SCATTERING THEORY FOR A NONLINEAR SYSTEM OF WAVE EQUATIONS WITH CRITICAL GROWTH

BY

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Abstract. We consider scattering properties of the critical nonlinear system of wave equations with Hamilton structure

\[
\begin{align*}
  u_{tt} - \Delta u &= -F_1(|u|^2, |v|^2)u, \\
  v_{tt} - \Delta v &= -F_2(|u|^2, |v|^2)v,
\end{align*}
\]

for which there exists a function \( F(\lambda, \mu) \) such that

\[
\frac{\partial F(\lambda, \mu)}{\partial \lambda} = F_1(\lambda, \mu), \quad \frac{\partial F(\lambda, \mu)}{\partial \mu} = F_2(\lambda, \mu).
\]

By using the energy-conservation law over the exterior of a truncated forward light cone and a dilation identity, we get a decay estimate for the potential energy. The resulting global-in-time estimates imply immediately the existence of the wave operators and the scattering operator.

1. Introduction. In this note, we continue our study from [3, 4] on the following nonlinear system of wave equations with Hamilton structure:

\[
\begin{align*}
  u_{tt} - \Delta u &= -F_1(|u|^2, |v|^2)u, \\
  v_{tt} - \Delta v &= -F_2(|u|^2, |v|^2)v, \\
  u(0) &= \varphi_1(x), \quad u_t(0) = \psi_1(x), \\
  v(0) &= \varphi_2(x), \quad v_t(0) = \psi_2(x), \\
  \left( \varphi_j, \psi_j \right) &\in \dot{H}^1 \times L^2, \quad j = 1, 2,
\end{align*}
\]

where we assume the existence of a function \( F(\lambda, \mu) \) such that

\[
\frac{\partial F(\lambda, \mu)}{\partial \lambda} = F_1(\lambda, \mu), \quad \frac{\partial F(\lambda, \mu)}{\partial \mu} = F_2(\lambda, \mu).
\]

To ensure that the potential energy of problem (1.1) tends to zero as \( t \to \infty \), which will play an important role in the proof of our result, we need to assume that \( F, F_1, F_2 \) satisfy the following assumptions similar to those

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in [3, 4]:

\( (H1) \quad |F_1| + |u^2 F_{11}| + |uv F_{12}| + |F_2| + |uv F_{21}| + |v^2 F_{22}| \leq C(|u|^{2^* - 2} + |v|^{2^* - 2}), \)

where \( F_{11} = \frac{\partial F_1}{\partial \lambda}, \) \( F_{12} = \frac{\partial F_1}{\partial \mu}, \) \( F_{21} = \frac{\partial F_2}{\partial \lambda}, \) \( F_{22} = \frac{\partial F_2}{\partial \mu}; \)

\( (H2) \quad F(|u|^2, |v|^2) \geq 0, \quad F(0,0) = 0; \)

\( (H3) \quad |u|^{2^*} + |v|^{2^*} \leq C_0 F(|u|^2, |v|^2); \)

\( (H4) \quad \frac{n-1}{2} |u|^2 F_1(|u|^2, |v|^2) + \frac{n-1}{2} |v|^2 F_2(|u|^2, |v|^2) \geq \frac{n+1}{2} F(|u|^2, |v|^2) \)

for \(|u|\) or \(|v|\) larger than a fixed constant \( M; \)

\( (H5) \quad |F_1(|u_1|^2, |v_1|^2)u_1 - F_1(|u_2|^2, |v_2|^2)u_2| + |F_2(|u_1|^2, |v_1|^2)v_1 - F_2(|u_2|^2, |v_2|^2)v_2| \)

\( \leq C(|u_1|^{2^* - 2} + |v_1|^{2^* - 2} + |u_2|^{2^* - 2} + |v_2|^{2^* - 2})(|u_1 - u_2| + |v_1 - v_2|). \)

Note that \((H1)\) and \((H2)\) imply an inequality which is reverse to \((H3)\):

\( (1.2) \quad F(|u|^2, |v|^2) \leq C(|u|^{2^*} + |v|^{2^*}). \)

It is easy to verify that e.g. the function \( F(|u|^2, |v|^2) = |u|^6 + |u|^4|v|^2 + |u|^2|v|^4 + |v|^6 \) satisfies \((H1)-(H5)\) in the space dimension \( n = 3 \). For the physical background and related research on the wave equation, we refer the reader to [1, 3–7] and the references therein.

Let us end this section by recalling what we have done in our previous papers. On the basis of a dilation identity derived through the Lagrangian associated with problem \((1.1)\), we prove in [3] that the “potential energy” cannot concentrate at any given point. We combine this fact with the Strichartz estimate to improve the regularity of a solution with finite-energy initial data. That reasoning is completed by standard energy estimates.

