COLLOQUIUM MATHEMATICUM

VOL. 106 2006 NO. 1

LIFTS FOR SEMIGROUPS OF ENDOMORPHISMS OF AN INDEPENDENCE ALGEBRA

BY

JOÃO ARAÚJO (Lisboa)

Dedicated to the memory of Professor Kazimierz Urbanik

Abstract. For a universal algebra $\mathcal{A}$, let $\mathrm{End}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{A})$ denote, respectively, the endomorphism monoid and the automorphism group of $\mathcal{A}$. Let S be a semigroup and let T be a characteristic subsemigroup of S. We say that $\phi \in \mathrm{Aut}(S)$ is a *lift* for $\psi \in \mathrm{Aut}(T)$ if $\phi|T = \psi$. For $\psi \in \mathrm{Aut}(T)$ we denote by $L(\psi)$ the set of lifts of ψ, that is,

$$L(\psi) = \{\phi \in \mathrm{Aut}(S) \mid \phi|_T = \psi\}.$$

Let $\mathcal{A}$ be an independence algebra of infinite rank and let S be a monoid of monomorphisms such that $G = \mathrm{Aut}(\mathcal{A}) \le S \le \mathrm{End}(\mathcal{A})$. It is obvious that G is characteristic in S. Fitzpatrick and Symons proved that if $\mathcal{A}$ is a set (that is, an algebra without operations), then $|L(\phi)| = 1$. The author proved in a previous paper that the analogue of this result does not hold for all monoids of monomorphisms of an independence algebra. The aim of this paper is to prove that the analogue of the result above holds for semigroups $S = \langle \mathrm{Aut}(\mathcal{A}) \cup E \cup R\rangle \le \mathrm{End}(\mathcal{A})$, where E is any set of idempotents and R is the empty set or a set containing a special monomorphism α and a special epimorphism α^*.

1. Introduction. We assume the reader to be familiar with both semigroup theory and universal algebra. We recommend as references [22] and [29]. Also we assume the reader to have a basic knowledge of the theory of independence algebras. We recommend [5], [14], [15] and [19] as references. Independence algebras were introduced as v^*-algebras by Narkiewicz [30] (see also [31] and [32]). For an excellent survey paper on v^*-algebras see Urbanik [41].

These algebras appeared in Poland as a result of research on different notions of independence valid in any universal algebra. This research, prompted by Marczewski [24], benefited from the contributions of Marczewski himself (e.g., [25]–[28]), Narkiewicz ([30]–[32]), Grätzer [20], Urbanik ([38]–[41]), etc. Such investigations led to many important results and to several questions that forty years later remain open. An excellent account and the main reference regarding all these investigations is Głazek [16], a comprehensive survey

2000 *Mathematics Subject Classification*: 08B20, 08A35, 20M10, 20M20.

Key words and phrases: universal algebra, independence algebra.

paper (containing the impressive number of more than eight hundred references). But when speaking about v^*-algebras, the most notable achievement is due to Urbanik: a series of deep papers leading to the classification of these algebras. For the full picture see [41]. (See also [9].)

Semigroup theorists rediscovered v^*-algebras (giving them the name of *independence algebras*) as a tool to provide unified proofs for results that *graphically* hold for both sets and vector spaces, or more precisely, hold for the endomorphism monoid of a set and for the endomorphism monoid of a vector space. Since the early 1990s, endomorphism monoids of independence algebras, and related semigroups, have been extensively studied and the topic continues to receive a great deal of attention. From the point of view of semigroups the pre-history of independence algebras might be in 1966, when Howie described the subsemigroup E_X of $T(X)$ generated by all the non-identity idempotents [21]. The corresponding result for $\mathrm{End}(V)$, where V is a finite-dimensional vector space, was proved by Erdos the following year [11], but it was not until nearly twenty years later that Reynolds and Sullivan [33] found the appropriate analogue in the infinite-dimensional case. Their work also uncovered a significant difference between the semigroups E_X and E_V where X is an infinite set, V is an infinite-dimensional vector space, and where for any algebra $\mathcal{A}$ we denote by $E_{\mathcal{A}}$ the subsemigroup of $\mathrm{End}(\mathcal{A})$ generated by the non-identity idempotents. Sullivan surveyed the parallels and distinctions between $T(X)$ and $\mathrm{End}(V)$ in an influential conference talk in 1990 which was published in [37].

Fountain and Lewin, having seen a preliminary version of Gould's paper [19], realized that independence algebras provided a suitable conceptual framework for unifying those results on products of idempotents which hold for both $T(X)$ and $\mathrm{End}(V)$. They described $E_{\mathcal{A}}$ for an independence algebra of finite rank in [14]. (For a direct proof see [4].) After that, independence algebras have been very useful to

(1) export results from semigroup theory to linear algebra (the other way, unfortunately, proved to be less fruitful);
(2) give unified proofs for results that hold for both sets and vector spaces;
(3) provide explanations for why sometimes vector spaces and sets behave differently.

Among others, the papers [7] and [8] are examples of (1); the papers [3] and [13] are examples of (3); the current paper is an example of (1) and (2).

The first step in the definition of independence algebras is the introduction of a notion of independence valid for universal algebras. Let $\mathcal{A}$ be an algebra with universe A and let X be a set contained in A. Then we denote by $\langle X \rangle$ the algebra generated by X. Now, a subset X of an algebra is said to be *independent* if $X = \emptyset$ or if, for every element $x \in X$, we have $x \notin \langle X \setminus \{x\} \rangle$; a set is *dependent* if it is not independent.

LEMMA 1.1. *For an algebra $\mathcal{A}$, the following conditions are equivalent*:

(1) *for every subset X of A and all elements u, v of A, if $u \in \langle X \cup \{v\} \rangle$ and $u \notin \langle X \rangle$, then $v \in \langle X \cup \{u\} \rangle$*;
(2) *for every subset X of A and every element $u \in A$, if X is independent and $u \notin \langle X \rangle$, then $X \cup \{u\}$ is independent*;
(3) *for every subset X of A, if Y is a maximal independent subset of X, then $\langle X \rangle = \langle Y \rangle$*;
(4) *for subsets X, Y of A with $Y \subseteq X$, if Y is independent, then there is an independent set Z with $Y \subseteq Z \subseteq X$ and $\langle Z \rangle = \langle X \rangle$.*

Proof. See [29, p. 50, Exercise 6]. ∎

An algebra $\mathcal{A}$ is said to have the *exchange property*, or to satisfy [EP], if it satisfies the equivalent conditions of Lemma 1.1. A *basis* for $\mathcal{A}$ is a subset of A which generates A and is independent. It is clear from Lemma 1.1 that any algebra with [EP] has a basis. Furthermore, for such an algebra, bases may be characterized as minimal generating sets or maximal independent sets, and all bases for $\mathcal{A}$ have the same cardinality [19, Proposition 3.3]. This cardinal is called the *rank* of $\mathcal{A}$ and is written $\operatorname{rank}(\mathcal{A})$.

We say that $\mathcal{A}$ is an *independence algebra* if $\mathcal{A}$ satisfies [EP] and the following property:

[F] for every basis X of $\mathcal{A}$ and mapping $f : X \to A$, there exists a morphism $F : \langle X \rangle \to \mathcal{A}$ extending f.

