FINITE-DIMENSIONAL MAPS AND DENDRITES WITH
DENSE SETS OF END POINTS

BY

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Abstract. The first author has recently proved that if \( f : X \to Y \) is a \( k \)-dimensional map between compacta and \( Y \) is \( p \)-dimensional \((0 \leq k, p < \infty)\), then for each \( 0 \leq i \leq p + k \), the set of maps \( g \) in the space \( C(X, I^{p+2k+1-i}) \) such that the diagonal product \( f \times g : X \to Y \times I^{p+2k+1-i} \) is an \((i+1)\)-to-1 map is a dense \( G_\delta \)-subset of \( C(X, I^{p+2k+1-i}) \).

In this paper, we prove that if \( f : X \to Y \) is as above and \( D_j (j = 1, \ldots, k) \) are superdendrites, then the set of maps \( h \) in \( C(X, \prod_{j=1}^k D_j \times I^{p+1-i}) \) such that \( f \times h : X \to Y \times (\prod_{j=1}^k D_j \times I^{p+1-i}) \) is \((i+1)\)-to-1 is a dense \( G_\delta \)-subset of \( C(X, \prod_{j=1}^k D_j \times I^{p+1-i}) \) for each \( 0 \leq i \leq p \).

1. Introduction. In this paper, all spaces are separable metric spaces and maps are continuous. We denote the unit interval by \( I \). A compact metric space is called a compactum, and continuum means a connected compactum. Let \( X \) and \( Y \) be compacta. Then \( C(X, Y) \) denotes the set of all continuous maps from \( X \) to \( Y \) endowed with the sup metric. A map \( f : X \to Y \) is called \( \sigma \)-closed if there exists a family \( \{F_i\}_{i=1}^\infty \) of closed subsets in \( X \) such that \( X = \bigcup_{i=1}^\infty F_i \) and \( f|F_i : F_i \to f(F_i) \) is a closed map for each \( i = 1, 2, \ldots \). A map \( f : X \to Y \) is called \( k \)-dimensional if \( \dim f^{-1}(y) \leq k \) for each \( y \in Y \), and \( k \)-to-1 if \( |f^{-1}(y)| \leq k \) for each \( y \in Y \). In [3] and [4], Pasynkov proved that if \( f : X \to Y \) is a \( k \)-dimensional map from a compactum \( X \) to a finite-dimensional compactum \( Y \), then there is a map \( g : X \to I^k \) such that \( \dim (f \times g) = 0 \). Also, he proved that if \( f : X \to Y \) is a \( k \)-dimensional map of compacta and \( \dim Y = p < \infty \), then the set of maps \( g \) in the space \( C(X, I^{p+2k+1}) \) such that the diagonal product \( f \times g : X \to Y \times I^{p+2k+1} \) is an embedding is a dense \( G_\delta \)-subset of \( C(X, I^{p+2k+1}) \). Furthermore, in [2] the first author proved the following theorem.

Theorem 1 ([2]). If \( f : X \to Y \) is a \( k \)-dimensional map of compacta and \( \dim Y = p < \infty \), then for each \( 0 \leq i \leq p + k \), the set of maps \( g \) in the space \( C(X, I^{p+2k+1-i}) \) such that \( f \times g : X \to Y \times I^{p+2k+1-i} \) is...
(i + 1)-to-1 is a dense $G_\delta$-subset of $C(X, I^{p+2k+1-i})$. Hence the restriction $g|f^{-1}(y): f^{-1}(y) \to I^{p+2k+1-i}$ is (i + 1)-to-1 for each $y \in Y$.

A locally connected continuum $D$ is called a dendrite if it contains no circle. A dendrite $D$ is called a superdendrite [5] if the set of all end points of $D$ is dense in $D$. The main aim of this paper is to prove the following theorem.

