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FINITE-DIMENSIONAL MAPS AND DENDRITES WITH DENSE SETS OF END POINTS

BY

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Abstract. The first author has recently proved that if $f: X \to Y$ is a k-dimensional map between compacta and Y is p-dimensional ($0 \le k, p < \infty$), then for each $0 \le i \le p+k$, the set of maps g in the space $C(X, I^{p+2k+1-i})$ such that the diagonal product $f \times g: X \to Y \times I^{p+2k+1-i}$ is an $(i+1)$-to-1 map is a dense G_δ-subset of $C(X, I^{p+2k+1-i})$. In this paper, we prove that if $f: X \to Y$ is as above and D_j ($j = 1, \dots, k$) are superdendrites, then the set of maps h in $C(X, \prod_{j=1}^k D_j \times I^{p+1-i})$ such that $f \times h: X \to Y \times (\prod_{j=1}^k D_j \times I^{p+1-i})$ is $(i+1)$-to-1 is a dense G_δ-subset of $C(X, \prod_{j=1}^k D_j \times I^{p+1-i})$ for each $0 \le i \le p$.

1. Introduction. In this paper, all spaces are separable metric spaces and maps are continuous. We denote the unit interval by I. A compact metric space is called a *compactum*, and *continuum* means a connected compactum. Let X and Y be compacta. Then $C(X,Y)$ denotes the set of all continuous maps from X to Y endowed with the sup metric. A map $f: X \to Y$ is called *σ-closed* if there exists a family $\{F_i\}_{i=1}^\infty$ of closed subsets in X such that $X = \bigcup_{i=1}^\infty F_i$ and $f|F_i : F_i \to f(F_i)$ is a closed map for each $i = 1, 2, \dots$. A map $f: X \to Y$ is called *k-dimensional* if $\dim f^{-1}(y) \le k$ for each $y \in Y$, and *k-to-1* if $|f^{-1}(y)| \le k$ for each $y \in Y$. In [3] and [4], Pasynkov proved that if $f: X \to Y$ is a k-dimensional map from a compactum X to a finite-dimensional compactum Y, then there is a map $g: X \to I^k$ such that $\dim(f \times g) = 0$. Also, he proved that if $f: X \to Y$ is a k-dimensional map of compacta and $\dim Y = p < \infty$, then the set of maps g in the space $C(X, I^{p+2k+1})$ such that the diagonal product $f \times g: X \to Y \times I^{p+2k+1}$ is an embedding is a dense G_δ-subset of $C(X, I^{p+2k+1})$. Furthermore, in [2] the first author proved the following theorem.

THEOREM 1 ([2]). *If $f: X \to Y$ is a k-dimensional map of compacta and $\dim Y = p < \infty$, then for each $0 \le i \le p+k$, the set of maps g in the space $C(X, I^{p+2k+1-i})$ such that $f \times g: X \to Y \times I^{p+2k+1-i}$ is*

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$(i+1)$-to-1 is a dense G_δ-subset of $C(X, I^{p+2k+1-i})$. Hence the restriction $g|f^{-1}(y) : f^{-1}(y) \to I^{p+2k+1-i}$ is $(i+1)$-to-1 for each $y \in Y$.

A locally connected continuum D is called a *dendrite* if it contains no circle. A dendrite D is called a *superdendrite* [5] if the set of all end points of D is dense in D. The main aim of this paper is to prove the following theorem.

THEOREM 2. *Let $f : X \to Y$ be a k-dimensional map of compacta and $\dim Y = p < \infty$, and let D_j $(j = 1, \ldots, k)$ be superdendrites. Then the set of maps h in the space $C(X, \prod_{j=1}^k D_j \times I^{p+1-i})$ such that $f \times h : X \to Y \times (\prod_{j=1}^k D_j \times I^{p+1-i})$ is $(i+1)$-to-1 is a dense G_δ-subset of $C(X, \prod_{j=1}^k D_j \times I^{p+1-i})$ for each $0 \le i \le p$. Hence $h|f^{-1}(y) : f^{-1}(y) \to \prod_{j=1}^k D_j \times I^{p+1-i}$ is $(i+1)$-to-1 for each $y \in Y$.*

This is a generalization of the following theorem of Bowers [1] (cf. [5]) : If X is an n-dimensional compactum and $D_1, \ldots, D_n$ are superdendrites, then the set $\{h \in C(X, \prod_{j=1}^n D_j \times I) \mid h$ is an embedding$\}$ is a dense G_δ-subset in $C(X, \prod_{j=1}^n D_j \times I)$. As a corollary, we have a representation theorem for finite-dimensional maps using superdendrites (see Theorem 15).