Suppose a is a partial endomorphism of $\mathcal{A}$. We denote the domain of a by Δa and the image of a by ∇a. Observe that both Δa and ∇a are subalgebras of $\mathcal{A}$. If $a \in \operatorname{End}(\mathcal{A})$, we denote by $\operatorname{rank}(a)$ the rank of the algebra ∇a. Recall that given a subalgebra $\mathcal{B} \leq \mathcal{A}$, and a basis B for $\mathcal{B}$, if $B \cup X$ and $B \cup Y$ are bases of $\mathcal{A}$, then $|X| = |Y|$. This cardinal is called the *corank* of $\mathcal{B}$ and is denoted by $\operatorname{corank}(\mathcal{B})$. This notation will be extended to endomorphisms as above: for $a \in \operatorname{End}(\mathcal{A})$, we write $\operatorname{corank}(a)$ to denote the corank of ∇a.

Let $\mathcal{A}$ be an independence algebra and let X, Y be two disjoint and independent subsets of A. Then $\mathcal{A}$ is said to be *strong* if $\langle X \rangle \cap \langle Y \rangle = \mathrm{Con}$ implies that $X \cup Y$ is an independent set (where Con denotes the constants of the algebra $\mathcal{A}$). Clearly, any subalgebra of a strong independence algebra is also a strong independence algebra. Particular cases of strong independence algebras are sets and vector spaces.

Let f be an endomorphism of an independence algebra $\mathcal{A}$. Then B is a *preimage basis* for f if Bf is a basis for the image of f and $f|_B$ is injective. (We observe that mappings act on the right; we write xf rather than $f(x)$.)

If $\mathcal{A}$ is a universal algebra, denote by $\mathrm{End}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{A})$, respectively, the endomorphism monoid and the automorphism group of $\mathcal{A}$. By $\mathrm{PEnd}(\mathcal{A})$ we denote the monoid of partial endomorphisms of $\mathcal{A}$.

When $\mathcal{A} = X$, where X is a set (that is, when the algebra has no operations), then $\mathrm{End}(\mathcal{A}) = T(X)$ and $\mathrm{Aut}(\mathcal{A}) = \mathrm{Sym}(X)$, respectively, the monoid of all transformations on X and the symmetric group on X.

Let S be a semigroup and let T be a subsemigroup of S. We say that T is *characteristic* in S if, for every $\phi \in \mathrm{Aut}(S)$, we have $\phi|_T \in \mathrm{Aut}(T)$, that is, if the restriction to T of every automorphism of S is an automorphism of T. Now suppose that T is a characteristic subsemigroup of S. We say that an automorphism $\phi \in \mathrm{Aut}(S)$ is a *lift* for $\psi \in \mathrm{Aut}(T)$ if $\phi|_T = \psi$. For $\phi \in \mathrm{Aut}(T)$ we denote by $L(\phi)$ the set of lifts of ϕ, that is,

$$L(\phi) = \{\psi \in \mathrm{Aut}(S) \mid \psi|_T = \phi\}.$$

It is well known that lifts play a crucial role in the description of the automorphism group of a semigroup. The general scheme goes as follows: if we want to describe the automorphism group of S, a good idea is to try to find a subsemigroup $T \leq S$ such that:

(1) T is characteristic in S;
(2) we have (or can find) a description of $\mathrm{Aut}(T)$, the automorphism group of T;
(3) for every $g \in \mathrm{Aut}(T)$ we can describe $L(g)$.

Since (1) implies that $\mathrm{Aut}(S) = \bigcup_{g\in \mathrm{Aut}(T)} L(g)$, from (3) we get a description of $\mathrm{Aut}(S)$. (Usually it is necessary to have (2) in order to find (3).)

For example, in [23] Mal'tsev described $\mathrm{Aut}(T(X))$. He considered the semigroup

$$T_1(X) = \{f \in T(X) \mid |(X)f| = 1\},$$

proved that $T_1(X)$ is characteristic in $T(X)$, described the automorphisms of $T_1(X)$ (which are the mappings $\tau^g : a \mapsto g^{-1}ag$, where $g \in \mathrm{Sym}(X)$ and $a \in T_1(X)$) and proved that $|L(\tau^g)| = 1$ for all $g \in \mathrm{Aut}(T_1(X))$. Thus $\mathrm{Aut}(T(X)) = \mathrm{Inn}(T(X))$, the inner automorphisms, that is, the automorphisms induced under conjugation by the elements $g \in \mathrm{Sym}(X)$.

Another example: in [12] Fitzpatrick and Symons considered semigroups S of injective mappings such that $\mathrm{Sym}(X) \leq S \leq T(X)$ (where X is an infinite set; see also [10]). In the most delicate part of their proof they showed that, given an $h \in \mathrm{Aut}(\mathrm{Sym}(X))$, we have $|L(h)| = 1$. Thus, since $\mathrm{Sym}(X)$ is characteristic in S (and all automorphisms of $\mathrm{Sym}(X)$ are inner), we have $\mathrm{Aut}(S) = \bigcup_{\tau^g\in \mathrm{Aut}(\mathrm{Sym}(X))} L(\tau^g)$ and hence for every semigroup S of injective mappings such that $\mathrm{Sym}(X) \leq S \leq T(X)$ we have

$$\mathrm{Aut}(S) = \{\tau^g : s \mapsto g^{-1}sg \mid g \in \mathrm{Sym}(X)\}.$$

In [17] Gluskin used the same general scheme to describe the automorphism group of $\mathrm{End}(V)$, where V is a vector space, thus proving the vector space analogue of the result proved by Mal'tsev for sets, as referred to above.

The linear analogue of the result of Fitzpatrick and Symons quoted above would read as follows: given a vector space V of infinite dimension and a semigroup S of injective linear transformations such that $\mathrm{Aut}(V) \leq S \leq \mathrm{End}(V)$, prove that for all $h \in \mathrm{Aut}(\mathrm{Aut}(V))$ we have

$$|L(h)| = |\{g \in \mathrm{Aut}(S) \mid g|_{\mathrm{Aut}(\mathrm{Aut}(V))} = h\}| = 1.$$

In [3] the author proved that this is not true and then characterized the independence algebras in which the analogue of this result holds. This was a negative result and the best that could be done was to describe the bounds of that negative answer.

The aim of this paper is to provide positive answers for many classes of semigroups, thus providing the description of the automorphisms of those semigroups, modulo a description of $\mathrm{Aut}(\mathrm{Aut}(\mathcal{A}))$, an open problem in group theory. More precisely, we want to identify large classes of semigroups S, with $\mathrm{Aut}(V) \leq S \leq \mathrm{End}(V)$, in which an analogue of the result of Fitzpatrick and Symons referred to above holds. The cornerstone of the results in this paper are the so-called *fundamental representations* of endomorphisms (introduced in [1] and [6]) and two endomorphisms α and α^* introduced in [2]. These two endomorphisms have the property that $\langle \mathrm{Aut}(\mathcal{A}) \cup E(\mathrm{End}(\mathcal{A})) \cup \{\alpha, \alpha^*\}\rangle = \mathrm{End}(\mathcal{A})$ (the symbol $E(\mathrm{End}(\mathcal{A}))$ denotes the set of idempotents of $\mathrm{End}(\mathcal{A})$).

In Section 2 we introduce some notation and basic results. The following two sections contain technical results. In Section 5 we state and prove our main result. The paper ends with a section of proposed problems.

2. Preliminaries. We start by introducing some notation, definitions and conventions. Let $\mathcal{A}$ be an algebra. To simplify the notation let $G = \mathrm{Aut}(\mathcal{A})$ and denote by Con the constants of $\mathcal{A}$. In this paper we assume that $\mathrm{Con} \subseteq \{0\}$, that is, Con is empty or has at most one element, denoted by 0.

Let B be a basis for an independence algebra $\mathcal{A}$ and let $\alpha : B \to A$ be a mapping. Since there exists one and only one morphism $\overline{\alpha} : \mathcal{A} \to \mathcal{A}$ extending α (that is, such that $\overline{\alpha}|_B = \alpha$) we will use the same letter α to denote the mapping and the endomorphism.