**Theorem 2.** Let $f : X \to Y$ be a $k$-dimensional map of compacta and $\dim Y = p < \infty$, and let $D_j$ $(j = 1, \ldots, k)$ be superdendrites. Then the set of maps $h$ in the space $C(X, \prod_{j=1}^k D_j \times I^{p+1-i})$ such that $f \times h : X \to Y \times (\prod_{j=1}^k D_j \times I^{p+1-i})$ is (i + 1)-to-1 is a dense $G_\delta$-subset of $C(X, \prod_{j=1}^k D_j \times I^{p+1-i})$ for each $0 \leq i \leq p$. Hence $h|f^{-1}(y) : f^{-1}(y) \to \prod_{j=1}^k D_j \times I^{p+1-i}$ is (i + 1)-to-1 for each $y \in Y$.

This is a generalization of the following theorem of Bowers [1] (cf. [5]): If $X$ is an $n$-dimensional compactum and $D_1, \ldots, D_n$ are superdendrites, then the set $\{h \in C(X, \prod_{j=1}^n D_j \times I) \mid h$ is an embedding $\}$ is a dense $G_\delta$-subset in $C(X, \prod_{j=1}^n D_j \times I)$. As a corollary, we have a representation theorem for finite-dimensional maps using superdendrites (see Theorem 15).

**2. Main theorem.** First we set up some notation and terminology (cf. [5] and [6]). Let $X$, $Y$ be compacta and let $A \subset X$ be a closed subset. If $f : X \to Y$ is a map, we set

$$S_f = \{x \in X \mid f^{-1}(x) = \{x\}, \quad R_{(X,A,Y)} = \{f \in C(X,Y) \mid A \subset S_f\}.$$

A set $S \subset X$ is said to be residual if $S$ contains a dense $G_\delta$-subset of $X$. A map $f : X \to Y$ is called a $(k, \varepsilon)$-map ($\varepsilon > 0$) if for each $y \in Y$, there are subsets $A_1, \ldots, A_k$ of $f^{-1}(y)$ such that $f^{-1}(y) = \bigcup_{i=1}^k A_i$ and $\text{diam} A_i < \varepsilon$. The main aim of this section is to prove Theorem 2. To do this we need the following results.

**Theorem 3 ([6]).** Let $X$, $Y$ be spaces with $\dim Y < \infty$ and let $f : X \to Y$ be a $\sigma$-closed $k$-dimensional map. Then there exists a 0-dimensional $F_\sigma$-subset $A \subset X$ such that $f|(X \setminus A)$ is $(k - 1)$-dimensional.

**Theorem 4 ([5]).** Let $X$ be a compactum and let $A \subset X$ be a 0-dimensional $F_\sigma$-subset. Then for each superdendrite $D$, $R_{(X,A,D)} = \{f \in C(X,D) \mid A \subset S_f\}$ is residual in $C(X,D)$.

**Proposition 5.** Let $X$, $Y$ and $Z$ be compacta and let $f : X \to Y$ be a map. Then $\{g \in C(X,Z) \mid f|(X \setminus S_g)$ is $k$-dimensional$\}$ is a $G_\delta$-subset in $C(X,Z)$. 
Proof. For $a$, $b > 0$ and $g \in C(X,Z)$, let
\[
F(g,a) = \{ x \in X \mid \text{diam}(g^{-1}g(x)) \geq a \},
\]
\[
U(a,b) = \{ g \in C(X,Z) \mid \text{diam}(g^{-1}g(x)) \geq a \text{ for each } y \in Y \},
\]
where $d_{n+1}(F) < b$ if there exists an open cover of $F$ with mesh $b$ and order $\leq n$.

CLAIM. $U(a,b)$ is an open subset in $C(X,Y)$.