2. Main theorem. First we set up some notation and terminology (cf. [5] and [6]). Let X, Y be compacta and let $A \subset X$ be a closed subset. If $f : X \to Y$ is a map, we set

$$S_f = \{x \in X \mid f^{-1}f(x) = \{x\}\}, \quad R_{(X,A,Y)} = \{f \in C(X,Y) \mid A \subset S_f\}.$$

A set $S \subset X$ is said to be *residual* if S contains a dense G_δ-subset of X. A map $f : X \to Y$ is called a *(k,ε)-map* $(\varepsilon > 0)$ if for each $y \in Y$, there are subsets $A_1, \ldots A_k$ of $f^{-1}(y)$ such that $f^{-1}(y) = \bigcup_{i=1}^k A_i$ and $\operatorname{diam} A_i < \varepsilon$. The main aim of this section is to prove Theorem 2. To do this we need the following results.

THEOREM 3 ([6]). *Let X, Y be spaces with $\dim Y < \infty$ and let $f : X \to Y$ be a σ-closed k-dimensional map. Then there exists a 0-dimensional F_σ-subset $A \subset X$ such that $f|(X \setminus A)$ is $(k-1)$-dimensional.*

THEOREM 4 ([5]). *Let X be a compactum and let $A \subset X$ be a 0-dimensional F_σ-subset. Then for each superdendrite D, $R_{(X,A,D)} = \{f \in C(X,D) \mid A \subset S_f\}$ is residual in $C(X,D)$.*

PROPOSITION 5. *Let X, Y and Z be compacta and let $f : X \to Y$ be a map. Then $\{g \in C(X,Z) \mid f|(X \setminus S_g)$ is k-dimensional$\}$ is a G_δ-subset in $C(X,Z)$.*

Proof. For a, $b > 0$ and $g \in C(X, Z)$, let

$F(g, a) = \{x \in X \mid \operatorname{diam}(g^{-1}g(x)) \geq a\}$,

$U(a, b) = \{g \in C(X, Z) \mid d_{k+1}(F(g, a) \cap f^{-1}(y)) < b$ for each $y \in Y\}$,

where $d_{n+1}(F) < b$ if there exists an open cover of F with mesh $< b$ and order $\leq n$.

Claim. *$U(a, b)$ is an open subset in $C(X, Y)$.*

Proof of claim. Assume that $U(a, b)$ is not open in $C(X, Y)$. Then there exist $g \in U(a, b)$ and $\{g_i\}_{i=1}^{\infty} \subset C(X, Y) \setminus U(a, b)$ such that $g_i \to g$. For each $i = 1, 2, \ldots$, there exists $y_i \in Y$ such that $d_{k+1}(f^{-1}(y_i) \cap F(g_i, a)) \geq b$. We may assume $y = \lim_{i\to\infty} y_i$. Since $g \in U(a, b)$, there exists a family $\mathcal{U}$ of open subsets in X such that $\operatorname{ord}\mathcal{U} \leq k$, $\operatorname{mesh}\mathcal{U} < b$ and $f^{-1}(y) \cap F(g, a) \subset \bigcup\mathcal{U}$. Since $d_{k+1}(f^{-1}(y_i) \cap F(g_i, a)) \geq b$, there exist $x_i \in X$ such that $\operatorname{diam}(g_i^{-1}g_i(x_i)) \geq a$ and $f^{-1}(y_i) \cap g_i^{-1}g_i(x_i) \not\subset \bigcup\mathcal{U}$ for each $i = 1, 2, \ldots$. In fact, we may choose $x_i \in (f^{-1}(y_i) \cap g_i^{-1}g_i(x_i)) \setminus \bigcup\mathcal{U}$. We may assume $x = \lim_{i\to\infty} x_i$. Since $\operatorname{diam} g^{-1}g(x) \geq a$, we have $f^{-1}(y) \cap g^{-1}g(x) \subset \bigcup\mathcal{U}$. So $f^{-1}(y_i) \cap g_i^{-1}g_i(x_i) \subset \bigcup\mathcal{U}$ for infinitely many i. This is a contradiction. This completes the proof of the claim.

