ON RINGS OF CONSTANTS OF DERIVATIONS
IN TWO VARIABLES IN POSITIVE CHARACTERISTIC

BY

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Abstract. Let $k$ be a field of characteristic $p > 0$. We describe all derivations of the polynomial algebra $k[x, y]$, homogeneous with respect to a given weight vector, in particular all monomial derivations, with the ring of constants of the form $k[x^p, y^p, f]$, where $f \in k[x, y] \setminus k[x^p, y^p]$.

Introduction. A. Nowicki and M. Nagata proved in [4] that if $d$ is a nonzero $k$-derivation of $k[x, y]$, where $k$ is a field of characteristic $p > 0$, then $k[x, y]^d$, the ring of constants of $d$, is a free $k[x^p, y^p]$-module. Moreover they showed that if $p = 2$, then $k[x, y]^d = k[x^p, y^p, f]$ for some $f \in k[x, y]$. W. Li proved in [2] that the rank of $k[x, y]^d$ as a free $k[x^p, y^p]$-module equals 1 or $p$.

It is natural to ask, for arbitrary $p$, when a $k$-derivation of $k[x, y]$ has the ring of constants of the form $k[x^p, y^p, f]$, where $f \in k[x, y] \setminus k[x^p, y^p]$. In this paper we answer this question for derivations which are homogeneous with respect to a given weight vector (Theorem 11, Corollary 12). This is a generalization of the results of [1].

In the last section we obtain a description, for arbitrary $p$, of all monomial derivations of $k[x, y]$ with rings of constants of the form $k[x^p, y^p, f]$, where $f \in k[x, y] \setminus k[x^p, y^p]$ (Theorem 16, Corollary 17). Note that the rings of constants of all monomial derivations for $p = 2$ and $p = 3$ were computed by S.-I. Okuda in [5], using his adaptation of van den Essen's algorithm for the case of positive characteristic.

1. Preliminaries. Throughout this paper $k$ is a field of characteristic $p > 0$. We denote by $k[X]$ the polynomial $k$-algebra $k[x_1, \ldots, x_n]$ and by $k[X^p]$ the $k$-subalgebra $k[x_1^p, \ldots, x_n^p]$. In the case of two variables we will just write $k[x, y]$.

A $k$-linear mapping $d: k[X] \to k[X]$ is called a $k$-derivation of $k[X]$ if $d(fg) = fd(g) + gd(f)$ for all $f, g \in k[X]$. Every $k$-derivation $d$ of $k[X]$ is

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of the form \( g_1 \cdot \partial / \partial x_1 + \cdots + g_n \cdot \partial / \partial x_n \) for some polynomials \( g_1, \ldots, g_n \in k[X] \), that is, \( d \) is uniquely determined by the conditions \( d(x_1) = g_1, \ldots, d(x_n) = g_n \). If \( d \) is a \( k \)-derivation of \( k[X] \), then we denote by \( k[X]^d \) the ring of constants of \( d \):

\[
k[X]^d = \{ f \in k[X] : d(f) = 0 \}.
\]

Note that \( k[X]^d \subseteq k[X]^d \), so \( k[X]^d \) is a \( k[X]^d \)-algebra.

We introduce the notion of \( \gamma \)-homogeneity analogously to [3, 2.1]. Consider a vector \( \gamma = (\gamma_1, \ldots, \gamma_n) \in k^n \setminus \{(0, \ldots, 0)\} \). For every \( r \in k \) denote by \( k[X]_{(r)}^\gamma \) the \( k \)-linear span of all monomials \( x_1^{l_1} \cdots x_n^{l_n} \) such that

\[
l_1 \gamma_1 + \cdots + l_n \gamma_n = r.
\]

If no monomial satisfies this equality, then \( k[X]_{(r)}^\gamma = 0 \). We obtain a grading of \( k[X] \) by the additive group of the field \( k \). Polynomials belonging to \( k[X]_{(r)}^\gamma \) are called \( \gamma \)-forms of degree \( r \). In particular, \( x_i \) is a \( \gamma \)-form of degree \( \gamma_i \) for \( i = 1, \ldots, n \). If \( \gamma_1 = \cdots = \gamma_n \), then the \( \gamma \)-forms are exactly the \( p \)-homogeneous polynomials in the sense of [1].