In [4], we study the well-posedness of problem \((1.1)\) in the energy space under assumptions on nonlinearities slightly more general than those in \((H1)-(H5)\). By showing through an approximation argument that the energy and the dilation identities hold true for weak solutions, we prove that problem \((1.1)\) has a unique solution \((u, v)\) such that

\( (1.3) \quad (u, v, u_t, v_t) \in C(\mathbb{R}; \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2) \)

\( \cap \ L^q_{\text{loc}}(\mathbb{R}; \dot{B}^{1/2}_{q} \times \dot{B}^{1/2}_{q} \times \dot{B}^{-1/2}_{q} \times \dot{B}^{-1/2}_{q}). \)

Here, the Besov space \( \dot{B}^s_{p,q} \) is defined as the set of those functions for which the following norm is finite:

\( \|f\|_{\dot{B}^s_{p,q}} \equiv \left\{ \int_0^\infty \sup_{|y| \leq t} \left| \frac{t^{-s} \|y\| f - f}{L^p} \right|^q \frac{dt}{t} \right\}^{1/q}, \)
where \( \tau_y \) denotes the space translation by \( y \in \mathbb{R}^n \) (cf. [2, p. 493, eq. (3.15)]).
We limit ourselves to the particular case of this space for \( p = q \) and we write \( \dot{B}^s_q(\mathbb{R}^n) = \dot{B}^s_{q,q}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \) for \( q = 2(n + 1)/(n - 1) \) and \( q^* = 2n(n + 1)/(n^2 - 2n - 1) \).

2. Global space-time estimate. Our first goal is to improve the result from [4] and to obtain global-in-time estimates of solutions to (1.1).

**Theorem 2.1.** Assume that \( F, F_1, F_2 \) satisfy (H1)–(H5). Then problem (1.1) has a unique solution satisfying

\[
(1.1) \quad (u, v, u_t, v_t) \in C(\mathbb{R}; \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2) \\
\cap L^q(\mathbb{R}; \dot{B}^{1/2}_q \times \dot{B}^{-1/2}_q \times \dot{B}^{-1/2}_q \times \dot{B}^{-1/2}_q),
\]

where \( \dot{B}^s_q(\mathbb{R}^n) = \dot{B}^s_{q,q}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \) with \( q = 2(n + 1)/(n - 1) \) and \( q^* = 2n(n + 1)/(n^2 - 2n - 1) \).

Note that, by [4], problem (1.1) has a unique solution satisfying (1.3). Hence, to prove Theorem 2.1, we only need to verify that there exists \( T_0 > 0 \) such that for \( I = [T_0, \infty) \), the following quantities are finite:

\[
\|u\|_{L^q(I; \dot{B}^{1/2}_q(\mathbb{R}^n))}, \quad \|v\|_{L^q(I; \dot{B}^{-1/2}_q(\mathbb{R}^n))}, \\
\|u_t\|_{L^q(I; \dot{B}^{-1/2}_q(\mathbb{R}^n))}, \quad \|v_t\|_{L^q(I; \dot{B}^{-1/2}_q(\mathbb{R}^n))}.
\]

As we shall see below, to this end, we should first prove that \( \|u(t)\|_{L^{2^*}(\mathbb{R}^n)} \) and \( \|v(t)\|_{L^{2^*}(\mathbb{R}^n)} \) tend to zero as \( t \to \infty \). However, it follows from our assumptions (1.2) and (H3) that it suffices to show the following result.

**Proposition 2.2.** Let \( (u, v) \) be a solution of (1.1), and let \( F, F_1, F_2 \) satisfy (H2)–(H4). Then

\[
(2.3) \quad g(t) = \lim_{t \to \infty} \frac{1}{2} \int_{\mathbb{R}^n} F(|u(x, t)|^2, |v(x, t)|^2) \, dx = 0.
\]

**Proof.** Since the initial data have finite energy, we obtain

\[
(2.4) \quad \int_{|x| \geq R} e(u, v)(x, 0) \, dx \to 0 \quad \text{as} \ R \to \infty,
\]

where

\[
(2.5) \quad e(u, v) = \frac{1}{2} (|u_t|^2 + |v_t|^2 + |\nabla u|^2 + |\nabla v|^2 + F).
\]

Applying the energy conservation law on the exterior of a truncated forward light cone, for every \( t \geq 0 \) one gets

\[
(2.6) \quad \int_{|x| > R + t} e(u, v) \, dx + \text{Flux}(u, v; M^t_0) \to 0 \quad \text{as} \ R \to \infty,
\]
where the Flux on the mantle is given by (cf. [7, p. 137])

\[
\text{Flux}(u, v; M^b_a) = \int_{M^b_a} \left\{ (-u_t \nabla u - v_t \nabla v) \cdot \frac{-x}{|x|} + e(u, v) \times 1 \right\} d\sigma \\
= \int_{M^b_a} \left\{ \frac{1}{2} \left| \frac{x}{|x|} u_t + \nabla u \right|^2 + \frac{1}{2} \left| v_t + \nabla v \right|^2 + \frac{1}{2} \right\} d\sigma
\]

with

\[
M^b_a = \{(x, t) \in \mathbb{R}^n \times [a, b] : |x| = R + t\}.
\]