It is easy to see that $\{g \in C(X, Z) \mid f|(X \setminus S_g)$ is k-dimensional$\} = \bigcap_{m,n\in\mathbb{N}} U(1/m, 1/n)$. This completes the proof.

The next result is due to Pasynkov. For completeness, we will give the proof.

Proposition 6 ([3]). *Let X, Y and Z be compacta and let $f : X \to Y$ be a map. Then $\{g \in C(X, Z) \mid f \times g$ is k-dimensional$\}$ is a G_δ-subset in $C(X, Z)$.*

Proof. It suffices to observe that

$$\{g \in C(X, Z) \mid d_{k+1}((f \times g)^{-1}(y, z)) < b \text{ for each } y \in Y \text{ and } z \in Z\}$$

is an open subset of $C(X, Z)$. The argument is similar to that in the proof of the preceding proposition. This completes the proof.

Theorem 7. *Let $f : X \to Y$ be a k-dimensional map of compacta with $\dim Y = p < \infty$, and let D_j $(j = 1, \ldots, k)$ be superdendrites. Then the set of maps g in the space $C(X, \prod_{j=1}^{k} D_j)$ such that $f|(X \setminus S_g)$ is 0-dimensional is a dense G_δ-subset of $C(X, \prod_{j=1}^{k} D_j)$. In particular, the set of maps g in the space $C(X, \prod_{j=1}^{k} D_j)$ such that the diagonal product $f \times g : X \to Y \times \prod_{j=1}^{k} D_j$ is 0-dimensional is a dense G_δ-subset of $C(X, \prod_{j=1}^{k} D_j)$.*

Proof. Let $\mathcal{G}(X, \prod_{j=1}^{k} D_j) = \{g \in C(X, \prod_{j=1}^{k} D_j) \mid f|(X \setminus S_g)$ is 0-dimensional$\}$. By Propositions 5 and 6, it is sufficient to show that

$\mathcal{G}(X, \prod_{j=1}^k D_j)$ is a dense subset of $C(X, \prod_{j=1}^k D_j)$. Let $r = r_1 \times \ldots \times r_k \in C(X, \prod_{j=1}^k D_j)$. We will find a map $g = g_1 \times \cdots \times g_k \in \mathcal{G}(X, \prod_{j=1}^k D_j)$ arbitrarily close to r. By Theorem 3, there exists a 0-dimensional F_σ-set $A_1 \subset X$ such that $f|(X \setminus A_1) : X \setminus A_1 \to Y$ is $(k-1)$-dimensional. By Theorem 4, $R_{(X,A_1,D_1)}$ is residual in $C(X, D_1)$. So we can take a map $g_1 \in R_{(X,A_1,D_1)}$ arbitrarily close to r_1. Note that $A_1 \subset S_{g_1}$. Let $B_1 = X \setminus S_{g_1}$. Then B_1 is an F_σ-subset in X because S_{g_1} is a G_δ-subset in X. Since $f|B_1 : B_1 \to Y$ is a σ-closed $(k-1)$-dimensional map, by Theorem 3 there exists a 0-dimensional F_σ-subset $A_2 \subset B_1$ such that $f|(B_1 \setminus A_2) : B_1 \setminus A_2 \to Y$ is $(k-2)$-dimensional. By Theorem 4, we can take a map $g_2 \in R_{(X,A_2,D_2)}$ arbitrarily close to r_2. Note that $A_2 \subset S_{g_2}$. Let $B_2 = B_1 \setminus S_{g_2}$. Note that $B_2 = X \setminus (S_{g_1} \cup S_{g_2})$. By induction, for each $i = 1, \ldots, k$, we can find a map $g_i : X \to D_i$, an F_σ-subset $B_i \subset X$ and a 0-dimensional F_σ-subset $A_i \subset B_{i-1}$ such that

(1) r_i and g_i are arbitrarily close to each other,
(2) $g_i \in R_{(X,A_i,D_i)}$,
(3) $f|(B_{i-1} \setminus A_i) : B_{i-1} \setminus A_i \to Y$ is $(k-i)$-dimensional,
(4) $B_i = B_{i-1} \setminus S_{g_i}$.