For a semigroup S we denote by $E(S)$ the set of idempotents of S, that is, the set of elements $s \in S$ such that $s^2 = s$.

We introduce some auxiliary results about strong independence algebras.

LEMMA 2.1. *Let $\mathcal{A}$ be a strong independence algebra. Let $\mathcal{B}$ and $\mathcal{C}$ be subalgebras of $\mathcal{A}$. If B is a basis for $\mathcal{B} \cap \mathcal{C}$, $B \cup C$ is a basis for $\mathcal{B}$ and $B \cup D$ is a basis for $\mathcal{C}$, then $B \cup C \cup D$ is a basis for the algebra generated by $\mathcal{B}$ and $\mathcal{C}$.*

Proof. See [15, Lemma 1.6]. ∎

DEFINITION 2.2. Let I be a set and, for a symbol $0 \notin I$, let $I_0 = I \cup \{0\}$. Moreover, let $\mathcal{A}$ be a strong independence algebra and let $(A_i)_{i\in I}$ be a partition of a basis of $\mathcal{A}$. Consider the endomorphism $\alpha \in \mathrm{End}(\mathcal{A})$ defined by $A_i\alpha = \{a_i\}$ for $i \in I$, where $\{a_i : i \in I\}$ is an independent set (and hence a basis for $\nabla(\alpha)$), and let $A_0\alpha = \{0\}$. An endomorphism $\alpha \in \mathrm{End}(\mathcal{A})$ under these conditions is represented by the matrix

$$\begin{bmatrix} A_0 & A_i \\ 0 & a_i \end{bmatrix}_{i\in I}.$$

This matrix is said to be a *fundamental representation* of α. The set A_0 in the fundamental representation is said to be the *constant component*.

If the algebra has no constants, then the constant component is the empty set and then the endomorphism can be defined by

$$\begin{bmatrix} A_i \\ a_i \end{bmatrix}_{i\in I}.$$

The importance of this concept lies in the following fact:

THEOREM 2.3. *Every endomorphism of a strong independence algebra admits a fundamental representation.*

Proof. This follows from [15, Lemma 2.8] and the observations following Corollary 2.10 of [15]. See also [1] and [6]. ∎

We observe that if $e \in E(\mathrm{End}(\mathcal{A}))$, then e has a fundamental representation

$$\begin{bmatrix} A_0 & A_i \\ 0 & a_i \end{bmatrix}_{i\in I},$$

where $a_i \in A_i$ for all $i \in I$. Moreover, if C is a basis for ∇e and $C_0 = C \cup \{0\}$, then there is a basis of $\mathcal{A}$, say $B = \bigcup_{c\in C_0} A_c$, such that $A_c e = c$ for all $c \in C_0$. Thus, e can be represented as (and is defined by)

$$\begin{bmatrix} A_c \\ c \end{bmatrix}_{c\in C_0}.$$

Let $X \neq \emptyset$ be a subset of $A \times A$. Then the congruence generated by X will be denoted by $\Theta(X)$.

LEMMA 2.4. *Let e be an idempotent endomorphism with the fundamental representation*

$$\begin{bmatrix} A_c \\ c \end{bmatrix}_{c\in C_0},$$

where $c \in A_c$ for all $c \in C$. Moreover, let $X = \bigcup_{c\in C_0}(A_c \times A_c)$. Then $\Theta(X)$ is equal to $\mathrm{Ker}(e)$.

Proof. Clearly, (x, xe) belongs to X for all $x \in B$. Thus for every term t and $x_1, \ldots, x_n \in B$, we have $(x_1, x_1e) \in X, \ldots, (x_n, x_ne) \in X$ and hence

$$(t(x_1, \ldots, x_n), t(x_1e, \ldots, x_ne)) \in \Theta(X).$$

Therefore, for all elements $u = t(x_1, \ldots, x_n) \in A$, we have $(u, ue) \in \Theta(X)$. Now, let $(u, v) \in \mathrm{Ker}(e)$. On the one hand, (u, ue) and (ve, v) both belong to $\Theta(X)$. On the other hand, $(u, v) \in \mathrm{Ker}(e)$ implies that $ue = ve$. By transitivity, (u, v) belongs to $\Theta(X)$. This proves that $\mathrm{Ker}(e)$ is contained in $\Theta(X)$. As the converse is obvious, the lemma follows. ∎

Throughout this paper, $\mathcal{A}$ always denotes a strong independence algebra of infinite rank with universe A (recall that $\mathrm{Con} \subseteq \{0\}$), and S denotes a semigroup of endomorphisms of $\mathcal{A}$ such that $G = \mathrm{Aut}(\mathcal{A}) \leq S \leq \mathrm{End}(\mathcal{A})$.

In the next two sections we prove auxiliary technical results.

3. $e\tau = e$. The aim of this section is the proof of the following theorem.

THEOREM 3.1. *Let $\mathcal{A}$ be a strong independence algebra with at most one constant, let $G \leq S \leq \mathrm{End}(\mathcal{A})$ and let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. Then $e\tau = e$ for all $e \in E(S)$.*

This result will be proved in a series of lemmas. We start by introducing some notation. Let X be a basis for $\mathcal{A}$ and let $x, y \in X$. Then we denote by $(xy)_X$ the automorphism of $\mathcal{A}$ that is induced by the transposition $(xy)_X$, the permutation of X that maps x to y, maps y to x and fixes all the remaining elements of X. We first prove the following lemma.

LEMMA 3.2. *Let $\mathcal{A}$ be a strong independence algebra, S be a semigroup such that $G \leq S \leq \mathrm{End}(\mathcal{A})$ and let $e \in E(S)$. Moreover, let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. Then $\mathrm{Ker}(e\tau) = \mathrm{Ker}(e)$.*

Proof. Let

$$\begin{bmatrix} A_c \\ c \end{bmatrix}_{c \in C_0}$$

be a fundamental representation for e and let $B = \bigcup_{c \in C_0} A_c$. Let $c \in C_0$ and suppose that $|A_c| > 1$. Then for all $x, y \in A_c$, we have $(xy)_B e = e$ and hence $(xy)_B(e\tau) = e\tau$. Thus $x(e\tau) = y(e\tau)$ and so $(x, y) \in \mathrm{Ker}(e\tau)$. We have proved that $A_c \times A_c \subseteq \mathrm{Ker}(e\tau)$. If, for some $c \in C_0$, we have $|A_c| = 1$, say $A_c = \{x\}$, then $A_c \times A_c = \{(x, x)\} \subseteq \mathrm{Ker}(e\tau)$. Thus for all $c \in C_0$ we have

$$A_c \times A_c \subseteq \mathrm{Ker}(e\tau).$$

Let $X = \bigcup_{c \in C_0} (A_c \times A_c)$. As $X \subseteq \mathrm{Ker}(e\tau)$ it follows that $\Theta(X) \subseteq \mathrm{Ker}(e\tau)$. Then, by Lemma 2.4, $\mathrm{Ker}(e) = \Theta(X) \subseteq \mathrm{Ker}(e\tau)$. By symmetry, $\mathrm{Ker}(e\tau) \subseteq \mathrm{Ker}(e)$ and the lemma follows. ∎

The following lemmas show that $\nabla(e) = \nabla(e\tau)$. Observe that this is sufficient for proving the main theorem of this section. In fact, if $\nabla(e) = \nabla(e\tau)$ then, as $\mathrm{Ker}(e) = \mathrm{Ker}(e\tau)$, the idempotent endomorphisms e and $e\tau$ are $\mathcal{H}^{\mathrm{End}(\mathcal{A})}$-related and hence $e = e\tau$ (see [22, Theorem 2.2.5]).