Proof of claim. Assume that $U(a,b)$ is not open in $C(X,Y)$. Then there exist $g \in U(a,b)$ and \( \{ g_i \}_{i=1}^\infty \subset C(X,Y) \setminus U(a,b) \) such that $g_i \to g$. For each $i = 1, 2, \ldots$, there exists $y_i \in Y$ such that $d_{k+1}(f^{-1}(y_i) \cap F(g_i,a)) \geq b$. We may assume $y = \lim_{i \to \infty} y_i$. Since $g \in U(a,b)$, there exists a family $\mathcal{U}$ of open subsets in $X$ such that $\text{ord} \mathcal{U} \leq k$, mesh$\mathcal{U} < b$ and $f^{-1}(y) \cap F(g,a) \subset \bigcup \mathcal{U}$. Since $d_{k+1}(f^{-1}(y_i) \cap F(g_i,a)) \geq b$, there exist $x_i \in X$ such that $\text{diam}(g_i^{-1}g_i(x_i)) \geq a$ and $f^{-1}(y_i) \cap g_i^{-1}g_i(x_i) \nsubseteq \bigcup \mathcal{U}$ for each $i = 1, 2, \ldots$. In fact, we may choose $x_i \in (f^{-1}(y_i) \cap g_i^{-1}g_i(x_i)) \setminus \bigcup \mathcal{U}$. We may assume $x = \lim_{i \to \infty} x_i$. Since $\text{diam} g^{-1}g(x) \geq a$, we have $f^{-1}(y) \cap g^{-1}g(x) \subset \bigcup \mathcal{U}$. So $f^{-1}(y_i) \cap g_i^{-1}g_i(x_i) \subset \bigcup \mathcal{U}$ for infinitely many $i$. This is a contradiction. This completes the proof of the claim.

It is easy to see that $\{ g \in C(X,Z) \mid f|(X \setminus S_g) \text{ is } k\text{-dimensional} \} = \bigcap_{m,n \in \mathbb{N}} U(1/m, 1/n)$. This completes the proof.

The next result is due to Pasynkov. For completeness, we will give the proof.

**Proposition 6** ([3]). Let $X$, $Y$ and $Z$ be compacta and let $f : X \to Y$ be a map. Then $\{ g \in C(X,Z) \mid f \times g \text{ is } k\text{-dimensional} \}$ is a $G_\delta$-subset in $C(X,Z)$.

*Proof.* It suffices to observe that
\[
\{ g \in C(X,Z) \mid d_{k+1}((f \times g)^{-1}(y,z)) < b \text{ for each } y \in Y \text{ and } z \in Z \}
\]
is an open subset of $C(X,Z)$. The argument is similar to that in the proof of the preceding proposition. This completes the proof.

**Theorem 7.** Let $f : X \to Y$ be a $k$-dimensional map of compacta with $\text{dim} Y = p < \infty$, and let $D_j$ ($j = 1, \ldots, k$) be superdendrites. Then the set of maps $g$ in the space $C(X, \prod_{j=1}^k D_j)$ such that $f|(X \setminus S_g)$ is 0-dimensional is a dense $G_\delta$-subset of $C(X, \prod_{j=1}^k D_j)$. In particular, the set of maps $g$ in the space $C(X, \prod_{j=1}^k D_j)$ such that the diagonal product $f \times g : X \to Y \times \prod_{j=1}^k D_j$ is 0-dimensional is a dense $G_\delta$-subset of $C(X, \prod_{j=1}^k D_j)$.

*Proof.* Let $\mathcal{G}(X, \prod_{j=1}^k D_j) = \{ g \in C(X, \prod_{j=1}^k D_j) \mid f|(X \setminus S_g)$ is 0-dimensional$\}$. By Propositions 5 and 6, it is sufficient to show that
\[ G(X, \prod_{j=1}^{k} D_j) \] is a dense subset of \( C(X, \prod_{j=1}^{k} D_j) \). Let \( r = r_1 \times \ldots \times r_k \in C(X, \prod_{j=1}^{k} D_j) \). We will find a map \( g = g_1 \times \cdots \times g_k \in G(X, \prod_{j=1}^{k} D_j) \) arbitrarily close to \( r \). By Theorem 3, there exists a 0-dimensional \( F_{\sigma} \)-set \( A_1 \subset X \) such that \( f|(X \setminus A_1) : X \setminus A_1 \rightarrow Y \) is \((k-1)\)-dimensional. By Theorem 4, \( R_{(X,A_1,D_1)} \) is residual in \( C(X,D_1) \). So we can take a map \( g_1 \in R_{(X,A_1,D_1)} \) arbitrarily close to \( r_1 \). Note that \( A_1 \subset S_{g_1} \). Let \( B_1 = X \setminus S_{g_1} \). Then \( B_1 \) is an \( F_{\sigma} \)-subset in \( X \) because \( S_{g_1} \) is a \( G_{\delta} \)-subset in \( X \). Since \( f|B_1 : B_1 \rightarrow Y \) is a \( \sigma \)-closed \((k-1)\)-dimensional map, by Theorem 3 there exists a 0-dimensional \( F_{\sigma} \)-subset \( A_2 \subset B_1 \) such that \( f|(B_1 \setminus A_2) : B_1 \setminus A_2 \rightarrow Y \) is \((k-2)\)-dimensional. By Theorem 4, we can take a map \( g_2 \in R_{(X,A_2,D_2)} \) arbitrarily close to \( r_2 \). Note that \( A_2 \subset S_{g_2} \). Let \( B_2 = B_1 \setminus S_{g_2} \). Then \( B_2 = X \setminus (S_{g_1} \cup S_{g_2}) \). By induction, for each \( i = 1, \ldots, k \), we can find a map \( g_i : X \rightarrow D_i \), an \( F_{\sigma} \)-subset \( B_i \subset X \) and a 0-dimensional \( F_{\sigma} \)-subset \( A_i \subset B_{i-1} \) such that