By identity (2.7), the Flux is nonnegative. Since \(e(u, v)\) contains the potential energy term \(\frac{1}{2}F\), it follows from (2.5)–(2.7) that

\[
\frac{1}{2} \int_{|x| > R+t} F dx \leq \int_{|x| > R+t} e(u, v) dx \\
\leq \int_{|x| > R+t} e(u, v) dx + \text{Flux}(u, v; M^t_0) \to 0 \quad \text{as } R \to \infty.
\]

Therefore, to complete the proof of Proposition 2.2, it suffices to show that

\[
\frac{1}{2} \int_{|x| \leq R+t} F dx \to 0 \quad \text{as } t \to \infty.
\]

If we replace \(t\) by \(t + R\), (2.6), (2.8) and (2.9) can be rewritten as

\[
\int_{|x| > t} e(u, v) dx + \text{Flux}(u, v; M^t_R) \to 0 \quad \text{as } R \to \infty,
\]

\[
M^b_a = \{(x, t) \in \mathbb{R}^n \times [a, b] : |x| = t\},
\]

\[
\frac{1}{2} \int_{|x| \leq t} F dx \to 0, \quad t \to \infty.
\]

To prove (2.9'), we use the following dilation identity obtained in [3, 4]:

\[
\text{div}_{x,t} \left( -tP_0, tQ_0 + \frac{n-1}{2} u_t u + \frac{n-1}{2} v_t v \right) - R_0 = 0,
\]

where

\[
Q_0 = \frac{1}{2} |u'|^2 + \frac{1}{2} |v'|^2 + \frac{1}{2} F + u_t \frac{x \cdot \nabla u}{t} + v_t \frac{x \cdot \nabla v}{t},
\]

\[
P_0 = \left( \frac{1}{2} |u_t|^2 + \frac{1}{2} |v_t|^2 - \frac{1}{2} |\nabla u|^2 - \frac{1}{2} |\nabla v|^2 - \frac{1}{2} F \right) \left( \frac{x}{t} \right),
\]

\[
R_0 = \frac{n-1}{2} F_1 |u|^2 + \frac{n-1}{2} F_2 |v|^2 - \frac{n+1}{2} F.
\]
Integrating identity (2.10) over \(K(T, S) = \{(x, t) \in \mathbb{R}^n \times [T, S] : T \leq t \leq S, \ |x| < t\}\), we obtain

\[
(2.11) \quad 0 = \int_{DS} \left( SQ_0 + \frac{n-1}{2} u_t u + \frac{n-1}{2} v_t v \right) dx \\
- \int_{DT} \left( TQ_0 + \frac{n-1}{2} u_t u + \frac{n-1}{2} v_t v \right) dx \\
- \frac{1}{\sqrt{2}} \int_{M_T^S} \left( tQ_0 + \frac{n-1}{2} u_t u + \frac{n-1}{2} v_t v + x \cdot P_0 \right) d\sigma \\
+ \int_{K(T, S)} \int_{K(T, S)} R_0 \, dx \, dt = I_1 + I_2 + I_3 + I_4,
\]

where

\[
D_T = \{(x, t) : |x| \leq T\}, \quad M_T^S = \{(x, t) : T \leq t \leq S, \ |x| = t\}.
\]

Note that \(t = |x|\) on \(M_T^S\), hence we rewrite the term \(I_3\) in (2.11) as

\[
(2.12) \quad I_3 = -\frac{1}{\sqrt{2}} \int_{M_T^S} \left( \frac{|x|}{2} |u'|^2 + \frac{|x|}{2} |v'|^2 + \frac{|x|}{2} F + u_t x \cdot \nabla u + v_t x \cdot \nabla v + \frac{n-1}{2} u_t u \\
+ \frac{n-1}{2} v_t v + \frac{n-1}{2} \frac{u}{|x|} x \cdot \nabla u + \frac{n-1}{2} \frac{v}{|x|} x \cdot \nabla v \right) \left( \frac{1}{|x|} (x \cdot \nabla u)^2 \\
+ \frac{n-1}{2} \frac{v}{|x|} x \cdot \nabla v + \frac{n-1}{2} \frac{v}{|x|} x \cdot \nabla v \right) \right) d\sigma \\
- \frac{|x|}{2} |\nabla u|^2 - \frac{|x|}{2} |\nabla v|^2 + \frac{|x|}{2} |u_t|^2 + \frac{|x|}{2} |v_t|^2 - \frac{|x|}{2} F \right) d\sigma \\
= -\frac{1}{\sqrt{2}} \int_{M_T^S} \left( \frac{|x|}{2} |u_t|^2 + \frac{|x|}{2} |v_t|^2 + 2u_t x \cdot \nabla u + 2v_t x \cdot \nabla v + \frac{n-1}{2} u_t u \\
+ \frac{n-1}{2} v_t v + \frac{1}{|x|} (x \cdot \nabla u)^2 + \frac{1}{|x|} (x \cdot \nabla v)^2 \\
+ \frac{n-1}{2} \frac{u}{|x|} x \cdot \nabla u + \frac{n-1}{2} \frac{v}{|x|} x \cdot \nabla v \right) d\sigma \\
= -\frac{1}{\sqrt{2}} \int_{M_T^S} \left[ |x| \left( \frac{x \cdot \nabla u}{|x|} + u_t \right)^2 + \frac{n-1}{2} \frac{x \cdot \nabla u}{|x|} + u_t \\
+ |x| \left( \frac{x \cdot \nabla v}{|x|} + v_t \right)^2 + \frac{n-1}{2} v \left( \frac{x \cdot \nabla v}{|x|} + v_t \right) \right] d\sigma,
\]
where $|u'|^2 = |\nabla u|^2 + |u_t|^2$. If we parameterize $M^S_T$ by $y \mapsto (y, |y|)$ and set $\bar{u}(y) = u(y, |y|), \bar{v}(y) = v(y, |y|)$, then