Lemma 3.3. *Let S be a semigroup such that $G \leq S \leq \mathrm{End}(\mathcal{A})$ and let $e \in E(S)$. Moreover, let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. Then $\nabla(e)$ cannot be strictly contained in $\nabla(e\tau)$.*

Proof. Suppose that $\nabla(e\tau) = \mathcal{A}$. Then $e\tau$ is a surjective idempotent endomorphism and hence $e\tau = \mathrm{id}_A$. This comes from the fact that $e\tau$ is idempotent and hence all the elements in $\nabla(e\tau)$ are fixed points. As $\nabla(e\tau) = \mathcal{A}$ it follows that $a(e\tau) = a$ for all $a \in A$. However, if $e\tau = \mathrm{id}_A = \mathrm{id}_A\tau$, it follows that $e = \mathrm{id}_A$ and hence $\nabla(e) = \nabla(e\tau)$.

Suppose now that $\nabla(e) \subseteq \nabla(e\tau)$ and $\nabla(e\tau)$ is strictly contained in A. Moreover, let B be a basis for $\nabla(e)$, let C be such that $B \cup C$ is a basis for $\nabla(e\tau)$, and let D be such that $B \cup C \cup D$ is a basis for $\mathcal{A}$. If $|C| \geq 1$ then there exists $g \in G$ such that $g|_B = \mathrm{id}_B$ and $g|_{C\cup D}$ has no fixed points. (This is possible because $|C \cup D| > 1$, as both C and D are non-empty.) Then $eg = e$ and hence $(e\tau)g = e\tau$. However, for $c \in C$, we have $c(e\tau) = c$, as $c \in \nabla(e\tau)$, but $cg \neq c$. Thus, $c(e\tau)g \neq c(e\tau)$. This contradiction follows from the supposition that $|C| \geq 1$. The lemma is proved. ∎

Corollary 3.4. *Let S be a semigroup such that $G \leq S \leq \mathrm{End}(\mathcal{A})$ and let $e \in E(S)$. Moreover, let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. Then $\nabla(e\tau)$ cannot be strictly contained in $\nabla(e)$.*

Proof. Certainly $e\tau \in E(S)$ and $\tau^{-1} \in \mathrm{Aut}(S)$ is such that $\tau^{-1}|_G = \mathrm{id}_G$. Hence $\nabla(e\tau)$ cannot be strictly contained in $\nabla((e\tau)\tau^{-1})$, that is, cannot be strictly contained in $\nabla(e)$. ∎

Lemma 3.5. *Let S be a semigroup such that $G \leq S \leq \mathrm{End}(\mathcal{A})$ and let $e \in E(S)$. Moreover, let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. If $\mathrm{corank}(e) > 1$, then $\nabla(e) = \nabla(e\tau)$.*

Proof. Consider the following sets:

1. B, a basis for $\nabla(e) \cap \nabla(e\tau)$;
2. $B \cup C$, a basis for $\nabla(e)$;
3. $B \cup D$, a basis for $\nabla(e\tau)$;
4. $B \cup C \cup D \cup W$, a basis for $\mathcal{A}$.

Suppose that $D \neq \emptyset$, say $y \in D$. Now let $g \in G$ be such that $g|_{B\cup C} = \mathrm{id}_{B\cup C}$ and $(D \cup W)g = D \cup W$, with $yg = z \neq y$. This is possible because $\mathrm{corank}(e) > 1$ implies $|D \cup W| > 1$. We have $eg = e$ and hence $(e\tau)g = e\tau$. But for $w \in A$ such that $w(e\tau) = y$, we have $w(e\tau)g = yg = z \neq y$, a contradiction. Thus D must be empty. It follows that $C = \emptyset$ as well. In fact, if $C \neq \emptyset$, then $\nabla(e\tau) = \langle B \rangle$ is strictly contained in $\nabla(e) = \langle B \cup C \rangle$, which is impossible by Corollary 3.4. Hence, $D = \emptyset = C$ and $\nabla(e) = \langle B \rangle = \nabla(e\tau)$ and the lemma follows. ∎

As noted before Lemma 3.3 we have the following corollary.

COROLLARY 3.6. *Let S be a semigroup such that $G \leq S \leq \mathrm{End}(\mathcal{A})$ and let $e \in E(S)$. Moreover, let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. If* $\mathrm{corank}(e) > 1$, *then $e = e\tau$.*

COROLLARY 3.7. *Let S be a semigroup such that $G \leq S \leq \mathrm{End}(\mathcal{A})$ and let $e \in E(S)$. Moreover, let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. If* $\nabla(e) > \mathrm{Con}$, *then* $\nabla(e\tau) \cap \nabla(e) \neq \mathrm{Con}$.

Proof. By the previous result, if $\mathrm{corank}(e) > 1$ then $e = e\tau$ and hence $\nabla(e) = \nabla(e\tau)$.

Suppose now that $\mathrm{corank}(e) = 1$ and $\nabla(e) \cap \nabla(e\tau) = \mathrm{Con}$. As $\mathcal{A}$ is strong, if B is a basis for $\nabla(e)$ and C is a basis for $\nabla(e\tau)$, then $B \cup C$ is independent, and hence there is a set W such that $B \cup C \cup W$ is a basis for $\mathcal{A}$. However, $\mathrm{corank}(e) = 1$ and hence $|C \cup W| = 1$. As we suppose $C \neq \emptyset$, we have $W = \emptyset$ and so $B \cup C$ is a basis for $\mathcal{A}$. This implies $\mathrm{corank}(e\tau) = |B| > 1$, as $|B| = \mathrm{rank}(\mathcal{A})$, which is supposed to be infinite. This contradicts the previous corollary as $\mathrm{corank}(e\tau) > 1$ implies $e\tau = (e\tau)\tau^{-1} = e$. ∎

Now let $e \in E(S)$ be such that $\mathrm{corank}(e) = 1$ and $b_1 \in \nabla(e) \cap \nabla(e\tau)$. Moreover, let $B \cup \{x\}$ be a basis for $\nabla(e)$ such that $b_1 \in B$. As $\mathrm{corank}(e) = 1$ there is an element $y \in A$ such that $B \cup \{x\} \cup \{y\}$ is a basis for $\mathcal{A}$. Under these conditions we have the following three lemmas.

LEMMA 3.8. *Suppose that e is defined by*

$$\begin{bmatrix} \{x, y\} & \{b\} \\ x & b \end{bmatrix}_{b \in B}.$$

Then $\nabla(e) = \nabla(e\tau)$.

Proof. For an element $b_1 \in B$, consider an automorphism $g \in G$ induced by the permutation $(x y b_1)_Y \in \mathrm{Sym}(Y)$, where $Y = B \cup \{x, y\}$. Clearly, ege is defined by

$$\begin{bmatrix} \{x, y, b_1\} & \{b\} \\ x & b \end{bmatrix}_{b \in B \setminus \{b_1\}}.$$

As ege is idempotent and $\mathrm{corank}(ege) = 2$, Corollary 3.6 implies $(ege)\tau = ege$. Thus $\nabla(ege) = \nabla((ege)\tau) = \nabla((e\tau)g(e\tau)) \subseteq \nabla(e\tau)$. However, $(B \setminus \{b_1\}) \cup \{x\} = \nabla(ege)$ and so $b_1 \in \nabla(e\tau)$, so that $B \cup \{x\} \subseteq \nabla(e\tau)$, and hence $\nabla(e) \subseteq \nabla(e\tau)$. Hence, by Lemma 3.3, we have $\nabla(e) = \nabla(e\tau)$. ∎

LEMMA 3.9. *Suppose that e is defined by*

$$\begin{bmatrix} \{x\} & \{b_1, y\} & \{b\} \\ x & b_1 & b \end{bmatrix}_{b\in B\setminus\{b_1\}}.$$

Then $\nabla(e) = \nabla(e\tau)$.