A \( k \)-derivation \( d \) of \( k[X] \) is called \( \gamma \)-homogeneous of degree \( s \), where \( s \in k \), if \( d(k[X]_{(r)}^\gamma) \subseteq k[X]_{(r+s)}^\gamma \) for every \( r \in k \), that is, \( d(x_i) \in k[X]_{(\gamma_i+s)}^\gamma \) for \( i = 1, \ldots, n \). Denote by \( E^\gamma \) the derivation of the form

\[
\gamma_1 x_1 \cdot \frac{\partial}{\partial x_1} + \cdots + \gamma_n x_n \cdot \frac{\partial}{\partial x_n},
\]

which is \( \gamma \)-homogeneous of degree 0. Observe that

\[
E^\gamma(x_1^{l_1} \cdots x_n^{l_n}) = (l_1 \gamma_1 + \cdots + l_n \gamma_n) \cdot x_1^{l_1} \cdots x_n^{l_n},
\]

so a polynomial \( f \) is a \( \gamma \)-form of degree \( r \) if and only if \( E^\gamma(f) = rf \). This is a weight analog of the Euler formula (compare [3, 2.1.1], [1, 1.4]). In particular, \( k[x, y]_{(0)}^\gamma \) is the ring of constants of \( E^\gamma \).

For every \( f \in k[X] \) let

\[
C_k(f) = k(x_1^p, \ldots, x_n^p)[f] \cap k[X],
\]

where \( k(x_1^p, \ldots, x_n^p) \) is the subfield of \( k(x_1, \ldots, x_n) \) generated by \( x_1^p, \ldots, x_n^p \). The following fact immediately follows from [4, Proposition 1.2].

**Proposition 1.** If \( d \) is a nonzero \( k \)-derivation of \( k[x, y] \) such that \( k[x, y]^d \neq k[x^p, y^p] \), then \( k[x, y]^d = C_k(f) \) for some \( (\text{and then for any}) \) \( f \in k[x, y]^d \setminus k[x^p, y^p] \). \( \blacksquare \)

We denote by \( \bar{f} \) the greatest common divisor of \( \partial f / \partial x_1, \ldots, \partial f / \partial x_n \) (defined up to a nonzero scalar factor). We write \( f \sim g \), where \( f, g \) are polynomials, if \( f = ag \) for some \( a \in k \setminus \{0\} \). We use the same convention for derivations, i.e. we write \( d_1 \sim d_2 \) if \( d_1 = ad_2 \) for some \( a \in k \setminus \{0\} \).

It is easy to verify that Corollary 2.4, Proposition 2.6, Theorem 3.2 and Corollary 3.3 from [1] hold true for \( \gamma \)-forms, so we obtain the following result.
Proposition 2. If \( f \in k[X] \setminus k[X^p] \) is a \( \gamma \)-form of a nonzero degree, then the following conditions are equivalent:

(i) \( C_k(f) = k[X^p][f] \),
(ii) \( f \) has no multiple factors and no factors from \( k[X^p] \setminus k \),
(iii) \( \bar{f} \sim 1 \). □

2. \( \gamma \)-homogeneous derivations of \( k[x, y] \). For a polynomial \( f \in k[x, y] \) we denote by \( d_f \) the jacobian derivation with respect to \( f \):

\[
d_f = \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial}{\partial x}.
\]

If \( f \) is a \( \gamma \)-form of degree \( r \), where \( \gamma = (\lambda, \mu) \), then \( d_f \) is a \( \gamma \)-homogeneous derivation of degree \( r - \lambda - \mu \). Note that \( d_f = d_g \) if and only if \( f - g \in k[x^p, y^p] \).