1. \( r_i \) and \( g_i \) are arbitrarily close to each other,
2. \( g_i \in R_{(X,A_i,D_i)} \),
3. \( f|(B_{i-1} \setminus A_i) : B_{i-1} \setminus A_i \rightarrow Y \) is \((k-i)\)-dimensional,
4. \( B_i = B_{i-1} \setminus S_{g_i} \).

Note that \( B_k = X \setminus \bigcup_{i=1}^{k} S_{g_i} \). Then \( r \) and \( g = g_1 \times \cdots \times g_k \) are arbitrarily close to each other and \( f|B_k : B_k \rightarrow Y \) is 0-dimensional. Note that \( \bigcup_{i=1}^{k} S_{g_i} \subset S_g \).

So \( g \) is the required map. This completes the proof.

Perhaps the next proposition is known. For completeness, we will give the proof.

**Proposition 8.** Let \( X \), \( Y \) and \( Z \) be compacta and let \( f : X \rightarrow Y \) be a map. Then for each \( k = 1, 2, \ldots \), the set \( H = \{ h \in C(X,Z) \mid f \times h \text{ is } k\text{-to-1} \} \) is a \( G_{\delta} \)-subset in \( C(X,Y) \).

**Proof.** For each \( n = 1, 2, \ldots \), let \( H_n = \{ h \in C(X,Z) \mid f \times h \text{ is a } (k,1/n)\text{-map} \} \). It is easy to see that \( H_n \) is an open subset in \( C(X,Z) \) and \( H = \bigcap_{n=1}^{\infty} H_n \). This completes the proof.

Now we prove Theorem 2.

**Proof of Theorem 2.** Let \( i = 0, 1, \ldots, p \) and

\[ v = (r_1 \times \cdots \times r_k) \times (u_1 \times \cdots \times u_{p+1-i}) \in C\left(X, \prod_{j=1}^{k} D_j \times I^{p+1-i}\right). \]

Let \( r = r_1 \times \cdots \times r_k \). By Proposition 8, it is sufficient to show that there exists a map \( h : X \rightarrow \prod_{j=1}^{k} D_j \times I^{p+1-i} \) arbitrarily close to \( v \), such that \( f \times h : X \rightarrow Y \times (\prod_{j=1}^{k} D_j \times I^{p+1-i}) \) is \((i+1)\)-to-1. By Theorem 7, there exists \( g : X \rightarrow \prod_{j=1}^{k} D_j \) arbitrarily close to \( r \), such that \( f|(X \setminus S_g) \) is 0-dimensional. Let \( X \setminus S_g = \bigcup_{i=1}^{\infty} F_i \), where \( F_i \) is closed in \( X \) and \( F_i \subset F_{i+1} \). 