Proof. Let $h \in G$ be the automorphism induced by $(xb_1)_Y$, where $Y = B \cup \{x, y\}$. Then $(x)heh = x$, $(y)heh = x$, $(b_1)heh = b_1$ and $(b)heh = b$ for the remaining elements of B. Let $g \in G$ be the automorphism induced by $(yb_1)_Y \in \mathrm{Sym}(Y)$ and let $heh = e^h$. Then $(x)e^hg = x$, $(y)e^hg = x$, $(b_1)e^hg = y$ and $(b)e^hg = b$ for the remaining $b \in B$. Thus e^hg is defined by

$$\begin{bmatrix} \{x, y\} & \{b_1\} & \{b\} \\ x & y & b \end{bmatrix}_{b\in B\setminus\{b_1\}}$$

and so e^hge^hge is defined by

$$\begin{bmatrix} \{x, y, b_1\} & \{b\} \\ x & b \end{bmatrix}_{b\in B\setminus\{b_1\}}.$$

As $\mathrm{corank}(e^hge^hge) = 2$ and e^hge^hge is idempotent, it follows that $(e^hge^hge)\tau = e^hge^hge$. Hence

$$\nabla(e^hge^hge) = \nabla((e^hge^hge)\tau) = \nabla((e^h\tau)ge^hg(e\tau)) \subseteq \nabla(e\tau).$$

Also $b_1 \in \nabla(e\tau)$ and so $B \cup \{x\} \subseteq \nabla(e\tau)$. Thus $\nabla(e) \subseteq \nabla(e\tau)$. It follows from Lemma 3.3 that $\nabla(e) = \nabla(e\tau)$. ∎

LEMMA 3.10. *Suppose that e is defined by*

$$\begin{bmatrix} \{y\} & \{x\} & \{b_1\} & \{b\} \\ 0 & x & b_1 & b \end{bmatrix}_{b\in B\setminus\{b_1\}}.$$

Then $\nabla(e) = \nabla(e\tau)$.

Proof. Let $Y = B \cup \{x, y\}$ and, for $z \in (B \cup \{x\}) \setminus \{b_1\}$, let $g_z \in G$ be the automorphism induced by $(b_1z)_Y$. Then $g_zeg_z = e$ since

$$\begin{aligned} yg_zeg_z &= yeg_z = 0g_z = 0 = ye, \\ b_1g_zeg_z &= zeg_z = zg_z = b_1 = b_1e, \\ zg_zeg_z &= b_1eg_z = b_1g_z = z = ze, \\ bg_zeg_z &= b = be \quad \text{for the remaining } b \in B. \end{aligned}$$

Now, $g_zeg_z = e$ implies $g_z(e\tau)g_z = e\tau$ and so

$$\begin{aligned} z(e\tau) &= (zg_z)(e\tau)g_z = b_1(e\tau)g_z \\ &= b_1g_z \quad (\text{as } b_1 \in \nabla(e\tau)) \\ &= z. \end{aligned}$$

Thus $z \in \nabla(e\tau)$ for all $z \in (B \cup \{x\}) \setminus \{b_1\}$ and, as $b_1 \in \nabla(e\tau)$, we have $B \cup \{x\} \subseteq \nabla(e\tau)$. Hence $\nabla(e) \subseteq \nabla(e\tau)$ and it follows from Lemma 3.3 that $\nabla(e) = \nabla(e\tau)$. ∎

We now conclude the proof of Theorem 3.1. Let S be a semigroup such that $G \leq S \leq \mathrm{End}(\mathcal{A})$ and $e \in E(S)$ be an idempotent endomorphism such that $\mathrm{corank}(e) = 1$. Moreover, let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$.

If $\mathrm{corank}(e\tau) > 1$, then it follows from Corollary 3.6 that $e\tau = (e\tau)\tau^{-1} = e$. Thus, we can assume that $\mathrm{corank}(e) = 1$ and that $\mathrm{corank}(e\tau) = 1$.

Recall the following sets defined in Lemma 3.5.

1. B is a basis for $\nabla(e) \cap \nabla(e\tau)$;
2. $B \cup C$ is a basis for $\nabla(e)$;
3. $B \cup D$ is a basis for $\nabla(e\tau)$;
4. $B \cup C \cup D \cup W$ is a basis for $\mathcal{A}$.

If $D = \emptyset$, then $\nabla(\alpha\tau) = \langle B \rangle \subseteq \nabla(e)$. If $C = \emptyset$ then $\nabla(e) \subseteq \nabla(e\tau)$. In either case, $\nabla(e) = \nabla(e\tau)$ by Lemma 3.3. Thus $W = \emptyset$ and $|D| = |C| = 1$ since $\mathrm{corank}(e) = \mathrm{corank}(e\tau) = 1$.

Suppose now that W is empty. Then, as $|D \cup W| = 1$, it follows that $|D| = 1$. By symmetry, we have $|C| = 1$, say $C = \{x\}$. Thus, $B \cup \{x\}$ is a basis for $\nabla(e)$. Let $\lambda = \mathrm{rank}(\mathcal{A})$ and $B = \{b_i \mid i \in I\}$.

First suppose that $0e^{-1} = \emptyset$ and let the following matrix be a fundamental representation for e:

$$\begin{bmatrix} A_x & A_b \\ x & b \end{bmatrix}_{b \in B}.$$

If $|A_x| = 1 = |A_b|$ for all $b \in B$, it follows that e is one-one and hence, as e is idempotent, $e = \mathrm{id}_A$. Thus $|A_x| > 1$ or $|A_b| > 1$ for some $b \in B$.

On the other hand, $\mathrm{corank}(e) = |D| = 1$ and if $B \cup T$ is a basis for $\mathcal{A}$, then $|T| = 2$. The set $A_x \cup \bigcup_{b \in B} A_b$ is a basis for $\mathcal{A}$. Hence $|A_x| \leq 2$ or $|A_b| \leq 2$ for some $b \in B$.

Suppose first that $|A_x| = 2$. Then the fundamental representation of e has the form

$$e \leftrightarrow \begin{bmatrix} \{x, y\} & \{b\} \\ x & b \end{bmatrix}_{b \in B}$$

and so, by Lemma 3.8, $\nabla(e) = \nabla(e\tau)$.

Suppose now that there is a $b \in B$ such that $|A_b| = 2$, say $|A_{b_1}| = 2$. Then e has the fundamental representation

$$e \leftrightarrow \begin{bmatrix} \{x\} & \{b_1, y\} & \{b\} \\ x & b_1 & b \end{bmatrix}_{b \in B \setminus \{b_1\}}$$

and so, by Lemma 3.9, $\nabla(e) = \nabla(e\tau)$.

Finally, suppose that $0e^{-1} \neq \{0\}$. Then e has a fundamental representation of the form

$$e \leftrightarrow \begin{bmatrix} A_0 & A_x & A_b \\ 0 & x & b \end{bmatrix}_{b\in B}.$$

Now $Y = A_0 \cup A_x \cup \bigcup_{b\in B} A_b$ is a basis for $\mathcal{A}$ which contains $B \cup \{x\}$. As corank $\langle B \cup \{x\}\rangle) = 1$ it follows that $|A_0| = 1$.

Thus $1 = |A_0| = |A_x| = |A_b|$ for all $b \in B$, and hence, by Lemma 3.10, $\nabla(e) = \nabla(e\tau)$.

This finishes the proof of Theorem 3.1.