We can reformulate Proposition 4.1 and generalize Proposition 4.3 from [1] in the following way.

Proposition 3. Let \( d \) be a nonzero \( k \)-derivation of \( k[x, y] \) such that \( k[x, y]^d \neq k[x^p, y^p] \), and let \( f \in k[x, y]^d \setminus k[x^p, y^p] \). Then

\[
\bar{f} \cdot d \sim \gcd(d(x), d(y)) \cdot d_f,
\]

where \( \bar{f} = \gcd(\partial f/\partial x, \partial f/\partial y) \). In particular, if \( d(x), d(y) \) are coprime and \( \bar{f} \sim 1 \), then \( d \sim d_f \). □

Corollary 4. Let \( d \) be a nonzero \( k \)-derivation of \( k[x, y] \). If \( d(f) = 0 \) for some \( f \in k[x, y]^\gamma(0) \setminus k[x^p, y^p] \), then \( k[x, y]^d = k[x, y]^\gamma(0) \).

Proof. If \( f \in [x, y]^\gamma(0) \), then \( E^\gamma(f) = 0 \), so \( k[x, y]^d = k[x, y]^E^\gamma = k[x, y]^\gamma(0) \), by Proposition 1. □

Corollary 5. Let \( \gamma = (\lambda, \mu) \) and let \( f \in k[x, y] \setminus k[x^p, y^p] \) be a \( \gamma \)-form of degree 0 such that \( \bar{f} \sim 1 \).

(a) If \( \lambda, \mu \neq 0 \), then \( d_f \sim E^\gamma \).
(b) If \( \lambda = 0, \mu \neq 0 \), then \( y d_f \sim E^\gamma \).
(c) If \( \lambda \neq 0, \mu = 0 \), then \( x d_f \sim E^\gamma \).

Proof. Applying Proposition 3 to \( d = E^\gamma \), we obtain the following formula:

\[
\gcd(\lambda x, \mu y) \cdot d_f \sim \bar{f} \cdot E^\gamma. \quad \blacksquare
\]

Recall Proposition 2.7 from [1] in the case of two variables.

Proposition 6. Let \( f, g \in k[x, y] \). Then \( k[x^p, y^p, f] = k[x^p, y^p, g] \) if and only if \( f - ag \in k[x^p, y^p] \) for some \( a \in k \setminus \{0\} \). □

The following proposition is a generalization of Proposition 4.4 from [1]. This proof is new; the proof in [1] was partially specific to homogeneity without weights.
Proposition 7. Let \( f \in k[x, y]_{(0)}^{\gamma} \setminus k[x^p, y^p] \), where \( \gamma = (\lambda, \mu) \). Then the following conditions are equivalent:

(i) \( k[x, y]_{(0)}^{\gamma} = k[x^p, y^p, f] \),

\[
\lambda + \mu = 0, \quad f = axy + g
\]

(ii) or \( \lambda = 0, \quad f = ax + g \)

\( \lambda = 0 \)

\( \mu = 0 \)

\( f = ay + g \)

for some \( a \in k \setminus \{0\} \) and \( g \in k[x^p, y^p] \),

(iii) \( \bar{f} \sim 1 \).

Proof. (i) \( \Rightarrow \) (ii). Assume that \( k[x, y]_{(0)}^{\gamma} = k[x^p, y^p, f] \). If \( \lambda + \mu = 0 \), then all monomials of degree 0 are of the form \( x^{mp+l}y^{np+l} \), where \( m, n, l \geq 0 \), so \( k[x, y]_{(0)}^{\gamma} = k[x^p, y^p, xy] \), and, by Proposition 6, \( f - axy \in k[x^p, y^p] \) for some \( a \in k \setminus \{0\} \). If \( \lambda = 0, \mu \neq 0 \), then all monomials of degree 0 are of the form \( x^ly^{np} \), where \( l, n \geq 0 \), and we have \( k[x, y]_{(0)}^{\gamma} = k[x^p, y^p, x] \), so (Proposition 6) \( f - ax \in k[x^p, y^p] \) for some \( a \in k \setminus \{0\} \). Analogously, if \( \lambda \neq 0, \mu = 0 \), then \( k[x, y]_{(0)}^{\gamma} = k[x^p, y^p, y] \), so \( f - ay \in k[x^p, y^p] \) for some \( a \in k \setminus \{0\} \).