Note that $B_k = X \setminus \bigcup_{i=1}^k S_{g_i}$. Then r and $g = g_1 \times \cdots \times g_k$ are arbitrarily close to each other and $f|B_k : B_k \to Y$ is 0-dimensional. Note that $\bigcup_{i=1}^k S_{g_i} \subset S_g$. So g is the required map. This completes the proof.

Perhaps the next proposition is known. For completeness, we will give the proof.

PROPOSITION 8. *Let X, Y and Z be compacta and let $f : X \to Y$ be a map. Then for each $k = 1, 2, \ldots$, the set $H = \{h \in C(X, Z) \mid f \times h$ is k-to-$1\}$ is a G_δ-subset in $C(X, Y)$.*

Proof. For each $n = 1, 2, \ldots$, let $H_n = \{h \in C(X, Z) \mid f \times h$ is a $(k, 1/n)$-map$\}$. It is easy to see that H_n is an open subset in $C(X, Z)$ and $H = \bigcap_{n=1}^\infty H_n$. This completes the proof.

Now we prove Theorem 2.

Proof of Theorem 2. Let $i = 0, 1, \ldots, p$ and

$$v = (r_1 \times \cdots \times r_k) \times (u_1 \times \cdots \times u_{p+1-i}) \in C\Big(X, \prod_{j=1}^k D_j \times I^{p+1-i}\Big).$$

Let $r = r_1 \times \cdots \times r_k$. By Proposition 8, it is sufficient to show that there exists a map $h : X \to \prod_{j=1}^k D_j \times I^{p+1-i}$ arbitrarily close to v and such that $f \times h : X \to Y \times (\prod_{j=1}^k D_j \times I^{p+1-i})$ is $(i+1)$-to-1. By Theorem 7, there exists $g : X \to \prod_{j=1}^k D_j$ arbitrarily close to r and such that $f|(X \setminus S_g)$ is 0-dimensional. Let $X \setminus S_g = \bigcup_{l=1}^\infty F_l$, where F_l is closed in X and $F_l \subset F_{l+1}$

for $l = 1, 2, \ldots$. For $l = 1, 2, \ldots$, let

$$S_l(X, I^{p+1-i}) \\ = \{s \in C(X, I^{p+1-i}) \mid (f \times s)|F_l : F_l \to Y \times I^{p+1-i} \text{ is } (i+1)\text{-to-1}\}.$$

As $f|F_l : F_l \to Y$ is 0-dimensional, by Theorem 1 the set $\{s \in C(F_l, I^{p+1-i}) \mid f|F_l \times s : F_l \to Y \times I^{p+1-i}$ is $(i+1)$-to-1$\}$ is a dense G_δ-subset in $C(F_l, I^{p+1-i})$ for $l = 1, 2, \ldots$. So it is easy to see that $S_l(X, I^{p+1-i})$ is a dense G_δ-subset in $C(X, I^{p+1-i})$ for $l = 1, 2, \ldots$. By Baire's theorem $\bigcap_{l=1}^{\infty} S_l(X, I^{p+1-i})$ is a dense G_δ-subset in $C(X, I^{p+1-i})$. So we can select $s \in \bigcap_{l=1}^{\infty} S_l(X, I^{p+1-i})$ arbitrarily close to $u = u_1 \times \cdots \times u_{p+1-i}$. Let $h = g \times s : X \to \prod_{j=1}^{k} D_j \times I^{p+1-i}$. Then h is as required. This completes the proof.

3. Finite-dimensional maps and compositions of maps parallel to the unit interval and superdendrites. Now we consider an application of Theorem 2. A map $f : X \to Y$ is said to be *embedded in a map* $f_0 : X_0 \to Y_0$ (see [3], [4]) if there exist embeddings $g : X \to X_0$ and $h : Y \to Y_0$ such that $h \circ f = f_0 \circ g$. A map $f : X \to Y$ is *parallel* to the space Z (see [3], [4]) if f can be embedded in the natural projection $p : Y \times Z \to Y$. In [3], [4], Pasynkov proved the following theorem: If $f : X \to Y$ has $\dim f = k$ and $\dim Y < \infty$, then f can be represented as the composition $f = h_k \circ \cdots \circ h_1 \circ g$ of a zero-dimensional map g and maps h_i $(i = 1, \ldots, k)$ parallel to the unit interval I. Furthermore the first author proved the following [2]: A map $f : X \to Y$ of compacta with $\dim Y = p < \infty$ is k-dimensional if and only if f can be represented as the composition $f = g_{p+2k+1} \circ \cdots \circ g_{p+k+1} \circ g_{p+k} \circ \cdots \circ g_1$ of maps parallel to the unit interval I such that g_i is $(i+1)$-to-1 for each $i = 1, \ldots, p+k$ and g_{p+k+1} is zero-dimensional. In this section we prove another representation theorem for finite-dimensional maps using superdendrites.

