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RATIONAL FUNCTIONS WITHOUT POLES IN A COMPACT SET

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Abstract. Let X be an irreducible nonsingular complex algebraic set and let K be a compact subset of X. We study algebraic properties of the ring of rational functions on X without poles in K. We give simple necessary conditions for this ring to be a regular ring or a unique factorization domain.

1. Introduction and main results. Throughout this note X stands for an irreducible nonsingular algebraic set in $\mathbb{C}^N$, for some N. We write $\mathcal{O}$ for the sheaf of regular functions on X and regard $\mathcal{O}(X)$ and $\mathcal{O}_x$, for any point x in X, as subrings of the field $\mathcal{K}(X)$ of rational functions on X. Thus

$$\mathcal{O}(X) = \bigcap_{x\in X} \mathcal{O}_x \subseteq \mathcal{K}(X),$$

and $\mathcal{K}(X)$ is the field of fractions of $\mathcal{O}(X)$.

Unless explicitly stated otherwise, henceforth we consider X endowed with the topology induced by the usual metric topology on $\mathbb{C}$. Given a compact subset K of X, let $\mathcal{O}(K)$ denote the subring of $\mathcal{K}(X)$ consisting of all rational functions on X with no poles in K. In other words,

$$\mathcal{O}(K) = \bigcap_{x\in K} \mathcal{O}_x.$$

Hence $\mathcal{O}(X) \subseteq \mathcal{O}(K) \subseteq \mathcal{K}(X)$, and $\mathcal{K}(X)$ is the field of fractions of $\mathcal{O}(K)$. In particular, $\mathcal{O}(K)$ has no zero divisors.

In this note we investigate algebraic properties of the ring $\mathcal{O}(K)$, addressing the following questions: Is it a ring of fractions, with suitable denominators, of $\mathcal{O}(X)$? Is it Noetherian? Under what assumptions is it a unique factorization domain?

We give complete answers for a large class of compact subsets of X. However, we do not know, for example, if for every compact subset K of X,

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the answer is "yes" to the first or second question. In order to state our results we need some preparation.

Let us put

$$\widehat{K} = \{x \in X \mid |f(x)| \leq \sup_{y \in K} |f(y)| \text{ for every } f \text{ in } \mathcal{O}(X)\}.$$

The set $\widehat{K}$ is compact and $K \subseteq \widehat{K}$. We say that K is *algebraically convex* if $K = \widehat{K}$. Of course, $\widehat{K}$ is algebraically convex.

We write $H^*(-, \mathbb{Z})$ to denote the Čech cohomology with coefficients in $\mathbb{Z}$. Let $H^2_{\mathrm{alg}}(X, \mathbb{Z})$ be the subgroup of $H^2(X, \mathbb{Z})$ generated by the cohomology classes represented by irreducible algebraic hypersurfaces of X. Equivalently, $H^2_{\mathrm{alg}}(X, \mathbb{Z})$ consists precisely of the first Chern classes of algebraic line bundles on X. We write $D(K)$ for the image of $H^2_{\mathrm{alg}}(X, \mathbb{Z})$ under the restriction homomorphism $H^2(X, \mathbb{Z}) \to H^2(K, \mathbb{Z})$ (that is, the homomorphism induced by the inclusion map $K \hookrightarrow X$). The restriction homomorphism $H^2(\widehat{K}, \mathbb{Z}) \to H^2(K, \mathbb{Z})$ gives rise to a homomorphism from $D(\widehat{K})$ into $D(K)$ written

$$\alpha_K : D(\widehat{K}) \to D(K).$$

By construction, α_K is surjective. We set $\widehat{D}(K) = \operatorname{Ker} \alpha_K$.

Most of our results will concern compact subsets K of X with $\widehat{D}(K) = 0$, while the group $\widehat{D}(K) = 0$ if K is algebraically convex. Moreover, $H^2_{\mathrm{alg}}(X, \mathbb{Z}) = 0$ (which is satisfied if $H^2(X, \mathbb{Z}) = 0$) implies $\widehat{D}(K) = 0$ and $D(K) = 0$ for every K.