Now, let \( \lambda, \mu \neq 0 \) and \( \lambda + \mu \neq 0 \). Note that \( \lambda, \mu \) are linearly dependent over the prime subfield \( \mathbb{F}_p \) of \( k \), because \( k[x, y]_{(0)}^{\gamma} \neq k[x^p, y^p, f] \). Consider integers \( j, l \in \{2, \ldots, p-1\} \) such that \( j \lambda + \mu = 0 \) and \( \lambda + l \mu = 0 \). In this case the monomials \( x^y \) and \( xy^l \) are \( \gamma \)-homogeneous of degree 0, so \( x^y, xy^l \in k[x^p, y^p, f] \).

Following the method from Example 4.3 in [4], we consider polynomials \( u(T), v(T) \in k[x^p, y^p][T] \) such that \( x^jy = u(f), xy^l = v(f) \). We obtain the following equalities:

\[
\begin{align*}
 jx^{j-1}y &= u'(f) \cdot \frac{\partial f}{\partial x}, \quad x^j = u'(f) \cdot \frac{\partial f}{\partial y}, \quad y^l = v'(f) \cdot \frac{\partial f}{\partial x},
\end{align*}
\]

from which we deduce that \( u'(f) = cx^{-1} \) for some \( c \in k \setminus \{0\} \), so \( x^{j-1} \in k[x^p, y^p, f] \). This is a contradiction, because \( E^\gamma(x^{-1}) \neq 0 \).

(ii) \( \Rightarrow \) (i). Consider arbitrary \( a \in k \setminus \{0\} \) and \( g \in k[x^p, y^p] \). If \( \lambda + \mu = 0 \), then \( k[x, y]_{(0)}^{\gamma} = k[x^p, y^p, xy] = k[x^p, y^p, f] \) for \( f = axy + g \). If \( \lambda = 0 \) and \( f = ax + g \), then \( k[x, y]_{(0)}^{\gamma} = k[x^p, y^p, x] = k[x^p, y^p, f] \). Analogously, if \( \mu = 0 \) and \( f = ay + g \), then \( k[x, y]_{(0)}^{\gamma} = k[x^p, y^p, y] = k[x^p, y^p, f] \).

(ii) \( \Rightarrow \) (iii). Obviously, in each case \( f \) belongs to \( k[x, y]_{(0)}^{\gamma} \setminus k[x^p, y^p] \) and \( \partial f/\partial x, \partial f/\partial y \) are coprime.

(iii) \( \Rightarrow \) (ii). If \( \lambda, \mu \neq 0 \), then, by Corollary 5, \( df = cE^\gamma \) for some \( c \in k \setminus \{0\} \), so we obtain a system of partial differential equations \( \partial f/\partial x = c\mu y \) and \( \partial f/\partial y = -c\lambda x \). Note that

\[
c\mu = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = -c\lambda,
\]

so \( \lambda + \mu = 0 \). In this case the general solution is of the form \( f = c\mu xy + g \), where \( g \in k[x^p, y^p] \).
If $\lambda = 0$, then (Corollary 5) $yd_f = cE^\gamma$ for some $c \in k \setminus \{0\}$, we have a system $\partial f/\partial x = c\mu$, $\partial f/\partial y = 0$, and the solution is $f = c\mu x + g$, where $g \in k[x^p, y^p]$. Analogously, if $\mu = 0$, then $\partial f/\partial x = 0$ and $\partial f/\partial y = -c\lambda$, so $f = -c\lambda y + g$, where $g \in k[x^p, y^p]$.