LEMMA 9 ([6]). *Let $\varepsilon > 0$. Suppose that $f : X \to Y$ is a map of compacta with $\dim f = 0$ and $\dim Y = p < \infty$. For each $i = 1, \ldots, l$, let K_i and L_i be closed disjoint subsets of X. Then there are open subsets E_i of X separating X between K_i and L_i such that $f|(\mathrm{Cl}(E_1) \cup \cdots \cup \mathrm{Cl}(E_l))$ is a (p, ε)-map.*

The next three results are essentially contained in [2]. For completeness, we give their proofs.

PROPOSITION 10 ([2]). *Let X, Y and Z be compacta and $0 \le k < \infty$. Let T be the set of maps $g = u \times v : X \to Y \times Z$ in $C(X, Y \times Z)$ such that $\dim v(u^{-1}(y)) \le k$ for each $y \in Y$. Then T is a G_δ-subset of $C(X, Y \times Z)$.*

Proof. Let $\varepsilon > 0$ and let T_ε be the set of maps $g = u \times v : X \to Y \times Z$ in $C(X, Y \times Z)$ such that for each $y \in Y$, $v(u^{-1}(y))$ is covered by a family

$\mathcal{U}$ of open sets of Z such that $\operatorname{mesh}\mathcal{U} < \varepsilon$ and $\operatorname{ord}\mathcal{U} \le k$. Then T_ε is open in $C(X, Y \times Z)$. Note that $T = \bigcap_{n=1}^{\infty} T_{1/n}$.

LEMMA 11 ([2], cf. [7], [9]). *Let $f : X \to Y$ be a 0-dimensional map from a compactum X to a p-dimensional compactum Y ($p < \infty$). Let T be the set of all maps $u : X \to I$ such that $\dim u(f^{-1}(y)) = 0$ for each $y \in Y$. Then T is a dense G_δ-subset of $C(X, I)$.*

Proof. Let W_b be the set of maps $u : X \to I$ such that for each $y \in Y$, $u(f^{-1}(y))$ is covered by disjoint open sets of diameters less than b. By an argument similar to that in the proof of Proposition 5, W_b is an open subset of $C(X, I)$. Since $T = \bigcap_{n=1}^{\infty} W_{1/n}$, it suffices to prove that W_b is dense in $C(X, I)$. Let $h \in C(X, I)$ and $\varepsilon > 0$. Choose $\delta > 0$ such that if $A \subset X$ and $\operatorname{diam} A < \delta$, then $\operatorname{diam} h(A) < \min\{2\varepsilon, b/(2p)\} = \varepsilon'$. Choose a finite family $\{(U_n, V_n) \mid n = 1, \ldots, m\}$, where U_n and V_n are open subsets of X such that $\{U_n \mid n = 1, \ldots, m\}$ is a cover of X with $\mathrm{Cl}(U_n) \subset V_n$ and $\operatorname{diam} V_n < \delta$ for all $n = 1, \ldots, m$. By Lemma 9, there are open subsets E_n separating X between $\mathrm{Cl}(U_n)$ and $X \setminus V_n$ such that $f|(\mathrm{Cl}(E_1) \cup \cdots \cup \mathrm{Cl}(E_m))$ is a (p, δ)-map. Note that $X \setminus (E_1 \cup \cdots \cup E_m) = \bigcup_{n=1}^{m} D_n$, where D_n ($n = 1, \ldots, m$) are disjoint closed subsets of X such that $\operatorname{diam} D_n < \delta$. There are points $x_n \in I$ such that $h(D_n) \subset (x_n - \varepsilon'/2, x_n + \varepsilon'/2)$. The function that maps each D_n to x_n has a continuous extension $u : X \to I$ whose supremum distance to h is less than $\varepsilon'/2 \le \varepsilon$. Let $y \in Y$. Since $f|(\mathrm{Cl}(E_1) \cup \cdots \cup \mathrm{Cl}(E_m))$ is a (p, δ)-map, there are closed subsets $A_1, \ldots, A_p$ of $f^{-1}(y)$ such that $f^{-1}(y) \cap (\mathrm{Cl}(E_1) \cup \cdots \cup \mathrm{Cl}(E_m)) = \bigcup_{i=1}^{p} A_i$ and $\operatorname{diam} A_i < \delta$ for $i = 1, \ldots, p$. Note that $\operatorname{diam} u(A_i) < 2\varepsilon'$ for each i, and $u(D_n) = \{x_n\}$ for each n. We can see that for each component C of $\bigcup_{i=1}^{p} u(A_i)$, we have $\operatorname{diam} C < 2p\varepsilon' \le b$ and