$$d\sigma = \sqrt{2} \, dy,$$

$$\bar{u}_r \equiv y \cdot \frac{\nabla \bar{u}}{|y|} = x \cdot \frac{\nabla u}{|x|} + u_t = u_r + u_t,$$

$$\bar{v}_r \equiv y \cdot \frac{\nabla \bar{v}}{|y|} = x \cdot \frac{\nabla v}{|x|} + v_t = v_r + v_t$$

where $\nabla \bar{u} = \sum_{j=0}^n \partial_j \bar{u}$ and $\nabla u = \sum_{j=1}^n \partial_j u$. Therefore

\begin{equation}
(2.13) \quad I_3 = - \int_T \int_{\Sigma^{n-1}} \left( r \bar{u}_r^2 + \frac{n-1}{2} \bar{u} \bar{u}_r + r \bar{v}_r^2 + \frac{n-1}{2} \bar{v} \bar{v}_r \right) r^{n-1} \, dr \, d\sigma(\omega)
\end{equation}

$$= - \int_T \int_{\Sigma^{n-1}} r \left( \frac{\bar{u}_r + \frac{n-1}{2r} \bar{u}}{2} \right)^2 + \left( \frac{\bar{v}_r + \frac{n-1}{2r} \bar{v}}{2} \right)^2 \right) r^{n-1} \, dr \, d\sigma(\omega)
\end{equation}

\begin{equation}
+ \int_T \int_{\Sigma^{n-1}} \frac{n-1}{2} (\bar{u} \bar{u}_r + \bar{v} \bar{v}_r) r^{n-1} \, dr \, d\sigma(\omega)
\end{equation}

\begin{equation}
+ \int_T \int_{\Sigma^{n-1}} \frac{(n-1)^2}{4} (\bar{u}^2 + \bar{v}^2) r^{n-2} \, dr \, d\sigma(\omega).
\end{equation}

Note that

\begin{equation}
\int_T \int_{\Sigma^{n-1}} \frac{n-1}{2} \bar{u} \bar{u}_r r^{n-1} \, dr \, d\sigma(\omega)
\end{equation}

$$= \frac{1}{2} \int_{\Sigma^{n-1}} \int_T \frac{n-1}{2} \partial_r (\bar{u}^2(r\omega)) r^{n-1} \, dr \, d\sigma(\omega)
\end{equation}

$$= \frac{1}{2} \int_{\Sigma^{n-1}} \frac{n-1}{2} \bar{u}^2 (S\omega) S^{n-1} \, d\sigma(\omega) - \frac{1}{2} \int_{\Sigma^{n-1}} \frac{n-1}{2} \bar{u}^2 (T\omega) T^{n-1} \, d\sigma(\omega)
\end{equation}

\begin{equation}
- \left( \frac{n-1}{2} \right)^2 \int_{\Sigma^{n-1}} \bar{u}^2 (r\omega) r^{n-2} \, dr \, d\sigma(\omega)
\end{equation}

$$= \frac{n-1}{4} \int_{\partial D_S} u^2 \, d\sigma - \frac{n-1}{4} \int_{\partial D_T} u^2 \, d\sigma
\end{equation}

$$- \frac{(n-1)^2}{4} \int_{\Sigma^{n-1}} \bar{u}^2 (r\omega) r^{n-2} \, dr \, d\sigma(\omega),$$

and
\[
\int_S \int_{T \Sigma^{n-1}} \frac{n-1}{2} v^2 r^{n-1} dr d\sigma(\omega) = \int_S \int_{T \Sigma^{n-1}} \frac{n-1}{4} v^2 d\sigma - \frac{n-1}{4} \int_{\partial D_T} v^2 d\sigma - \frac{(n-1)^2}{4} \int_{\Sigma^{n-1}} v^2 r^{n-2} dr d\sigma(\omega).
\]

Hence, the expression in (2.13) reduces to

\[
(2.14) \quad I_3 = - \int_S \int_{T \Sigma^{n-1}} r \left( \left| \frac{u}{r} + \frac{n-1}{2r} \bar{u} \right|^2 + \left| \frac{v}{r} + \frac{n-1}{2r} \bar{v} \right|^2 \right) r^{n-1} dr d\sigma(\omega)
+ \frac{n-1}{4} \int_{\partial D_S} (u^2 + v^2) d\sigma - \frac{n-1}{4} \int_{\partial D_T} (u^2 + v^2) d\sigma.
\]