The set

$$S(K) = \{h \in \mathcal{O}(X) \mid h^{-1}(0) \subseteq X \setminus K\}$$

is a multiplicatively closed subset of $\mathcal{O}(X)$; furthermore, the ring of fractions $S(K)^{-1}\mathcal{O}(X)$, regarded as a subring of $\mathcal{K}(X)$, satisfies

$$S(K)^{-1}\mathcal{O}(X) \subseteq \mathcal{O}(K).$$

We do not know whether or not these two rings are always equal. Note that $S(K)^{-1}\mathcal{O}(X)$ shares with $\mathcal{O}(X)$ some nice algebraic properties. For example, since $\mathcal{O}(X)$ is a regular ring (a Noetherian ring whose localization with respect to each maximal ideal is a regular local ring), so is $S(K)^{-1}\mathcal{O}(X)$ [8, Corollary 2.6, p. 209]. Below we give a criterion for the equality $S(K)^{-1}\mathcal{O}(X) = \mathcal{O}(K)$.

THEOREM 1.1. *Let K be a compact subset of X with $\widehat{D}(K) = 0$. Then $S(K)^{-1}\mathcal{O}(X) = \mathcal{O}(K)$ and, in particular, $\mathcal{O}(K)$ is a regular ring.*

Given a ring A (commutative with identity), we denote by $\operatorname{Pic}(A)$ the *Picard group* of A (that is, the group of isomorphism classes of finitely generated projective A-modules of rank 1). Assuming A is a regular ring with no zero divisors, one has $\mathrm{Pic}(A) = 0$ if and only if A is a unique factorization domain [3] (more precisely, see the references to [3] given in [2, pp. 306, 307] and [1, Theorem 7.2.1, p. 147].

THEOREM 1.2. *Let K be a compact subset of X with $\widehat{D}(K) = 0$. Then the groups $\mathrm{Pic}(\mathcal{O}(K))$ and $D(K)$ are canonically isomorphic. The ring $\mathcal{O}(K)$ is a unique factorization domain if and only if $D(K) = 0$.*

Theorem 1.2 is essential in the proof of our next result.

THEOREM 1.3. *The following conditions are equivalent*:

(a) *For every compact subset K of X, the ring $\mathcal{O}(K)$ is a unique factorization domain.*
(b) *For every algebraically convex compact subset K of X, the ring $\mathcal{O}(K)$ is a unique factorization domain.*
(c) $H^2_{\mathrm{alg}}(X, \mathbb{Z}) = 0$.

Let us now specialize to $\dim X = 1$.

COROLLARY 1.4. *If $\dim X = 1$, then for every compact subset K of X, the ring $\mathcal{O}(K)$ is regular and a unique factorization domain.*

Proof. Since $\dim X = 1$, we have $H^2(X, \mathbb{Z}) = 0$. Hence for every compact subset K of X, the groups $\widehat{D}(K)$ and $D(K)$ are 0. The conclusion follows from Theorems 1.1 and 1.2. ∎

REMARK 1.5. In connection with Corollary 1.4 let us mention that, assuming $\dim X = 1$, the ring $\mathcal{O}(X)$ is a unique factorization domain if and only if X is of genus 0 (by the genus of X we mean the genus of its unique, up to isomorphism, nonsingular projectivization).

Remark 1.5 is certainly well known, but we give a short proof of the assertion contained in it at the end of Section 2. Proofs of Theorems 1.1, 1.2, and 1.3 are also given in Section 2. The reader may consult [2, Section 12.2, Propositions 12.4.14 and 12.6.2] and [11] for related results concerning functions on real algebraic varieties.