**Corollary 8.** Let $d$ be a nonzero $k$-derivation of $k[x,y]$ such that $k[x,y]^d \neq k[x^p, y^p]$, and let $f \in k[x,y]^d \setminus k[x^p, y^p]$ be a $\gamma$-form. Then $k[x,y]^d = k[x^p, y^p, f]$ if and only if $\bar{f} \sim 1$.

**Proof.** This follows from Propositions 1 and 2 if $f$ is a $\gamma$-form of a nonzero degree, and from Proposition 7 and Corollary 4 if $f$ is a $\gamma$-form of degree 0.

The next two propositions explain some relations between $\gamma$-homogeneity of derivations and $\gamma$-homogeneity of polynomials.

**Lemma 9.** Let $f \in k[x,y] \setminus k[x^p, y^p]$. If $d_f$ is a $\gamma$-homogeneous $k$-derivation of $k[x,y]$, then there exists a $\gamma$-form $h \in k[x,y] \setminus k[x^p, y^p]$ such that $f - h \in k[x^p, y^p]$.

**Proof.** Assume that $d_f$ is $\gamma$-homogeneous of degree $s$. This means that $\partial f/\partial x$ and $\partial f/\partial y$ are $\gamma$-forms of degrees $s + \mu$ and $s + \lambda$, respectively.

If $f_r$ is the $\gamma$-homogeneous component of $f$ of degree $r \in k$, then $\partial f_r/\partial x$ is the $\gamma$-homogeneous component of $\partial f/\partial x$ of degree $r - \lambda$, so $\partial f_r/\partial x = 0$ for $r \neq s + \lambda + \mu$. Analogously, $\partial f_r/\partial y$ is the $\gamma$-homogeneous component of $\partial f/\partial y$ of degree $r - \mu$, so $\partial f_r/\partial y = 0$ for $r \neq s + \lambda + \mu$. This implies that $f_r \in k[x^p, y^p]$ for $r \neq s + \lambda + \mu$, and we may put $h = f_{s+\lambda+\mu}$.

**Lemma 10.** If $d$ is a nonzero $\gamma$-homogeneous $k$-derivation of $k[x,y]$ such that $k[x,y]^d = k[x^p, y^p, f]$, where $f \in k[x,y] \setminus k[x^p, y^p]$, then there exists a $\gamma$-form $h \in k[x,y]^d \setminus k[x^p, y^p]$ such that $f - h \in k[x^p, y^p]$.

**Proof.** By $\gamma$-homogeneity of $d$, all $\gamma$-homogeneous components of $f$ belong to $k[x,y]^d$. If the $\gamma$-homogeneous component of $f$ of degree 0 does not belong to $k[x^p, y^p]$, then $k[x,y]^d = k[x,y]^0$ by Corollary 4, so $f \in k[x,y]^0$, and we may apply the implication (i)$\Rightarrow$(ii) from Proposition 7.

Now assume that the $\gamma$-homogeneous component of $f$ of degree 0 belongs to $k[x^p, y^p]$. Let $f_r$ be the $\gamma$-homogeneous component of $f$ of degree $r \neq 0$, so $f_r \in k[x,y]^d$, and, by the assumption, $f_r = u(f)$ for some polynomial $u(T) \in k[x^p, y^p][T]$. Then $r f_r = E^\gamma(f_r) = E^\gamma(f) \cdot u'(f)$.

Assume that $f_r \neq 0$. Then $\deg f_r \leq \deg E^\gamma(f)$, where $\deg$ denotes the ordinary degree of a polynomial, so the above equality implies that $r f_r = cE^\gamma(f)$ for some $c \in k \setminus \{0\}$. Hence $E^\gamma(f)$ is a $\gamma$-form of degree $r$ and $f_r$ is the only nonzero $\gamma$-homogeneous component of $f$ of a nonzero degree, so we may put $h = f_r$.