Next using the fact that \( |\nabla \mu|^2 - \mu_r^2 = \frac{|\nabla \omega \mu|^2}{r^2} \), we obtain

\[
(2.15) \quad I_1 = \int_{D_S} \left( SQ_0 + \frac{n-1}{2} u_t u + \frac{n-1}{2} v_t v \right) dx
= \int_{D_S} \left\{ S \left[ \frac{1}{2} |u_t|^2 + \frac{1}{2} \left( u_r + \frac{n-1}{2r} u \right)^2 + \frac{1}{2r^2} |\nabla \omega u|^2 \right]
+ \frac{1}{2} |v_t|^2 + \frac{1}{2} \left( v_r + \frac{n-1}{2r} v \right)^2 + \frac{1}{2r^2} |\nabla \omega v|^2 + \frac{1}{2} F \right\] + r \left( u_r + \frac{n-1}{2r} u \right) u_t + r \left( v_r + \frac{n-1}{2r} v \right) v_t \} \right\} dx
- \frac{n-1}{4} \int_{\partial D_S} (u^2 + v^2) d\sigma + \frac{(n-1)(n-3)}{8} \int_{D_S} S |u|^2 + |v|^2 \frac{dx}{r^2}.
\]

Similarly, we have

\[
(2.16) \quad I_2 = - \int_{D_T} \left( TQ_0 + \frac{n-1}{2} u_t u + \frac{n-1}{2} v_t v \right) dx
= - \int_{D_T} \left\{ T \left[ \frac{1}{2} |u_t|^2 + \frac{1}{2} \left( u_r + \frac{n-1}{2r} u \right)^2 + \frac{1}{2r^2} |\nabla \omega u|^2 \right]
+ \frac{1}{2} |v_t|^2 + \frac{1}{2} \left( v_r + \frac{n-1}{2r} v \right)^2 + \frac{1}{2r^2} |\nabla \omega v|^2 + \frac{1}{2} F \right\] + r \left( u_r + \frac{n-1}{2r} u \right) u_t + r \left( v_r + \frac{n-1}{2r} v \right) v_t \} \right\} dx
+ \frac{n-1}{4} \int_{\partial D_T} (u^2 + v^2) d\sigma - \frac{(n-1)(n-3)}{8} \int_{D_T} T |u|^2 + |v|^2 \frac{dx}{r^2}.
\]

Finally, assumption (H4) means \( I_4 \geq 0 \).
Now, let $T = \varepsilon S$ for some $0 < \varepsilon < 1$. Substituting (2.14)–(2.16) into (2.11) and using Hardy’s inequality
\[
\int \frac{|\mu|^2}{|x|^2} \, dx \leq C \int |\nabla \mu|^2 \, dx
\]
we deduce that
\[
(2.17) \quad S \int_{D_S} \frac{1}{2} F \, dx \leq C \varepsilon S E_0
\]
\[
+ \int_{\Sigma_{n-1}} \int_{\varepsilon S} r \left( \left| \frac{n-1}{2} \frac{u}{r} \right|^2 + \left| \frac{n-1}{2} \frac{v}{r} \right|^2 \right) r^{n-1} \, dr \, d\sigma(\omega).
\]
Observe that by direct computation, we have
\[
\leq \sqrt{2} \int_{M_{S}^\varepsilon} r \left( |u_r + u_t|^2 + |v_r + v_t|^2 \right) \, d\sigma
\]
\[
+ \frac{2}{\sqrt{2}} \left( \frac{n-1}{2} \right)^2 \int_{M_{S}^\varepsilon} r \left( \left| \frac{u}{r} \right|^2 + \left| \frac{v}{r} \right|^2 \right) \, d\sigma
\]
\[
\leq \sqrt{2} S \int_{M_{S}^\varepsilon} \left( \frac{x}{|x|} u_t + \nabla u \right)^2 + \frac{x}{|x|} v_t + \nabla v \right)^2 \, d\sigma
\]
\[
+ \frac{(n-1)^2}{2\sqrt{2}} \int_{M_{S}^\varepsilon} \left( \frac{u^2}{|x|} + \frac{v^2}{|x|} \right) \, d\sigma
\]
\[
\equiv I + II.
\]
It is easy to see (cf. equation (2.7)) that
\[
(2.18) \quad I = \sqrt{2} S \int_{M_{S}^\varepsilon} \left( \frac{x}{|x|} u_t + \nabla u \right)^2 + \frac{x}{|x|} v_t + \nabla v \right)^2 \, d\sigma
\]
\[
\leq C S \text{[Flux}(u, v; M_{S}^\varepsilon)],
\]
and
\begin{equation}
II = \frac{(n-1)^2}{2\sqrt{2}} \int_{M^S_{\varepsilon S}} \left( \frac{u^2}{t} + \frac{v^2}{t} \right) d\sigma = \frac{(n-1)^2}{2\sqrt{2}} \left( \int_{M^S_{\varepsilon S}} t^{-n/2} d\sigma \right)^{2/n}
\times \left\{ \left( \int_{M^S_{\varepsilon S}} u^{*2} d\sigma \right)^{(n-2)/n} + \left( \int_{M^S_{\varepsilon S}} v^{*2} d\sigma \right)^{(n-2)/n} \right\}
\leq C \left( \int_0^S \int_{\Sigma^{n-1}} t^{-n/2} t^{n-1} dt d\sigma(\omega) \right)^{2/n} \left( \int_{M^S_{\varepsilon S}} (u^{*2} + v^{*2}) d\sigma \right)^{(n-2)/n}
\leq CS \left\{ \int_{M^S_{\varepsilon S}} \frac{F}{2} d\sigma \right\}^{(n-2)/n} \leq CS [\text{Flux}(u,v; M^S_{\varepsilon S})]^{(n-2)/n}.
\end{equation}