for \( l = 1, 2, \ldots \). For \( l = 1, 2, \ldots, \) let
\[
S_l(X, I^{p+1-i}) = \{ s \in C(X, I^{p+1-i}) \mid (f \times s)|F_l : F_l \to Y \times I^{p+1-i} \text{ is } (i+1)-\text{to}-1 \}.
\]
As \( f|F_l : F_l \to Y \) is 0-dimensional, by Theorem 1 the set \( \{ s \in C(F_l, I^{p+1-i}) \mid f|F_l \times s : F_l \to Y \times I^{p+1-i} \text{ is } (i+1)-\text{to}-1 \} \) is a dense \( G_\delta \)-subset in \( C(F_l, I^{p+1-i}) \) for \( l = 1, 2, \ldots \). So it is easy to see that \( S_l(X, I^{p+1-i}) \) is a dense \( G_\delta \)-subset in \( C(X, I^{p+1-i}) \) for \( l = 1, 2, \ldots \). By Baire’s theorem \( \bigcap_{l=1}^{\infty} S_l(X, I^{p+1-i}) \) is a dense \( G_\delta \)-subset in \( C(X, I^{p+1-i}) \). So we can select \( s \in \bigcap_{l=1}^{\infty} S_l(X, I^{p+1-i}) \) arbitrarily close to \( u = u_1 \times \cdots \times u_{p+1-i} \). Let \( h = g \times s : X \to \prod_{j=1}^k D_j \times I^{p+1-i} \).
Then \( h \) is as required. This completes the proof.

3. Finite-dimensional maps and compositions of maps parallel to the unit interval and superdendrites. Now we consider an application of Theorem 2. A map \( f : X \to Y \) is said to be embedded in a map \( f_0 : X_0 \to Y_0 \) (see [3], [4]) if there exist embeddings \( g : X \to X_0 \) and \( h : Y \to Y_0 \) such that \( h \circ f = f_0 \circ g \). A map \( f : X \to Y \) is parallel to the space \( Z \) (see [3], [4]) if \( f \) can be embedded in the natural projection \( p : Y \times Z \to Y \). In [3], [4], Pasynkov proved the following theorem: If \( f : X \to Y \) has \( \dim f = k \) and \( \dim Y < \infty \), then \( f \) can be represented as the composition \( f = h_k \circ \cdots \circ h_1 \circ g \) of a zero-dimensional map \( g \) and maps \( h_i \) (\( i = 1, \ldots, k \)) parallel to the unit interval \( I \). Furthermore the first author proved the following [2]: A map \( f : X \to Y \) of compacta with \( \dim Y = p < \infty \) is \( k \)-dimensional if and only if \( f \) can be represented as the composition \( f = g_{p+2k+1} \circ \cdots \circ g_{p+k+1} \circ g_{p+k} \circ \cdots \circ g_1 \) of maps parallel to the unit interval \( I \) such that \( g_i \) is \( (i+1)\text{-to-1} \) for each \( i = 1, \ldots, p+k \) and \( g_{p+k+1} \) is zero-dimensional. In this section we prove another representation theorem for finite-dimensional maps using superdendrites.

Lemma 9 ([6]). Let \( \varepsilon > 0 \). Suppose that \( f : X \to Y \) is a map of compacta with \( \dim f = 0 \) and \( \dim Y = p < \infty \). For each \( i = 1, \ldots, l \), let \( K_i \) and \( L_i \) be closed disjoint subsets of \( X \). Then there are open subsets \( E_i \) of \( X \) separating \( X \) between \( K_i \) and \( L_i \) such that \( f|(\text{Cl}(E_1) \cup \cdots \cup \text{Cl}(E_l)) \) is a \((p, \varepsilon)\)-map.

The next three results are essentially contained in [2]. For completeness, we give their proofs.

Proposition 10 ([2]). Let \( X, Y \) and \( Z \) be compacta and \( 0 \leq k < \infty \). Let \( T \) be the set of maps \( g = u \times v : X \to Y \times Z \) in \( C(X, Y \times Z) \) such that \( \dim v(u^{-1}(y)) \leq k \) for each \( y \in Y \). Then \( T \) is a \( G_\delta \)-subset of \( C(X, Y \times Z) \).