$$u(f^{-1}(y)) \subset \{x_1, \ldots, x_m\} \cup \bigcup_{i=1}^{p} u(A_i).$$

Hence, each component of $u(f^{-1}(y))$ has a neighbourhood that is closed and open in $u(f^{-1}(y))$ and has diameter less than b. It follows that $u \in W_b$, which completes the proof.

LEMMA 12 ([2]). *Let $f : X \to Y$, $g : X \to Z$ and $u : X \to I$ be maps of compacta such that $\dim Z = k$ and $\dim u((f \times g)^{-1}(y, z)) = 0$ for each $y \in Y$ and $z \in Z$. Then $\dim(g \times u)(f^{-1}(y)) \le k$ for each $y \in Y$.*

Proof. Let $y \in Y$. Consider the natural projection $p : Y \times Z \times I \to Y \times Z$. Then $p|(f \times g \times u)(X) : (f \times g \times u)(X) \to (f \times g)(X)$ is a 0-dimensional map, because for $(y, z) \in (f \times g)(X) \subset Y \times Z$,

$$p^{-1}(y, z) \cap (f \times g \times u)(X) = \{(y, z)\} \times u((f \times g)^{-1}(y, z)).$$

Also, note that $(f \times g \times u)(f^{-1}(y)) = \{y\} \times (g \times u)(f^{-1}(y))$, and hence the set

$$p(\{y\} \times (g \times u)(f^{-1}(y)) = (f \times g)(f^{-1}(y)) = \{y\} \times g(f^{-1}(y)) \subset \{y\} \times Z$$

is at most k-dimensional. Since

$$p|\{y\} \times (g \times u)(f^{-1}(y)) : \{y\} \times (g \times u)(f^{-1}(y)) \to \{y\} \times g(f^{-1}(y))$$

is a zero-dimensional map, by a theorem of Hurewicz we conclude that $\dim(g \times u)(f^{-1}(y)) \leq k$.

By Theorem 7, Propositions 6 and 10, and Lemmas 11 and 12, we obtain the next result.

Proposition 13 ([2], cf. [7], [9]). *Let $f : X \to Y$ be a k-dimensional map of compacta and $\dim Y = p < \infty$. Let T be the set of all maps $h = g \times u : X \to \prod_{j=1}^k D_j \times I$ in $C(X, \prod_{j=1}^k D_j \times I)$ such that $\dim h(f^{-1}(y)) \leq k$, $\dim u((f \times g)^{-1}(y,t)) = 0$ for all $y \in Y$ and $t \in \prod_{j=1}^k D_j$, and $\dim(f \times g) = 0$. Then T is a dense G_δ-subset of $C(X, \prod_{j=1}^k D_j \times I)$.*

Proposition 14. *Let $f : X \to Y$ be a k-dimensional map of compacta and $\dim Y = p < \infty$. For $i = 0, 1, \ldots, p+1$, let $p_i : \prod_{j=1}^k D_j \times I^{p+1} \to \prod_{j=1}^k D_j \times I^{p+1-i}$ be the natural projection, where $p_0 : \prod_{j=1}^k D_j \times I^{p+1} \to \prod_{j=1}^k D_j \times I^{p+1}$ is the identity. Let $\widetilde{E}(X, \prod_{j=1}^k D_j \times I^{p+1})$ be the set of maps g in $C(X, \prod_{j=1}^k D_j \times I^{p+1})$ such that*