Substituting estimates (2.18) and (2.19) into (2.17) and dividing by $S$, we obtain
\begin{equation}
\int_{D_S} \frac{1}{2} F dx \leq C \varepsilon E_0 + C [\text{Flux}(u,v; M^S_{\varepsilon S})] + C [\text{Flux}(u,v; M^S_{\varepsilon S})]^{(n-2)/n}.
\end{equation}

From (2.6'), letting $S \to \infty$ and then $\varepsilon \to 0$, we get (2.9').

Assumption (H3) and Proposition 2.2 immediately imply the following result.

**Proposition 2.3.** Let $(u,v)$ be a solution of (1.1), and let $F,F_1,F_2$ satisfy (H2)–(H4). Then
\[ \lim_{|t| \to \infty} \int_{\mathbb{R}^n} (|u(x,t)|^{2*} + |v(x,t)|^{2*}) dx = 0. \]

**Proof of Theorem 2.1.** We ought to show that $u,v \in L^q([T_0, \infty); \dot{B}^1_q)$ for some $T_0$. By Proposition 2.3, for any fixed $\varepsilon_0 > 0$ one can choose $T_0$ such that
\[ \int_{\mathbb{R}^n} (|u(x,t)|^{2*} + |v(x,t)|^{2*}) dx \leq \varepsilon_0, \quad \forall t > T_0. \]

As in [3, proof of Proposition 3.1], for every $T > T_0$, we can derive the inequalities
\begin{align*}
\|u\|_{q,T_0,T} + \|v\|_{q,T_0,T} & \leq CE_0^{1/2} + C \sup_{T_0 \leq t \leq T} \|u\|_{L^2(\mathbb{R}^n)}^{\gamma} + \|u\|_{q,T_0,T}^{\gamma} \\
& \quad + C \sup_{T_0 \leq t \leq T} \|v\|_{L^2(\mathbb{R}^n)}^{\beta} + \|v\|_{q,T_0,T}^{\beta} \\
& \leq CE_0^{1/2} + C \varepsilon_{0}^{\beta/2*} (\|u\|_{q,T_0,T} + \|v\|_{q,T_0,T}) \\
& \leq CE_0^{1/2} + C \varepsilon_{0}^{\beta/2*} (\|u\|_{q,T_0,T} + \|v\|_{q,T_0,T})^{\gamma},
\end{align*}
where \( \|u\|_{q,T_0,T} = \left( \int_{T_0}^T \|u(t)\|_{B^{q/2}_q}^q \, dt \right)^{1/q} \) and
\[ \beta = (1 - \alpha)(2^* - 2) > 0, \quad \gamma = \alpha(2^* - 2) + 1 > 1, \quad \alpha = (n - 2)/(n - 1). \]
For \( \varepsilon_0 \) sufficiently small, the above inequality implies
\[ \|u\|_{q,T_0,T} + \|v\|_{q,T_0,T} \leq 2CE_0 \]
for all \( T > T_0 \). Letting \( T \to \infty \) we complete the proof of Theorem 2.1. ■

3. Scattering theory. As we have proved the global-in-time existence of solutions to problem (1.1), the following questions arise. What is the asymptotic behavior of the solution \((u,v)\) as \( t \to \pm \infty \)? Does it converge to a solution of the corresponding free system
\[
\begin{aligned}
&u_{tt} - \Delta u = 0, \\
v_{tt} - \Delta v = 0,
\end{aligned}
\]
in the sense of \( \dot{H}^1 \times \dot{H}^1 \) norm? These questions will be discussed in this section; in other words, we will construct the scattering operator for problem (1.1) and we shall study its properties.

For simplicity of exposition, let \((u^\pm, v^\pm)\) be the solutions of system (3.1) with the initial data \((\varphi^\pm_1, \varphi^\pm_2, \psi^\pm_1, \psi^\pm_2)\), respectively. We also denote by \((u, v)\) the solution to problem (1.1) with the initial data \((\varphi_1, \varphi_2, \psi_1, \psi_2)\).