2. Proofs. The following fact will be useful.

PROPOSITION 2.1 (see [7, Corollary 5]). *Let E be an algebraic vector bundle on X. Let K be a compact subset of X and let s be a holomorphic section of E defined in some neighborhood of K in X. If K is algebraically convex, then s can be approximated on K by global algebraic sections of E (the approximation is with respect to the compact-open topology on the space of continuous sections of E over K).*

LEMMA 2.2. *Let E be an algebraic line bundle on X and let K be an algebraically convex compact subset of X. If the restriction $E|K$ is topolog-*

ically trivial on K, *then there exists an algebraic section* $v : X \to E$ *with* $v(x) \neq 0$ *for all* x *in* K.

Proof. One can find an open neighborhood U of K in X such that $E|U$ is topologically trivial. Since K is algebraically convex, it is also holomorphically convex in X, and hence there exists a Stein neighborhood of K in U [5, Proposition 3, p. 211]. Shrinking U if necessary, we may assume that U itself is a Stein neighborhood. It follows from Grauert's theorem [4] that $E|U$ is holomorphically trivial. Thus one can choose a holomorphic section $u : U \to E$ with $u(x) \neq 0$ for all x in U. The existence of v with the required properties is now a consequence of Proposition 2.1. ∎

Proof of Theorem 1.1. It suffices to show that $\mathcal{O}(K)$ is contained in $S(K)^{-1}\mathcal{O}(X)$. Let φ be a nonzero element of $\mathcal{O}(K)$ and let V be the set of poles of φ. One can find an algebraic line bundle L on X and an algebraic section $s : X \to L$ such that $s^{-1}(0) = V$ (just take the algebraic line bundle corresponding to the divisor of poles of φ and the section determined by some local equations of this divisor). We have

$$K \subseteq X \setminus V = X \setminus s^{-1}(0).$$

We claim that there is an algebraic section $v : X \to L^\vee$, of the dual bundle $L^\vee$ of L, with $v(x) \neq 0$ for all x in K.

Indeed, note that $L|X \setminus V$ is algebraically trivial, and hence so is $L^\vee|X \setminus V$. In particular, $L^\vee$ is topologically trivial on K and therefore $c_1(L^\vee|K) = 0$ in $H^2(K, \mathbb{Z})$, where $c_1(-)$ stands for the first Chern class. Since $c_1(L^\vee|\widehat{K})$ is in $D(\widehat{K})$, $\alpha_K(c_1(L^\vee|\widehat{K}) = c_1(L^\vee|K)$, and $\operatorname{Ker}\alpha_K = \widehat{D}(K) = 0$, we obtain $c_1(L^\vee|\widehat{K}) = 0$. The last equality implies that $L^\vee|\widehat{K}$ is topologically trivial. Since $\widehat{K}$ is algebraically convex and $K \subseteq \widehat{K}$, the claim follows from Lemma 2.2.

Define $h : X \to \mathbb{C}$ by

$$h(x) = v(x)(s(x)) \quad \text{for all } x \text{ in } X.$$

By construction, h is a regular function satisfying

$$K \subseteq X \setminus h^{-1}(0) \subseteq X \setminus V.$$

It follows that φ is a regular function on $X \setminus h^{-1}(0)$, and hence $\varphi = f/h^n$ in $\mathcal{K}(X)$ for some regular function $f : X \to \mathbb{C}$ and some nonnegative integer n [10, p. 50]. Thus φ is in $S(K)^{-1}\mathcal{O}(X)$ and the proof is complete. ∎

Given an algebraic line bundle L on X, we let $L(X)$ denote the $\mathcal{O}(X)$-module of global algebraic sections of L. It is well known that $L(X)$ is a finitely generated projective $\mathcal{O}(X)$-module of rank 1, and the correspondence $L \to L(X)$ gives rise to an isomorphism from the Picard group $\operatorname{Pic}(X)$ of X onto $\operatorname{Pic}(\mathcal{O}(X))$ (cf. [9]). In turn the correspondence $L \to S(K)^{-1}L(X)$

determines a homomorphism from $\operatorname{Pic}(X)$ into $\operatorname{Pic}(S(K)^{-1}\mathcal{O}(X))$, written

$$\beta_K : \operatorname{Pic}(X) \to \operatorname{Pic}(S(K)^{-1}\mathcal{O}(X)).$$

By [1, Proposition 7.17, p. 144, Theorem 7.2.1, p. 147], β_K is surjective.