4. $\tau|_R = \mathrm{id}_R$**.** We start with a definition. Let $S_0 = \langle E_0 \cup G\rangle$, where $E_0 \subseteq E(\mathrm{End}(\mathcal{A}))$. We say that $R = \{\alpha, \alpha^*\} \subseteq \mathrm{End}(\mathcal{A})$ is a *regular set* in S_0 if it has the following two properties:

(1) for some $\omega \in S_0$ we have $\alpha\omega\alpha^* = \mathrm{id}_A$;
(2) for some $h \in G$ we have $\alpha^*\alpha\omega\alpha = \alpha^*\alpha\omega h$.

Now we can state the main theorem in this section.

THEOREM 4.1. *Let* $R = \{\alpha, \alpha^*\}$ *be a regular set in* $S_0 = \langle E_0 \cup G\rangle$, *where* $E_0 \subseteq E(\mathrm{End}(\mathcal{A}))$. *Let* $S = \langle S_0 \cup R\rangle$ *and let* $\tau \in \mathrm{Aut}(S)$ *be such that* $\tau|G = \mathrm{id}_G$. *Then* $\tau = \mathrm{id}_S$.

Proof. Since R is regular, there exists an element $\omega \in S_0$ such that $\alpha\omega\alpha^* = \mathrm{id}_A$. The main tool in this proof will be the element $\alpha^*\alpha\omega\alpha \in S$. Let $\varepsilon = \alpha^*\alpha\omega$. Then

$$\varepsilon^2 = \alpha^*\alpha\omega\alpha^*\alpha\omega = \alpha^*(\alpha\omega\alpha^*)\alpha\omega = \alpha^*(\mathrm{id}_A)\alpha\omega = \alpha^*\alpha\omega = \varepsilon.$$

This proves that ε is idempotent.

Observe that α must be a monomorphism since $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\alpha\omega\alpha^*) = \mathrm{Ker}(\mathrm{id}_A)$. Since $\mathrm{Ker}(\mathrm{id}_A)$ is the identity relation so is $\mathrm{Ker}(\alpha)$.

Now, for some $h \in G$, we have $\varepsilon\alpha = \varepsilon h$ (since R is regular) so that $(\varepsilon\alpha)\tau = (\varepsilon h)\tau$. As ε is idempotent, by the main theorem of the previous section, $\varepsilon\tau = \varepsilon$; and by hypothesis $h\tau = h$. Therefore

$$(\varepsilon\alpha)\tau = (\varepsilon h)\tau \;\Rightarrow\; \varepsilon(\alpha\tau) = \varepsilon h = \varepsilon\alpha.$$

Let $y \in \nabla(\varepsilon)$. Since ε is idempotent, $y\varepsilon = y$. Thus

$$y(\alpha\tau) = y\varepsilon(\alpha\tau) = y\varepsilon\alpha = y\alpha.$$

Hence, as α is injective, we can apply Lemma 3.4 of [3] to conclude that $\alpha\tau = \alpha$.

To prove that $\alpha^*\tau = \alpha^*$ we have the following chain of equalities:

$$\begin{aligned}
\alpha^* &= \alpha^* \operatorname{id}_A \\
&= \alpha^*\alpha\omega\alpha^* && (\text{as } \alpha\omega\alpha^* = \operatorname{id}_A) \\
&= (\alpha^*\alpha\omega)\tau\alpha^* && (\text{as } \alpha^*\alpha\omega = \varepsilon \text{ and } \varepsilon\tau = \varepsilon) \\
&= (\alpha^*\tau)(\alpha\tau)(\omega\tau)\alpha^* \\
&= (\alpha^*\tau)\alpha\omega\alpha^* && (\text{as } \alpha\tau = \alpha \text{ and } \omega\tau = \omega) \\
&= (\alpha^*\tau) \operatorname{id}_A && (\text{as } \alpha\omega\alpha^* = \operatorname{id}_A) \\
&= \alpha^*\tau.
\end{aligned}$$

The theorem is proved. ■

5. The main result. The results of the previous two sections provide the necessary background to prove the main result of this paper:

THEOREM 5.1. *Let $\mathcal{A}$ be a strong independence algebra of infinite rank with at most one constant. Let $S_0 = \langle G \cup E_0 \rangle$, where $E_0 \subseteq E(\operatorname{End}(\mathcal{A}))$, and let $S = \langle S_0 \cup R_0 \rangle$ with $R_0 \in \{\emptyset, \{\alpha, \alpha^*\}\}$ where $\{\alpha, \alpha^*\}$ is a regular set in S_0. Then every $\psi \in \operatorname{Aut}(G)$ admits at most one lift, that is,*

$$|L(\psi)| = |\{\phi \in \operatorname{Aut}(S) \mid \phi|_S = \psi\}| \le 1.$$

Proof. Let $\psi \in \operatorname{Aut}(G)$ and suppose that $\phi_1, \phi_2 \in L(\psi)$. Therefore $\phi_1|_G = \psi = \phi_2|_G$ so that $\tau|_G = (\phi_1\phi_2^{-1})|_G = \operatorname{id}_G$ and $\tau|_G \in \operatorname{Aut}(S)$. The main theorem in Section 3 implies that $e\tau = e$ for all $e \in E_0$ (and all $\tau \in \operatorname{Aut}(S)$ such that $\tau|_G = \operatorname{id}_G$), so that $e\phi_1 = e\phi_2$. In the same way the main theorem in Section 4 implies that $a\tau = a$ for all $a \in R_0$, and hence $a\phi_1 = a\phi_2$. We have proved that ϕ_1 and ϕ_2 coincide on a generating set of S and hence $\phi_1 = \phi_2$. ■

In [3] it was proved that there are some semigroups S such that $G \le S \le \operatorname{End}(\mathcal{A})$ and $|L(\psi)| \ge 1$ for some $\psi \in \operatorname{Aut}(G)$. But nothing was said about $\operatorname{End}(\mathcal{A})$ itself. Theorem 5.1 together with the main result of [2] gives the following corollary.

COROLLARY 5.2. *Let $\mathcal{A}$ be a strong independence algebra of infinite rank with at most one constant. Let $S = \operatorname{End}(\mathcal{A})$. Then every $\psi \in \operatorname{Aut}(G)$ admits at most one lift, that is,*

$$|L(\psi)| = |\{\phi \in \operatorname{Aut}(S) \mid \phi|_S = \psi\}| \le 1.$$

Proof. We start by introducing two special endomorphisms. Let B be a basis of $\mathcal{A}$ and let $\alpha \in \operatorname{End}(\mathcal{A})$ be such that $\alpha : B \to Y$ is a one-one mapping and $|B| = |Y| = \operatorname{corank}(\alpha)$. Hence $\operatorname{rank}(\alpha) = \operatorname{corank}(\alpha) = \operatorname{rank}(\mathcal{A})$. Let $Y \cup W$ be a basis of $\mathcal{A}$.

Define $\alpha^{-1} : Y \to B$ such that for all $x\alpha \in Y$ we have $(x\alpha)\alpha^{-1} = x$. Moreover, let

$$\alpha^* : Y \cup W \to B, \quad y \in Y \mapsto y\alpha^{-1}, \quad x \in W \mapsto b_x \in B.$$

Thus $\alpha^*|_Y = \alpha^{-1}$ and α^* is onto.

As said before, we will use the same letter α to represent the mapping $\alpha : B \to Y$ and the unique morphism $\overline{\alpha} : \langle B \rangle \to \langle Y \rangle$ that extends the mapping α. A similar convention will be adopted for α^{-1} and α^*. Moreover, the letters B and Y are reserved for the independent sets related to α. The letter W will represent a fixed set such that $Y \cup W$ is a basis of $\mathcal{A}$.