**Definition.**

(a) If for any \((\varphi^\pm_1, \varphi^\pm_2, \psi^\pm_1, \psi^\pm_2) \in X = \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2(\mathbb{R}^n)\) there exists \((\varphi_1, \varphi_2, \psi_1, \psi_2) \in X\) such that
\[
\| (u,v,u_t,v_t) - (u^\pm,v^\pm,u^\pm_t,v^\pm_t) \|_X \to 0 \quad \text{as} \quad t \to \pm \infty,
\]
then problem (1.1) is said to have the wave operator. The functions \((\varphi^\pm_1, \varphi^\pm_2, \psi^\pm_1, \psi^\pm_2)\) are called the asymptotic states of \((u, v, u_t, v_t)\) at \( t = \pm \infty \).

(b) If for any \((\varphi_1, \varphi_2, \psi_1, \psi_2) \in X\), there exist \((\varphi^\pm_1, \varphi^\pm_2, \psi^\pm_1, \psi^\pm_2) \in X\) such that (3.2) holds true, then problem (1.1) is said to be asymptotically complete.

If the conditions in both (a) and (b) hold true, then the wave operators \(W_\pm\) are
\[ W_+(\varphi^+_1, \varphi^+_2, \psi^+_1, \psi^+_2) = W_-(\varphi^-_1, \varphi^-_2, \psi^-_1, \psi^-_2) = (\varphi_1, \varphi_2, \psi_1, \psi_2). \]

The main result of this section reads as follows.

**Theorem 3.1.** The wave operators \(W_\pm\) and the scattering operator \(S \equiv W^{-1}_+ \circ W_-\) for problem (1.1) exist and are isomorphisms of \(X = \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2(\mathbb{R}^n)\).
Hence, it is well known that the solution to the free system associated with (1.1),

\[\begin{align*}
\mu_{tt} - \Delta \mu &= 0, \\
\nu_{tt} - \Delta \nu &= 0, \\
\mu(0) &= \varphi_1(x), \quad \mu_t(0) = \psi_1(x), \\
\nu(0) &= \varphi_2(x), \quad \nu_t(0) = \psi_2(x),
\end{align*}\]

is given by

\[\begin{align*}
(\mu, \nu) &= (\cos(\Delta t)\varphi_1 + A^{-1}\sin(\Delta t)\psi_1, \cos(\Delta t)\varphi_2 + A^{-1}\sin(\Delta t)\psi_2), \\
&\quad - A\sin(\Delta t)\varphi_1 + \cos(\Delta t)\psi_1, -A\sin(\Delta t)\varphi_2 + \cos(\Delta t)\psi_2).
\end{align*}\]

Proof. We set \(A = (-\Delta)^{1/2}\) and define

\[U_0(t)(\varphi_1, \varphi_2, \psi_1, \psi_2) \equiv (\cos(\Delta t)\varphi_1 + A^{-1}\sin(\Delta t)\psi_1, \cos(\Delta t)\varphi_2 + A^{-1}\sin(\Delta t)\psi_2, \\
- A\sin(\Delta t)\varphi_1 + \cos(\Delta t)\psi_1, -A\sin(\Delta t)\varphi_2 + \cos(\Delta t)\psi_2).\]

It is well known that the solution to the free system associated with (1.1),

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\mu_{tt} - \Delta \mu &= 0, \\
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\mu(0) &= \varphi_1(x), \quad \mu_t(0) = \psi_1(x), \\
\nu(0) &= \varphi_2(x), \quad \nu_t(0) = \psi_2(x),
\end{align*}\]

is given by

\[\begin{align*}
(\mu, \nu) &= (\cos(\Delta t)\varphi_1 + A^{-1}\sin(\Delta t)\psi_1, \cos(\Delta t)\varphi_2 + A^{-1}\sin(\Delta t)\psi_2).
\end{align*}\]

Hence,

\[\begin{align*}
(\mu, \nu, \mu_t, \nu_t) &= U_0(t)(\varphi_1, \varphi_2, \psi_1, \psi_2).
\end{align*}\]

**STEP 1: Asymptotic completeness.** For any \((\varphi_1, \varphi_2, \psi_1, \psi_2) \in X\), let

\[\begin{align*}
(u^\pm(t), v^\pm(t), u_t^\pm(t), v_t^\pm(t)) &= U_0(t)(\varphi_1, \varphi_2, \psi_1, \psi_2) \\
&\quad - \int_0^\infty U_0(t-\tau)(0, 0, F_1(|u|^2, |v|^2)u, F_2(|u|^2, |v|^2)v) \, d\tau.
\end{align*}\]

Combining the Strichartz estimates, the nonlinear estimates from [3, Proposition 3.1], and Proposition 2.3, we obtain

\[\begin{align*}
\|((u, v, u_t, v_t) - (u^\pm, v^\pm, u_t^\pm, v_t^\pm))\|_X \\
&\leq \left\| \int_0^\infty \left( A^{-1}\sin A(t-\tau)F_1(|u|^2, |v|^2)u, A^{-1}\sin A(t-\tau)F_2(|u|^2, |v|^2)v, \\
\cos A(t-\tau)F_1(|u|^2, |v|^2)u, \cos A(t-\tau)F_2(|u|^2, |v|^2)v \right) \, d\tau \right\|_X \\
&\leq C \sup_{\tau \in [t, \pm\infty)} \|u\|_{L^2}^\beta \|u\|_{L^q(\tau; \dot{B}^{1/2}_q)}^\gamma \\
&\quad + C \sup_{\tau \in [t, \pm\infty)} \|v\|_{L^2}^\beta \|v\|_{L^q(\tau; \dot{B}^{1/2}_q)}^\gamma \rightarrow 0 \quad \text{as } t \to \pm\infty
\end{align*}\]