Proof. Let \( \varepsilon > 0 \) and let \( T_\varepsilon \) be the set of maps \( g = u \times v : X \to Y \times Z \) in \( C(X, Y \times Z) \) such that for each \( y \in Y \), \( v(u^{-1}(y)) \) is covered by a family
\( \mathcal{U} \) of open sets of \( Z \) such that \( \text{mesh } \mathcal{U} < \varepsilon \) and \( \text{ord } \mathcal{U} \leq k \). Then \( T_\varepsilon \) is open in \( C(X, Y \times Z) \). Note that \( T = \bigcap_{n=1}^{\infty} T_{1/n} \).

**Lemma 11** ([2], cf. [7], [9]). Let \( f : X \to Y \) be a \( 0 \)-dimensional map from a compactum \( X \) to a \( p \)-dimensional compactum \( Y \) (\( p < \infty \)). Let \( T \) be the set of all maps \( u : X \to I \) such that \( \dim u(f^{-1}(y)) = 0 \) for each \( y \in Y \). Then \( T \) is a dense \( G_\delta \)-subset of \( C(X, I) \).

**Proof.** Let \( W_b \) be the set of maps \( u : X \to I \) such that for each \( y \in Y \), \( u(f^{-1}(y)) \) is covered by disjoint open sets of diameters less than \( b \). By an argument similar to that in the proof of Proposition 5, \( W_b \) is an open subset of \( C(X, I) \). Since \( T = \bigcap_{n=1}^{\infty} W_{1/n} \), it suffices to prove that \( W_b \) is dense in \( C(X, I) \). Let \( h \in C(X, I) \) and \( \varepsilon > 0 \). Choose \( \delta > 0 \) such that if \( A \subset X \) and \( \text{diam } A < \delta \), then \( \text{diam } h(A) < \min\{2\varepsilon, b/(2p)\} = \varepsilon' \).

Choose a finite family \( \{(U_n, V_n)\mid n = 1, \ldots, m\} \), where \( U_n \) and \( V_n \) are open subsets of \( X \) such that \( \{U_n \mid n = 1, \ldots, m\} \) is a cover of \( X \) with \( \text{Cl}(U_n) \subset V_n \) and \( \text{diam } V_n < \delta \) for all \( n = 1, \ldots, m \). By Lemma 9, there are open subsets \( E_n \) separating \( X \) between \( \text{Cl}(U_n) \) and \( X \setminus V_n \) such that \( f|(\text{Cl}(E_1) \cup \cdots \cup \text{Cl}(E_m)) \) is a \((p, \delta)\)-map. Note that \( X \setminus (E_1 \cup \cdots \cup E_m) = \bigcup_{n=1}^{m} D_n \), where \( D_n \) \( (n = 1, \ldots, m) \) are disjoint closed subsets of \( X \) such that \( \text{diam } D_n < \delta \). There are points \( x_n \in I \) such that \( h(D_n) \subset (x_n - \varepsilon'/2, x_n + \varepsilon'/2) \). The function that maps each \( D_n \) to \( x_n \) has a continuous extension \( u : X \to I \) whose supremum distance to \( h \) is less than \( \varepsilon'/2 \leq \varepsilon \). Let \( y \in Y \). Since \( f|(\text{Cl}(E_1) \cup \cdots \cup \text{Cl}(E_m)) \) is a \((p, \delta)\)-map, there are closed subsets \( A_1, \ldots, A_p \) of \( f^{-1}(y) \) such that \( f^{-1}(y) \cap (\text{Cl}(E_1) \cup \cdots \cup \text{Cl}(E_m)) = \bigcup_{i=1}^{p} A_i \) and \( \text{diam } A_i < \delta \) for \( i = 1, \ldots, p \). Note that \( \text{diam } u(A_i) < 2\varepsilon' \) for each \( i \), and \( u(D_n) = \{x_n\} \) for each \( n \). We can see that for each component \( C \) of \( \bigcup_{i=1}^{p} u(A_i) \), we have \( \text{diam } C < 2p\varepsilon' \leq b \) and

\[
u(f^{-1}(y)) \subset \{x_1, \ldots, x_m\} \cup \bigcup_{i=1}^{p} u(A_i).
\]

Hence, each component of \( u(f^{-1}(y)) \) has a neighbourhood that is closed and open in \( u(f^{-1}(y)) \) and has diameter less than \( b \). It follows that \( u \in W_b \), which completes the proof.