1. *for each $0 \leq i \leq p$, $f \times (p_i \circ g) : X \to Y \times \prod_{j=1}^k D_j \times I^{p+1-i}$ is an $(i+1)$-to-1 map,*
2. *the map $h = p_p \circ g = g' \times u : X \to \prod_{j=1}^k D_j \times I$ satisfies $\dim h(f^{-1}(y)) \leq k$, $\dim u((f \times g')^{-1}(y,t)) = 0$ for all $y \in Y$ and $t \in \prod_{j=1}^k D_j$, and $\dim(f \times g') = 0$.*

Then $\widetilde{E}(X, \prod_{j=1}^k D_j \times I^{p+1})$ is a dense G_δ-subset of $C(X, \prod_{j=1}^k D_j \times I^{p+1})$.

Proof. Note that if $q : A \to B$ is an open map and C is a dense subset of B, then $q^{-1}(C)$ is dense in A. The natural projection $p_i : \prod_{j=1}^k D_j \times I^{p+1} \to \prod_{j=1}^k D_j \times I^{p+1-i}$ induces the open map $P_i : C(X, \prod_{j=1}^k D_j \times I^{p+1}) \to C(X, \prod_{j=1}^k D_j \times I^{p+1-i})$ defined by $P_i(h) = p_i \circ h$ for $h \in C(X, \prod_{j=1}^k D_j \times I^{p+1})$. For $i = 0, 1, \ldots, p$, let E_i be the set of $g \in C(X, \prod_{j=1}^k D_j \times I^{p+1-i})$ such that $f \times g : X \to Y \times \prod_{j=1}^k D_j \times I^{p+1-i}$ is $(i+1)$-to-1. Also, let T be the subset of $C(X, \prod_{j=1}^k D_j \times \mathrm{I})$ as in Proposition 13. By Theorem 2 and Proposition 13,

$$\widetilde{E}\Big(X, \prod_{j=1}^{k} D_j \times I^{p+1}\Big) = \bigcap_{i=0}^{p} P_i^{-1}(E_i) \cap P_p^{-1}(T)$$

is a dense G_δ-subset of $C(X, \prod_{j=1}^{k} D_j \times I^{p+1})$. This completes the proof.

Now, we have the following representation theorem for finite-dimensional maps.

THEOREM 15. *Let $f : X \to Y$ be a map of compacta such that $0 \leq k < \infty$ and $\dim Y = p < \infty$. Then f is k-dimensional if and only if f can be represented as the composition*

$$f = g_{p+k+1} \circ \cdots \circ g_{p+1} \circ g_p \circ \cdots \circ g_1$$

such that

(1) *g_i is an $(i+1)$-to-1 map for each $i = 1, \ldots, p$ and g_{p+1} is a zero-dimensional map,*
(2) *g_i is parallel to I for $i = 1, \ldots, p+1$,*
(3) *g_i is parallel to a superdendrite for $i = p+2, \ldots, p+k+1$.*

Proof. Let $\widetilde{E}(X, \prod_{j=1}^{k} D_j \times I^{p+1})$ be as in Proposition 14. Choose $g \in \widetilde{E}(X, \prod_{j=1}^{k} D_j \times I^{p+1})$. Let

$$Z_i = \begin{cases} \prod_{j=1}^{k} D_j \times I^{p+1-i} & \text{for } i = 0, 1, \ldots, p+1, \\ \prod_{j=1}^{p+k+1-i} D_j & \text{for } i = p+2, \ldots, p+k. \end{cases}$$

For $i = 0, 1, \ldots, p+k$, let $p_i : Z_0 \to Z_i$ be the natural projection. For $i = 0, 1, \ldots, p+k$, put $X_i = (f \times (p_i \circ g))(X)$ and put $X_{p+k+1} = Y$. Let $g_1 = q_1 \circ (f \times g)$ and for $i = 2, \ldots, p+k+1$, let $g_i = q_i | X_{i-1} : X_{i-1} \to X_i$, where $q_i : Y \times Z_{i-1} \to Y \times Z_i$ is the natural projection for $i = 1, \ldots, p+k$ and $q_{p+k+1} : Y \times D_1 \to Y$ is the natural projection. By Propositon 14, we see that g_{p+1} is a 0-dimensional map. Hence the maps g_i are as desired. This completes the proof.

REMARK. After the paper [2] had been submitted for publication, the paper of Tuncali and Valov [8] was published. They obtained a more general result in the class of all metrizable spaces.

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