where

\[\frac{1}{q} = \frac{n-1}{2(n+1)}, \quad \alpha = \frac{n-2}{n-1}, \quad \beta = (1 - \alpha)(2^* - 2) > 0,
\]

\[\gamma = \alpha(2^* - 2) + 1 > 1.\]
If we introduce the notation
\[(3.9) \quad (\Phi_1^\pm, \Phi_2^\pm, \Psi_1^\pm, \Psi_2^\pm)\]
\[= \pm\infty \int_0^\infty (-A^{-1}\sin(A\tau)F_1(|u|^2, |v|^2)u, -A^{-1}\sin(A\tau)F_2(|u|^2, |v|^2)v, \cos(A\tau)F_1(|u|^2, |v|^2)u, \cos(A\tau)F_2(|u|^2, |v|^2)v)\, d\tau,\]
then (3.7) reduces to
\[(u^\pm(t), v^\pm(t), u_t^\pm(t), v_t^\pm(t)) = U_0(t)(\varphi_1 - \Phi_1^\pm, \varphi_2 - \Phi_2^\pm, \psi_1 - \Psi_1^\pm, \psi_2 - \Psi_2^\pm).\]
Therefore, we can define the operator \(\tilde{W}_{\pm}^{-1}\) on \(X\) by the formula
\[(3.10) \quad (\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm) = \tilde{W}_{\pm}^{-1}(\varphi_1, \varphi_2, \psi_1, \psi_2)\]
\[\equiv (\varphi_1 - \Phi_1^\pm, \varphi_2 - \Phi_2^\pm, \psi_1 - \Psi_1^\pm, \psi_2 - \Psi_2^\pm).\]

**Step 2: Wave operator.** For any \((\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm) \in X\), the existence of the wave operators is equivalent to the existence of solutions to the integral equation
\[(3.11) \quad (u, v, u_t, v_t) = U_0(t)(\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm)\]
\[+ \pm\infty \int_t^\infty U_0(t - \tau)(0, 0, F_1(|u|^2, |v|^2)u, F_2(|u|^2, |v|^2)v)\, d\tau\]
which satisfy
\[\lim_{t \to \pm\infty} \| \pm\infty \int_t^\infty U_0(t - \tau)(0, 0, F_1(|u|^2, |v|^2)u, F_2(|u|^2, |v|^2)v)\, d\tau \|_X = 0.\]
To deal with (3.11), consider the space
\[\mathcal{Y}(I) = \{(u, v, u_t, v_t) \in C(I; X) : (u, v, u_t, v_t) \in L^q(I; \dot{B}^{1/2}_q \times \dot{B}^{1/2}_q \times \dot{B}^{-1/2}_q \times \dot{B}^{-1/2}_q)\}\]
as well as its closed subset
\[B = \{(u, v, u_t, v_t) \in \mathcal{Y}(I) : \|(u, v, u_t, v_t)\|_{\mathcal{Y}(I)} \leq C_{t_0}\},\]
where either \(I = [t_0, \infty)\) or \(I = (-\infty, -t_0]\) and \(\lim_{|t_0| \to \infty} C_{t_0} = 0\) for
\[C_{t_0} = \|U_0(t_0)(\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm)\|_{L^q(I; \dot{B}^{1/2}_q \times \dot{B}^{1/2}_q \times \dot{B}^{-1/2}_q \times \dot{B}^{-1/2}_q)}.\]
By a standard argument, we can get the local well-posedness of (3.11) in \(B \subset \mathcal{Y}(I)\). Therefore, if we define the wave operators by
\[W_{\pm} : (\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm) \mapsto (\varphi_1, \varphi_2, \psi_1, \psi_2) = (u(0), v(0), u_t(0), v_t(0)),\]
then $W^{-1}_\pm$ exists and is equal to $\tilde{W}^{-1}_\pm$ in (3.10). In fact, the initial data of equation (3.11) are given by
\[
(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\psi}_1, \tilde{\psi}_2) = (\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm)
\]
\[
+ \int_0^{\pm \infty} U_0(-\tau)(0,0, F_1(|u|^2, |v|^2)u, F_2(|u|^2, |v|^2)v) \, d\tau.
\]
Hence, we obtain from (3.9) the identity
\[
(\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm) = (\tilde{\varphi}_1 - \Phi_1^\pm, \tilde{\varphi}_2 - \Phi_2^\pm, \tilde{\psi}_1 - \Psi_1^\pm, \tilde{\psi}_2 - \Psi_2^\pm),
\]
which means $W_\pm$ is invertible. Thus $W^{-1}_\pm = \tilde{W}^{-1}_\pm$ are isomorphisms on $X = \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2$. Consequently, the scattering operator $S = W^{-1}_+ \circ W_-$ is also an isomorphism on $X$.

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