The main result in [2] states that

$$\mathrm{End}(\mathcal{A}) = \langle G \cup E(\mathrm{End}(\mathcal{A})) \cup R \rangle,$$

where $R = \{\alpha, \alpha^*\}$. Therefore, to prove the theorem, it remains to prove that $R = \{\alpha, \alpha^*\}$ is a regular set in $\mathrm{End}(\mathcal{A})_0 = \langle G \cup E(\mathrm{End}(\mathcal{A})) \rangle$. It is obvious that $\alpha\alpha^* = \mathrm{id}_A$. Since $\mathrm{id}_A \in G$, it follows that there exists $\omega = \mathrm{id}_A \in \mathrm{End}(\mathcal{A})_0$ such that $\alpha\omega\alpha^* = \mathrm{id}_A$. Finally, we have to prove that $\alpha^*\alpha\alpha = \alpha^*\alpha h$ for some $h \in G$. Let $\alpha^*\alpha = \varepsilon$ and recall that $|Y\alpha| = |Y| = |B|$ and $|W\alpha| = |W| = |B|$. Let Z be a set such that $Y\alpha \cup W\alpha \cup Z$ is a basis for $\mathcal{A}$. The mapping $\alpha^*\alpha\alpha$ is determined by the following composition of mappings:

$$Y \cup W \to B \to Y \cup W \to Y\alpha \cup W\alpha \cup Z.$$

As $|W\alpha \cup Z| = |B| = |W|$ and $|Y\alpha| = |Y|$, there is a bijection $g : Y\alpha \cup W\alpha \cup Z \to Y \cup W$ such that $g|_{Y\alpha} = \alpha^{-1}|_{Y\alpha}$ and $(W\alpha \cup Z)g = W$.

We claim that $\varepsilon\alpha g = \varepsilon$. In fact, let $a \in A$. Then, since $\nabla(\varepsilon) = \langle Y \rangle$, we have $a\varepsilon = t(y_1, \ldots, y_n)$ for some term t and $y_1, \ldots, y_n \in Y$. Hence

$$\begin{aligned}(a)\varepsilon\alpha g &= t(y_1, \ldots, y_n)\alpha g = t(y_1\alpha, \ldots, y_n\alpha)g = t(y_1\alpha g, \ldots, y_n\alpha g)\\ &= t(y_1, \ldots, y_n) \quad (\text{as } g_{|_{Y\alpha}} = \alpha^{-1}|_{Y\alpha})\\ &= a\varepsilon.\end{aligned}$$

This proves that $\varepsilon\alpha g = \varepsilon$ and hence, for $h = g^{-1} \in G$, we have $\varepsilon\alpha = \varepsilon h$ as required. Therefore R is a regular set in $\mathrm{End}(\mathcal{A})_0$ and hence, by the previous theorem,

$$|L(\psi)| = |\{\phi \in \mathrm{Aut}(S) \mid \phi|_S = \psi\}| \leq 1. \blacksquare$$

Theorem 5.1 yields a description of $\mathrm{Aut}(S)$, for every semigroup S under the hypothesis of the theorem, provided we know a description of $\mathrm{Aut}(G)$. Since this is the case when $\mathcal{A}$ is a set or a vector space, we finish this paper with two corollaries. We start by introducing some notation. Let $S \leq T(X)$. Then the *normalizer* of S in $\mathrm{Sym}(X)$ is

$$N_{\mathrm{Sym}(X)}(S) = \{g \in \mathrm{Sym}(X) \mid g^{-1}Sg = S\}.$$

Let T be a semigroup, let $S \leq T$ be a subsemigroup and $H \leq T$ be a group. For $h \in H$, let $\tau^h : S \to T$ be the mapping defined by $s\tau^h = h^{-1}sh$ for all $s \in S$. Now we can state the first corollary.

Corollary 5.3. *Let X be an infinite set. Let $S_0 = \langle \mathrm{Sym}(X) \cup E_0 \rangle$, where $E_0 \subseteq E(T(X))$, and let $S = \langle S_0 \cup R_0 \rangle$ with $R_0 \in \{\emptyset, \{\alpha, \alpha^*\}\}$ where $\{\alpha, \alpha^*\}$ is a regular set in S_0. Then*

$$\mathrm{Aut}(S) = \{\tau^g \mid g \in \mathrm{Sym}(X)\}.$$

In particular $\mathrm{Aut}(T(X)) = \{\tau^g \mid g \in \mathrm{Sym}(X)\}$.

Proof. It is well known (see [35]) that $\mathrm{Aut}(\mathrm{Sym}(X))$ is the group of all mappings induced under conjugation by the permutations $g \in \mathrm{Sym}(X)$, that is, if $\phi \in \mathrm{Aut}(\mathrm{Sym}(X))$ then, for some $g \in \mathrm{Sym}(X)$, $\phi = \tau^g : \mathrm{Sym}(X) \to \mathrm{Sym}(X)$, where $a\tau^g = g^{-1}ag$.

By Theorem 5.1, $|L(\tau^g)| \leq 1$ and it is obvious that $|L(\tau^g)| = 1$ if and only if $g \in N_{\mathrm{Sym}(X)}(S)$. Since $\mathrm{Sym}(X) \leq S$, it follows that $N_{\mathrm{Sym}(X)}(S) = \mathrm{Sym}(X)$ and hence

$$\mathrm{Aut}(S) = \{\tau^g : S \to S \mid g \in \mathrm{Sym}(X)\}.$$

In particular $\mathrm{Aut}(T(X)) = \{\tau^g : T(X) \to T(X) \mid g \in \mathrm{Sym}(X)\}$. ∎

Before stating our final corollary, we introduce some notation and definitions about vector spaces. Let V be an infinite vector space (over a field F). A bijection γ of V is said to be *semilinear* if there is an automorphism α of F such that for all $a, b \in V$ and $\lambda \in F$ we have

$$(a+b)\gamma = (a)\gamma + (b)\gamma \quad \text{and} \quad (\lambda a)\gamma = \lambda^\alpha (a\gamma).$$

The group of semilinear transformations is usually denoted by $\Gamma(V)$. It is well known (see [34]) that

$$\mathrm{Aut}(\mathrm{Aut}(V)) = \{\tau^g : \mathrm{Aut}(V) \to \mathrm{Aut}(V) \mid g \in \Gamma(V)\}.$$

Repeating the same arguments used in the proof of the previous corollary we have the following.

Corollary 5.4. *Let V be a vector space of infinite dimension over a field F. Let $S_0 = \langle \mathrm{Aut}(V) \cup E_0 \rangle$, where $E_0 \subseteq E(\mathrm{End}(V))$, and let $S = \langle S_0 \cup R_0 \rangle$ with $R_0 \in \{\emptyset, \{\alpha, \alpha^*\}\}$ where $\{\alpha, \alpha^*\}$ is a regular set in S_0. Then*

$$\mathrm{Aut}(S) = \{\tau^g \mid g \in \Gamma(V) \text{ and } g^{-1}Sg = S\}.$$

In particular $\mathrm{Aut}(\mathrm{End}(V)) = \{\tau^g \mid g \in \Gamma(V)\}$.

6. Problems. The previous sections suggest a number of problems that we now state.

1. Given $S_0 = \langle G \cup E_0 \rangle$ classify all the regular sets in S_0, where $\mathcal{A}$ is an independence algebra of infinite rank, $G = \mathrm{Aut}(\mathcal{A})$ and $E_0 \subseteq E(\mathrm{End}(\mathcal{A}))$.
2. Given a semigroup S such that $S = \langle G \cup E_0 \cup \{\alpha, \alpha^*\} \rangle$, where α is injective, α^* is onto, but $\{\alpha, \alpha^*\}$ is not regular, describe the lifts of every $\psi \in \mathrm{Aut}(G)$.