Henceforth we will slightly abuse notation and make no distinction between an algebraic line bundle on X and its isomorphism class in $\operatorname{Pic}(X)$.

LEMMA 2.3. *For any compact subset K of X, there is a unique homomorphism*

$$\gamma_K : \operatorname{Pic}(S(K)^{-1}\mathcal{O}(X)) \to D(K)$$

such that $\gamma_K(\beta_K(L)) = c_1(L|K)$ for all L in $\operatorname{Pic}(X)$. The homomorphism γ_K is surjective and $\operatorname{Ker}\gamma_K$ consists precisely of all elements of the form $\beta_K(L)$, where L is in $\operatorname{Pic}(X)$ and $c_1(L|K) = 0$.

Proof. Since β_K is surjective, in order to prove the existence and uniqueness of γ_K it suffices to show that $c_1(L|K) = 0$ for every L in $\operatorname{Ker}\beta_K$. Suppose then that L is in $\operatorname{Ker}\beta_K$. This means that there is an isomorphism

$$\lambda : S(K)^{-1}\mathcal{O}(X) \to S(K)^{-1}L(X)$$

of $S(K)^{-1}\mathcal{O}(X)$-modules. Write $\lambda(1)$ and $\lambda(1) = s/h$ for some s in $L(X)$ and h in $S(K)$. We claim that

$$K \subseteq X \setminus s^{-1}(0).$$

Indeed, let x be a point in K. Choose a section u in $L(X)$ with $u(x) \neq 0$. We have $u/1 = \lambda(f/g) = fs/gh$ in $S(K)^{-1}L(X)$ for some f in $\mathcal{O}(X)$ and g in $S(K)$. Then there is ℓ in $S(K)$ for which $\ell ghu = \ell fs$ in $L(X)$. Since $(\ell ghu)(x) \neq 0$, we get $s(x) \neq 0$, that is, x belongs to $X \setminus s^{-1}(0)$ as claimed. The claim implies that $L|K$ is topologically trivial, and hence $c_1(L|K) = 0$ as required.

It follows from the definition of $D(K)$ that the homomorphism γ_K is surjective. The assertion about $\operatorname{Ker}\gamma_K$ is obvious. ∎

Proof of Theorem 1.2. We first show that $\widehat{D}(K) = 0$ implies that the canonical homomorphism

$$\gamma_K : \operatorname{Pic}(S(K)^{-1}\mathcal{O}(X)) \to D(K)$$

of Lemma 2.3 is an isomorphism. Since γ_K is always surjective, we only have to prove that γ_K is injective. This can be done as follows. Every element of $\operatorname{Ker}\gamma_K$ is of the form $\beta_K(L) = S(K)^{-1}L(X)$ for some L in $\operatorname{Pic}(X)$ with $c_1(L|K) = 0$. Recalling that the homomorphism $\alpha_K : D(\widehat{K}) \to D(K)$ satisfies

$$\alpha_K(c_1(L|\widehat{K})) = c_1(L|K) \quad \text{and} \quad \operatorname{Ker}\alpha_K = \widehat{D}(K) = 0,$$

we get $c_1(L|\widehat{K}) = 0$. The last equality implies that $L|\widehat{K}$ is topologically trivial. In view of Lemma 2.2 and the fact that $\widehat{K}$ is algebraically convex, there is a section s in $L(X)$ with $s(x) \neq 0$ for all x in K. Define

$$\mu : S(K)^{-1}\mathcal{O}(X) \to S(K)^{-1}L(X)$$

by $\mu(f/g) = fs/g$. Clearly, μ is a homomorphism of $S(K)^{-1}\mathcal{O}(X)$-modules. Our goal now is to prove that μ is an isomorphism, thereby demonstrating $\beta_K(L) = 0$ in $\mathrm{Pic}(S(K)^{-1}\mathcal{O}(X))$, as required.