Now we are ready to prove the following theorem.
Theorem 11. Let $k$ be a field of characteristic $p > 0$, let $d$ be a nonzero $\gamma$-homogeneous $k$-derivation of $k[x, y]$ such that $d(x)$ and $d(y)$ are coprime, and let $f \in k[x, y] \setminus k[x^p, y^p]$. Then

$$k[x, y]^d = k[x^p, y^p, f]$$

if and only if $d \sim d_f$.

Proof. ($\Rightarrow$) If $k[x, y]^d = k[x^p, y^p, f]$ for some $f \in k[x, y] \setminus k[x^p, y^p]$, then (Lemma 10) there exists a $\gamma$-form $h \in k[x, y]$ such that $f - h \in k[x^p, y^p]$, that is, $k[x, y]^d = k[x^p, y^p, h]$. Then $h \sim 1$ by Corollary 8, so, by Proposition 3, $d \sim d_h \sim d_f$.

($\Leftarrow$) If $d \sim d_f$, then (Lemma 9) $d \sim d_h$ for some $\gamma$-form $h \in k[x, y] \setminus k[x^p, y^p]$ such that $f - h \in k[x^p, y^p]$. Since $d(x)$ and $d(y)$ are coprime, that is, $h \sim 1$, we deduce by Corollary 8 that $k[x, y]^d = k[x^p, y^p, f]$.

Corollary 12. Let $d$ be a nonzero $\gamma$-homogeneous $k$-derivation of $k[x, y]$ such that $d(x)$ and $d(y)$ are coprime. Then $k[x, y]^d = k[x^p, y^p, f]$ for some $f \in k[x, y] \setminus k[x^p, y^p]$ if and only if $d$ is a jacobian derivation.

3. Monomial derivations of $k[x, y]$. A $k$-derivation $d: k[x, y] \to k[x, y]$ is called *monomial* if $d(x) = x^t y^u$ and $d(y) = x^v y^w$ for some integers $t, u, v, w \geq 0$. We will consider a slightly more general case:

\[
\begin{align*}
\{ & d(x) = \alpha x^t y^u, \\
& d(y) = \beta x^v y^w, 
\end{align*}
\]

where $\alpha, \beta \in k$.

Now consider an arbitrary nonzero $k$-derivation $d$ of $k[x, y]$ and a polynomial $f \in k[x, y] \setminus k[x^p, y^p]$. By Corollary 8, if $\partial f/\partial x$ and $\partial f/\partial y$ are coprime, $d(f) = 0$ and $f$ is a $\gamma$-form for some $\gamma$, then $k[x, y]^d = k[x^p, y^p, f]$. This is the way one can easily verify the following fact.

Example 13. Let $m, n, r, s$ be nonnegative integers, $m, n \neq -1 \mod p$, let $\alpha, \beta \in k \setminus \{0\}$. The following $k$-derivations of $k[x, y]$ have the rings of contants of the form $k[x^p, y^p, f]$, where $f \in k[x, y] \setminus k[x^p, y^p]$:

\[
\begin{align*}
\{ & d_1(x) = \alpha x^{rp}, \\
& d_1(y) = \beta y^{sp}, \\
& d_2(x) = \alpha x, \\
& d_2(y) = -\alpha y, \\
& d_3(x) = \alpha y^n, \\
& d_3(y) = \beta x^m, \\
& d_4(x) = \alpha x^{rp} y^n, \\
& d_4(y) = \beta,
\end{align*}
\]

