2, Lemma 2.1] and the fact that $H^i_\Phi(\cdot)$ commutes with direct limits, we deduce that $(H^i_\Phi(R))_\mathfrak{p} = 0$, since $\dim R_\mathfrak{p} = \operatorname{ht}\mathfrak{p} < i$. Thus

$$\operatorname{Supp} H^i_\Phi(R) \subseteq \{\mathfrak{q} \in V(\Phi) : \operatorname{ht}\mathfrak{q} \geq i\}.$$

Now, in view of the definition of S and Theorems 2.8 and 3.3, $(H^{\operatorname{ht}\mathfrak{q}}_\Phi(R))_\mathfrak{q} = 0$ for all $\mathfrak{q} \in V(\Phi)$ with $\mathfrak{q} \cap S \neq \emptyset$. Hence for all $I \in \Phi$ and i with $\operatorname{ht} I < i \leq d$,

$$\begin{aligned}\operatorname{Supp} H^i_\Phi(R) \subseteq \{&\mathfrak{q} \in V(\Phi) : \operatorname{ht}\mathfrak{q} = i \text{ and } \mathfrak{q} \cap S = \emptyset\}\\ &\cup \{\mathfrak{q} \in V(\Phi) : \operatorname{ht}\mathfrak{q} \geq i+1\}.\end{aligned}$$

Now, if $\mathfrak{q} \in V(\Phi)$ and $\mathfrak{q} \cap S = \emptyset$, then, since every element of Φ is an unmixed ideal, it follows that $\operatorname{ht}\mathfrak{q} = \operatorname{ht} J$ for some $J \in \Phi$. Hence $\{\mathfrak{q} \in V(\Phi) : \operatorname{ht}\mathfrak{q} = i$ and $\mathfrak{q} \cap S = \emptyset\} = \emptyset$ for all $I \in \Phi$ and i with $\operatorname{ht} I < i \leq d$. Consequently,

$$\operatorname{Supp} H^i_\Phi(R) \subseteq \{\mathfrak{q} \in V(\Phi) : \operatorname{ht}\mathfrak{q} \geq i+1\}$$

for all $I \in \Phi$ and i with $\operatorname{ht} I < i \leq d$, as required.

In order to prove (ii)⇒(i), in view of Theorem 3.3, it is sufficient to show that $H^{\operatorname{ht}\mathfrak{p}}_{\Phi R_\mathfrak{p}}(R_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in V(\Phi)$ with $\mathfrak{p} \cap S \neq \emptyset$. Indeed, for $\mathfrak{p} \in V(\Phi)$ there is $I \in \Phi$ such that $I \subseteq \mathfrak{p}$. Since $\mathfrak{p} \cap S \neq \emptyset$, it follows that $\mathfrak{p} \notin \operatorname{mAss}_R R/I$, and so $\operatorname{ht} I < \operatorname{ht}\mathfrak{p}$. Therefore, by assumption (ii), we have $\mathfrak{p} \notin \operatorname{Supp} H^{\operatorname{ht}\mathfrak{p}}_\Phi(R)$. Consequently, $(H^{\operatorname{ht}\mathfrak{p}}_\Phi(R))_\mathfrak{p} = 0$. Now, the result follows from [5, Theorem 4.3.2] and [2, Lemma 2.1]. ∎

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