**Lemma 12** ([2]). Let \( f : X \to Y, g : X \to Z \) and \( u : X \to I \) be maps of compacta such that \( \dim Z = k \) and \( \dim u((f \times g)^{-1}(y, z)) = 0 \) for each \( y \in Y \) and \( z \in Z \). Then \( \dim (g \times u)(f^{-1}(y)) \leq k \) for each \( y \in Y \).

**Proof.** Let \( y \in Y \). Consider the natural projection \( p : Y \times Z \times I \to Y \times Z \).

Then \( p|((f \times g) \times u)(X) : ((f \times g) \times u)(X) \to (f \times g)(X) \) is a \( 0 \)-dimensional map, because for \( (y, z) \in (f \times g)(X) \subset Y \times Z \), \( p^{-1}(y, z) \cap (f \times g \times u)(X) = \{(y, z)\} \times u((f \times g)^{-1}(y, z)) \).
Also, note that \((f \times g \times u)(f^{-1}(y)) = \{y\} \times (g \times u)(f^{-1}(y))\), and hence the set
\[
p(\{y\} \times (g \times u)(f^{-1}(y))) = (f \times g)(f^{-1}(y)) = \{y\} \times g(f^{-1}(y)) \subset \{y\} \times Z
\]
is at most \(k\)-dimensional. Since
\[
p|\{y\} \times (g \times u)(f^{-1}(y)) : \{y\} \times (g \times u)(f^{-1}(y)) \to \{y\} \times g(f^{-1}(y))
\]
is a zero-dimensional map, by a theorem of Hurewicz we conclude that \(\dim(g \times u)(f^{-1}(y)) \leq k\).

By Theorem 7, Propositions 6 and 10, and Lemmas 11 and 12, we obtain the next result.

**Proposition 13** ([2], cf. [7], [9]). Let \(f : X \to Y\) be a \(k\)-dimensional map of compacta and \(\dim Y = p < \infty\). Let \(T\) be the set of all maps \(h = g \times u : X \to \prod_{j=1}^{k} D_j \times I\) in \(C(X, \prod_{j=1}^{k} D_j \times I)\) such that \(\dim(h^{-1}(y)) \leq k\), \(\dim u((f \times g)^{-1}(y,t)) = 0\) for all \(y \in Y\) and \(t \in \prod_{j=1}^{k} D_j\), and \(\dim(f \times g) = 0\). Then \(T\) is a dense \(G_\delta\)-subset of \(C(X, \prod_{j=1}^{k} D_j \times I)\).

**Proposition 14.** Let \(f : X \to Y\) be a \(k\)-dimensional map of compacta and \(\dim Y = p < \infty\). For \(i = 0,1, \ldots, p+1\), let \(p_i : \prod_{j=1}^{k} D_j \times I^{p+1} \to \prod_{j=1}^{k} D_j \times I^{p+1-i}\) be the natural projection, where \(p_0 : \prod_{j=1}^{k} D_j \times I^{p+1} \to \prod_{j=1}^{k} D_j \times I^{p+1}\) is the identity. Let \(\hat{E}(X, \prod_{j=1}^{k} D_j \times I^{p+1})\) be the set of maps \(g\) in \(C(X, \prod_{j=1}^{k} D_j \times I^{p+1})\) such that

1. for each \(0 \leq i \leq p\), \((f \times (p_i \circ g)) : X \to Y \times \prod_{j=1}^{k} D_j \times I^{p+1-i}\) is an \((i+1)\)-to-1 map,
2. the map \(h = p_p \circ g = g' \times u : X \to \prod_{j=1}^{k} D_j \times I\) satisfies \(\dim h(f^{-1}(y)) \leq k\), \(\dim u((f \times g')^{-1}(y,t)) = 0\) for all \(y \in Y\) and \(t \in \prod_{j=1}^{k} D_j\), and \(\dim(f \times g') = 0\).