$k[x, y]^d = k[x^p, y^p, \beta xy^{sp} - \alpha x^{rp} y],

k[x, y]^d_1 = k[x^p, y^p, \beta x^p y^p, \beta xy^{sp} - \alpha x^{rp} y],

k[x, y]^d_2 = k[x^p, y^p, xy],

k[x, y]^d_3 = k[x^p, y^p, x^m, \beta x^p y^p, (n + 1)\beta x^{m+1} - (m + 1)\alpha y^{n+1}],

k[x, y]^d_4 = k[x^p, y^p, (n + 1)\beta x - \alpha x^{rp} y^{n+1}],

k[x, y]^d_4 = k[x^p, y^p, (n + 1)\beta x - \alpha x^{rp} y^{n+1}].$
\[
\begin{aligned}
\{ d_5(x) = 0, \\
d_5(y) = \beta, \\
d_6(x) = \alpha, \\
d_6(y) = \beta x^m y^p, \\
d_7(x) = \alpha, \\
d_7(y) = 0, \\
\} \\
k[x, y]^{d_5} = k[x^p, y^p, x], \\
k[x, y]^{d_6} = k[x^p, y^p, \beta x^{m+1} y^p - (m + 1)\alpha y], \\
k[x, y]^{d_7} = k[x^p, y^p, y]. \\
\end{aligned}
\]

We will show in Theorem 16 that the above derivations are, up to multiplication by a monomial, all derivations of the form (*) such that \( k[x, y]^d = k[x^p, y^p, f] \), where \( f \in k[x, y] \setminus k[x^p, y^p] \). Note the following adaptation of Proposition 2.1.6 from [3]. The original proof remains valid in our situation.

**Lemma 14.** Let \( d \) be a \( k \)-derivation of \( k[x, y] \) of the form (*). Then there exists a vector \( \gamma \in k^2 \setminus \{(0,0)\} \) such that \( d \) is a \( \gamma \)-homogeneous derivation. \( \blacksquare \)

Recall that if \( d \) is a \( k \)-derivation of \( k[x, y] \), then the polynomial
\[
d^* = \frac{\partial(d(x))}{\partial x} + \frac{\partial(d(y))}{\partial y}
\]
is called the divergence of \( d \), and recall Lemma 5.1 from [1].

**Lemma 15.** Let \( d \) be a \( k \)-derivation of \( k[x, y] \) and let
\[
d(x) = \sum_{0 \leq j, l < p} a_{jl} x^j y^l, \quad d(y) = \sum_{0 \leq j, l < p} b_{jl} x^j y^l,
\]
where \( a_{jl}, b_{jl} \in k[x^p, y^p] \). Then \( d \) is a jacobian derivation if and only if
\[(**) \quad d^* = 0, \quad a_{0,p-1} = 0, \quad b_{p-1,0} = 0. \quad \blacksquare \]

Finally, we can prove the following theorem.

**Theorem 16.** Let \( k \) be a field of characteristic \( p > 0 \), and let \( d \) be a \( k \)-derivation of \( k[x, y] \) of the form (*). Then
\[
k[x, y]^d = k[x^p, y^p, f]
\]
for some \( f \in k[x, y] \setminus k[x^p, y^p] \) if and only if \( d = x^j y^l \cdot d_i \), where \( j, l \geq 0, i \in \{1, \ldots, 7\} \) and \( d_i \) is a derivation from Example 13 with \( m, n, r, s \geq 0, m, n \neq -1 \)(mod \( p \)), \( \alpha, \beta \in k \setminus \{0\} \).

**Proof.** We may assume that \( d \) is a nonzero derivation. If \( \alpha, \beta \neq 0 \), we put \( j = \min(t, v) \) and \( l = \min(u, w) \), if \( \alpha = 0 \), we put \( j = v, l = w \), and if \( \beta = 0 \), we put \( j = t, l = u \). Then \( d = x^j y^l \cdot d_0 \), where \( d_0 \) is a \( k \)-derivation of \( k[x, y] \) such that \( d_0(x) \) and \( d_0(y) \) are coprime. By Lemma 14 the derivation \( d_0 \) is \( \gamma \)-homogeneous for some \( \gamma \in k^2 \setminus \{(0,0)\} \), so, by Corollary 12, the ring of constants of \( d \) is of the form \( k[x^p, y^p, f] \), where \( f \in k[x, y] \setminus k[x^p, y^p] \), if and only if \( d_0 \) is a jacobian derivation. We verify the conditions (***) from
Lemma 15 for all possible forms of \( d_0 \):