3. Given a semigroup of monomorphisms $G < S \leq \mathrm{End}(\mathcal{A})$, describe the lifts of every $\psi \in \mathrm{Aut}(G)$.
4. Given a semigroup of epimorphisms $G < S \leq \mathrm{End}(\mathcal{A})$, describe the lifts of every $\psi \in \mathrm{Aut}(G)$.
5. Characterize the semigroups S such that $\mathrm{Aut}(\mathcal{A}) \leq S \leq \mathrm{PEnd}(\mathcal{A})$ and S has the unique extension property.
6. Let S be a semigroup such that $\mathrm{Aut}(\mathcal{A}) \leq S \leq \mathrm{PEnd}(\mathcal{A})$. Describe the group $\mathrm{Aut}(S)$.

The description of $\mathrm{Aut}(\mathrm{Aut}(\mathcal{A}))$ is an open problem. We conjecture that for an independence algebra $\mathcal{A}$ of infinite rank,

$$\mathrm{Aut}(\mathrm{Aut}(\mathcal{A})) = \{\tau^g \mid g \in \mathrm{WAut}(\mathcal{A})\}$$

where $\mathrm{WAut}(\mathcal{A})$ is the group of weak automorphisms of $\mathcal{A}$ (see [18] and [36]).

Acknowledgements. I would like to thank my supervisor, Professor John Fountain, and Professors Victoria Gould and Peter M. Higgins, for their comments on a previous draft of this paper. Also I thank for the support of POCTI-ISFL-1-143 of Centro de Algebra da Universidade de Lisboa, financed by FCT and FEDER.

REFERENCES

[1] J. Araújo, *Normal semigroups of endomorphisms of proper independence algebras are idempotent generated*, Proc. Edinburgh Math. Soc. 45 (2002), 205–217.
[2] —, *Generators for the semigroup of endomorphisms of an independence algebra*, Algebra Colloq. 9 (2002), 375–382.
[3] —, *Lifts for semigroups of monomorphisms of an independence algebra*, Colloq. Math. 97 (2003), 277–284.
[4] —, *Idempotent generated endomorphisms of an independence algebra*, Semigroup Forum 67 (2003), 464–467.
[5] J. Araújo and J. Fountain, *The origins of independence algebras*, in: Semigroups and Languages (Lisboa, 2002), World Sci., 2004, 54–67.
[6] —, —, *A description of normal semigroups of endomorphisms of proper independence algebras*, Comm. Algebra 33 (2005), 2705–2711.
[7] J. Araújo and F. C. Silva, *Semigroups of linear endomorphisms closed under conjugation*, ibid. 28 (2000), 3679–3689.
[8] —, —, *Semigroups of matrices closed under conjugation by normal linear groups*, JP J. Algebra Number Theory 5 (2005), 535–545.
[9] P. J. Cameron and Cs. Szabó, *Independence algebras*, J. London Math. Soc. (2) 61 (2000), 321–334.
[10] S. Chantip and G. R. Wood, *Automorphisms of the semigroup of all onto mappings of a set*, Bull. Austral. Math. Soc. 14 (1976), 399–403.
[11] J. A. Erdos, *On products of idempotent matrices*, Glasgow Math. J. 8 (1967), 118–122.

[12] S. P. Fitzpatrick and J. S. Symons, *Automorphisms of transformation semigroups*, Proc. Edinburgh Math. Soc. 19 (1974/75), 327–329.

[13] J. Fountain, *The depth of the semigroup of balanced endomorphisms*, Mathematika 41 (1994), 199–208.

[14] J. Fountain and A. Lewin, *Products of idempotent endomorphisms of an independence algebra of finite rank*, Proc. Edinburgh Math. Soc. 35 (1992), 493–500.

[15] —, —, *Products of idempotent endomorphisms of an independence algebra of infinite rank*, Math. Proc. Cambridge Philos. Soc. 114 (1993), 303–319.

[16] K. Głazek, *Some old and new problems in the independence theory*, Colloq. Math. 42 (1979), 127–189.

[17] L. M. Gluskin, *Semigroups and rings of endomorphisms of linear spaces I*, Amer. Math. Soc. Transl. 45 (1965), 105–137.

[18] A. Goetz, *On weak isomorphisms and weak homomorphisms of abstract algebras*, Colloq. Math. 14 (1966), 163–167.

[19] V. A. R. Gould, *Independence algebras*, Algebra Universalis 33 (1995), 294–318.

[20] G. Grätzer, *A theorem on doubly transitive permutation groups with application to universal algebra*, Fund. Math. 52 (1963), 25–41.

[21] J. M. Howie, *The subsemigroup generated by the idempotents of a full transformation semigroup*, J. London Math Soc. 41 (1966), 707–716.

[22] —, *Fundamentals of Semigroup Theory*, Oxford Univ. Press, New York, 1995.

[23] A. I. Mal'cev [A. I. Mal'tsev], *Symmetric groupoids*, Mat. Sb. (N.S.) 31 (1952), 136–151 (in Russian); English transl.: Amer. Math. Soc. Transl. 113 (1979), 235–250.

[24] E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 6 (1958), 731–736.

[25] —, *Independence in some abstract algebras*, ibid. 7 (1959), 611–616.

[26] —, *Independence in algebras of sets and Boolean algebras*, Fund. Math. 48 (1960), 135–145.

[27] —, *Independence and homomorphisms in abstract algebras*, ibid. 50 (1961), 45–61.

[28] E. Marczewski and K. Urbanik, *Abstract algebras in which all elements are independent*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), 291–293.

[29] R. N. McKenzie, G. F. McNulty and W. F. Taylor, *Algebra, Lattices, Varieties*, Vol. I, Wadsworth, Monterey, CA, 1987.

[30] W. Narkiewicz, *Independence in a certain class of abstract algebras*, Fund. Math. 50 (1962), 333–340.

[31] —, *A note on v^*-algebras*, ibid. 52 (1963), 289–290.

[32] —, *On a certain class of abstract algebras*, ibid. 54 (1964), 115–124.

[33] M. A. Reynolds and R. P. Sullivan, *Products of idempotent linear transformations*, Proc. Roy. Soc. Edinburgh Sect. A 100 (1985), 123–138.

[34] C. E. Rickart, *Isomorphisms of infinite-dimensional analogues of the classical groups*, Bull. Amer. Math. Soc. 57 (1951), 435–448.

[35] W. R. Scott, *Group Theory*, Dover, New York, 1987.

[36] J. R. Senft, *On weak automorphisms of universal algebras*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), 115–118.

[37] R. P. Sullivan, *Transformation semigroups and linear algebra*, in: T. E. Hall, P. R. Jones and J. C. Meakin (eds.), Monash Conf. on Semigroup Theory, World Sci., 1991, 290–295.

[38] K. Urbanik, *A representation theorem for Marczewski's algebras*, Fund. Math. 48 (1960), 147–167.

[39] —, *A representation theorem for v^*-algebras*, ibid. 52 (1963), 291–317.

[40] K. Urbanik, *A representation theorem for two-dimensional v^*-algebras*, ibid. 57 (1965), 215–236.
[41] —, *Linear independence in abstract algebras*, Colloq. Math. 14 (1966), 233–255.

Centro de Álgebra
Universidade de Lisboa
Av. Gama Pinto, 2
1649-003 Lisboa, Portugal
E-mail: jaraujo@ptmat.fc.ul.pt

Universidade Aberta
Rua da Escola Politécnica, no. 141–147
1269-001 Lisboa, Portugal

Received 13 October 2005;
revised 13 January 2006 (4678)