Suppose $\mu(f/g) = fs/g = 0$ in $S(K)^{-1}L(X)$. Then $pfs = 0$ in $L(X)$ for some p in $S(K)$. Hence $f(x) = 0$ for all x in $X \setminus (ps)^{-1}(0)$, which in turn implies $f = 0$ in $\mathcal{O}(X)$ (recall that X is irreducible). It follows that $f/g = 0$ in $S(K)^{-1}\mathcal{O}(X)$, and therefore μ is injective.

In order to show that μ is surjective, let us take any element u/h in $S(K)^{-1}L(X)$, where u is in $L(X)$ and h is in $S(K)$. There is a unique rational function φ in $\mathcal{K}(X)$ such that $\varphi s = u/h$ as rational sections of L. Since $s(x) \neq 0$ for all x in K, it follows that φ is in $\mathcal{O}(K)$. By Theorem 1.1, φ belongs to $S(K)^{-1}\mathcal{O}(X)$. We have $\mu(\varphi) = u/h$, and hence μ is surjective.

We have just finished the proof that γ_K is an isomorphism. Since, by Theorem 1.1,

$$S(K)^{-1}\mathcal{O}(X) = \mathcal{O}(K),$$

the groups $\mathrm{Pic}(\mathcal{O}(K))$ and $D(K)$ are canonically isomorphic. The ring $\mathcal{O}(K)$ is regular (Theorem 1.1 again), and hence it is a unique factorization domain if and only if $\mathrm{Pic}(\mathcal{O}(K)) = 0$. The proof is complete, the last equality being equivalent to $D(K) = 0$. ∎

Proof of Theorem 1.3. If (c) holds, then $\widehat{D}(K) = 0$ and $D(K) = 0$ for every compact subset K of X, and hence (a) is satisfied in view of Theorem 1.2. It is obvious that (a) implies (b). It remains then to show that (b) implies (c).

Suppose (b) holds. Choose a compact subset C of X such that the inclusion map $C \hookrightarrow X$ is a homotopy equivalence (cf. for example [2, Corollary 9.3.7]). The restriction homomorphism $H^2(X,\mathbb{Z}) \to H^2(C,\mathbb{Z})$ is an isomorphism. It follows that for $K = \widehat{C}$, the restriction homomorphism $H^2(X,\mathbb{Z}) \to H^2(K,\mathbb{Z})$ is injective (note $C \subseteq K$). This implies that $H^2_{\mathrm{alg}}(X,\mathbb{Z})$ is isomorphic to a subgroup of $D(K)$. Since K is algebraically convex, we have $\widehat{D}(K) = 0$ and hence, by Theorem 1.2 and condition (b), $D(K) = 0$. Therefore $H^2_{\mathrm{alg}}(X,\mathbb{Z}) = 0$ and (c) is satisfied. The proof is complete. ∎

Justification of Remark 1.5. Let $\overline{X}$ be a nonsingular projectivization of X. The restriction homomorphism $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(X)$ is surjective, and since $\overline{X} \setminus X$ is a finite set, its kernel is a finitely generated Abelian group (cf. [6, Proposition 6.5, p. 133, Corollary 6.16, p. 145]). Write $g(X)$ for the genus of X. If $g(X) \geq 1$, then $\mathrm{Pic}(\overline{X})$ is not finitely generated, and hence $\mathrm{Pic}(X) \neq 0$. If $g(X) = 0$, then $\mathrm{Pic}(X) = 0$. Since $\mathrm{Pic}(X)$ is isomorphic to $\mathrm{Pic}(\mathcal{O}(X))$, we have $\mathrm{Pic}(\mathcal{O}(X)) = 0$ if and only if $g(X) = 0$. The argument is complete since $\mathrm{Pic}(\mathcal{O}(X)) = 0$ is equivalent to $\mathcal{O}(X)$ being a unique factorization domain.

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