Then \(\hat{E}(X, \prod_{j=1}^{k} D_j \times I^{p+1})\) is a dense \(G_\delta\)-subset of \(C(X, \prod_{j=1}^{k} D_j \times I^{p+1})\).

**Proof.** Note that if \(q : A \to B\) is an open map and \(C\) is a dense subset of \(B\), then \(q^{-1}(C)\) is dense in \(A\). The natural projection \(p_i : \prod_{j=1}^{k} D_j \times I^{p+1} \to \prod_{j=1}^{k} D_j \times I^{p+1-i}\) induces the open map \(P_i : C(X, \prod_{j=1}^{k} D_j \times I^{p+1}) \to C(X, \prod_{j=1}^{k} D_j \times I^{p+1-i})\) defined by \(P_i(h) = p_i \circ h\) for \(h \in C(X, \prod_{j=1}^{k} D_j \times I^{p+1})\). For \(i = 0,1, \ldots, p\), let \(E_i\) be the set of \(g\) in \(C(X, \prod_{j=1}^{k} D_j \times I^{p+1-i})\) such that \((f \times g) : X \to Y \times \prod_{j=1}^{k} D_j \times I^{p+1-i}\) is \((i+1)\)-to-1. Also, let \(T\) be the subset of \(C(X, \prod_{j=1}^{k} D_j \times I)\) as in Proposition 13. By Theorem 2 and Proposition 13,
\[
\tilde{E}(X, \prod_{j=1}^{k} D_j \times I^{p+1}) = \bigcap_{i=0}^{p} P_i^{-1}(E_i) \cap P_p^{-1}(T)
\]
is a dense $G_\delta$-subset of $C(X, \prod_{j=1}^{k} D_j \times I^{p+1})$. This completes the proof.

Now, we have the following representation theorem for finite-dimensional maps.

**Theorem 15.** Let $f : X \to Y$ be a map of compacta such that $0 \leq k < \infty$ and $\dim Y = p < \infty$. Then $f$ is $k$-dimensional if and only if $f$ can be represented as the composition

\[f = g_{p+k+1} \circ \cdots \circ g_{p+1} \circ g_p \circ \cdots \circ g_1\]
such that

1. $g_i$ is an $(i+1)$-to-1 map for each $i = 1, \ldots, p$ and $g_{p+1}$ is a zero-dimensional map,
2. $g_i$ is parallel to $I$ for $i = 1, \ldots, p+1$,
3. $g_i$ is parallel to a superdendrite for $i = p+2, \ldots, p+k+1$.

**Proof.** Let $\tilde{E}(X, \prod_{j=1}^{k} D_j \times I^{p+1})$ be as in Proposition 14. Choose $g \in \tilde{E}(X, \prod_{j=1}^{k} D_j \times I^{p+1})$. Let

\[Z_i = \begin{cases} 
\prod_{j=1}^{k} D_j \times I^{p+1-i} & \text{for } i = 0, 1, \ldots, p+1, \\
\prod_{j=1}^{p+k+1-i} D_j & \text{for } i = p+2, \ldots, p+k.
\end{cases}\]

For $i = 0, 1, \ldots, p+k$, let $p_i : Z_0 \to Z_i$ be the natural projection. For $i = 0, 1, \ldots, p+k$, put $X_i = (f \times (p_i \circ g))(X)$ and put $X_{p+k+1} = Y$. Let $g_1 = q_1 \circ (f \times g)$ and for $i = 2, \ldots, p+k+1$, let $g_i = |q_i| X_{i-1} : X_{i-1} \to X_i$, where $q_i : Y \times Z_{i-1} \to Y \times Z_i$ is the natural projection for $i = 1, \ldots, p+k$ and $q_{p+k+1} : Y \times D_1 \to Y$ is the natural projection. By Proposition 14, we see that $g_{p+1}$ is a 0-dimensional map. Hence the maps $g_i$ are as desired. This completes the proof.

**Remark.** After the paper [2] had been submitted for publication, the paper of Tuncali and Valov [8] was published. They obtained a more general result in the class of all metrizable spaces.

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