\[
\begin{align*}
\begin{cases}
  d_0(x) = \alpha x^m, \\
  d_0(y) = \beta y^n,
\end{cases}
\end{align*}
\]

where \( m, n \geq 0, \alpha, \beta \neq 0 \). We have \( d_0^* = \max x^{m-1} + n \beta y^{n-1}, a_{0,p-1} = 0 \) and \( b_{p-1,0} = 0 \). The conditions \((**))\) hold in two cases:

\begin{itemize}
  \item \( m \equiv 0 \pmod{p} \) and \( n \equiv 0 \pmod{p} \), that is, \( d_0 = d_1 \),
  \item \( m = 1, n = 1 \) and \( \alpha + \beta = 0 \), that is, \( d = d_2 \).
\end{itemize}

(b) \[
\begin{align*}
\begin{cases}
  d_0(x) = \alpha y^n, \\
  d_0(y) = \beta x^m,
\end{cases}
\end{align*}
\]

where \( m, n \geq 0, \alpha, \beta \neq 0 \). In this case \( d_0^* = 0 \). The conditions \((**))\) are equivalent to \( m, n \not\equiv -1 \pmod{p} \), that is, \( d_0 = d_3 \).

(c) \[
\begin{align*}
\begin{cases}
  d_0(x) = \alpha x^m y^n, \\
  d_0(y) = \beta,
\end{cases}
\end{align*}
\]

where \( m, n \geq 0, \beta \neq 0 \). We have \( d_0^* = \max x^{m-1} y^n, b_{p-1,0} = 0 \). The conditions \((**))\) hold in two cases:

\begin{itemize}
  \item \( m \equiv 0 \pmod{p} \) and \( n \not\equiv -1 \pmod{p} \), when \( d_0 = d_4 \),
  \item \( \alpha = 0 \), when \( d_0 = d_5 \).
\end{itemize}

(d) \[
\begin{align*}
\begin{cases}
  d_0(x) = \alpha, \\
  d_0(y) = \beta x^m y^n,
\end{cases}
\end{align*}
\]

where \( m, n \geq 0, \alpha \neq 0 \). We have \( d_0^* = n \beta x^m y^{n-1}, a_{0,p-1} = 0 \). The conditions \((**))\) hold in two cases:

\begin{itemize}
  \item \( m \not\equiv -1 \pmod{p} \) and \( n \equiv 0 \pmod{p} \), when \( d_0 = d_6 \),
  \item \( \beta = 0 \), when \( d_0 = d_7 \).
\end{itemize}

Note that in each case a polynomial \( f \) such that \( k[x,y]^d = k[x^p, y^p, f] \) can be easily obtained from the condition \( d_0 = d_f \), that is, \( \partial f / \partial x = d_0(y) \) and \( \partial f / \partial y = -d_0(x) \). \( \blacksquare \)

**Corollary 17.** All monomial \( k \)-derivations of \( k[x,y] \) such that \( k[x,y]^d = k[x^p, y^p, f] \) for some \( f \in k[x,y] \setminus k[x^p, y^p] \) are of the form \( x^j y^l \cdot d_i \), where:

1. \( j, l \geq 0 \),
2. \( i \in \{1, \ldots, 7\} \), but \( i = 2 \) only in the case of \( p = 2 \),
3. \( d_1, \ldots, d_7 \) are derivations from Example 13 with \( m, n, r, s \geq 0, m, n \not\equiv -1 \pmod{p} \) and \( \alpha = \beta = 1 \). \( \blacksquare \)

**REFERENCES**


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