COLLOQUIUM MATHEMATICUM

VOL. 106 2006 NO. 1

OUTER AUTOMORPHISMS OF ENDOMORPHISM ALGEBRAS

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Abstract. Commutative rings over which no endomorphism algebra has an outer automorphism are studied.

Let R always denote a nontrivial commutative ring with identity. If R is a field, then it is well known that the endomorphism algebra of any vector space over R has only inner automorphisms. Is it reasonable to expect that this, with minor variations, is the only such occurrence of this property? More precisely, what are the commutative rings R such that for every R-module M, the endomorphism algebra $\mathrm{End}_R M$ has no outer automorphisms?

Let $\mathcal{INN}$ denote the class of all such R. Thus $R \in \mathcal{INN}$ if and only if for every R-module M, $\mathrm{End}_R M$ has only inner automorphisms. In this paper, we shall see that the noetherian rings R which belong to $\mathcal{INN}$ are precisely the principal ideal rings of dimension 0, and that no ring of dimension > 0 belongs to $\mathcal{INN}$. Moreover, whenever we establish that a ring R does not belong to $\mathcal{INN}$, then we shall see that there exist arbitrarily large modules M for which $\mathrm{End}_R M$ has outer automorphisms. However, we cannot give a complete description of $\mathcal{INN}$; it remains a question whether or not there is any nonnoetherian ring R belonging to $\mathcal{INN}$. For results when only certain modules are considered, see the discussions for complete discrete valuation rings in [5, 6].

It will be useful to consider two subclasses of $\mathcal{INN}$. Let $\mathcal{ISO}$ consist of all R such that every isomorphism of the endomorphism algebras of two R-modules must be induced by an isomorphism of the modules. Clearly $\mathcal{ISO}$ is a subclass of $\mathcal{INN}$. The relationship of the second class to $\mathcal{INN}$ will not be so apparent. Let $\mathcal{SUM}$ be the class of all R such that for every R-module M, there exists $x \in M$ such that the cyclic submodule generated by x is a direct summand of M, and such that the annihilators $\mathrm{Ann}_R(x)$ and $\mathrm{Ann}_R M$ are equal. To see that $\mathcal{SUM}$ is a subclass of $\mathcal{ISO}$, one can

2000 *Mathematics Subject Classification*: Primary 16S50; Secondary 13C99.

Key words and phrases: endomorphism algebra, outer automorphism, module.

easily adapt part of Kaplansky's arguments in [4, Theorem 28] applying to bounded modules. We sketch this in a lemma.

Lemma 1. *$\mathcal{SUM}$ is contained in $\mathcal{ISO}$.*

Proof. Let $R \in \mathcal{SUM}$, let M and N be two R-modules, and let $\Phi : \operatorname{End}_R M \to \operatorname{End}_R N$ be an algebra isomorphism. Let I denote the common annihilator $I = \operatorname{Ann}_R M = \operatorname{Ann}_R N$. Choose $m_0 \in M$ and an idempotent $\varepsilon \in \operatorname{End}_R M$ such that $\operatorname{Ann}_R(m_0) = I$ and $\varepsilon(M) = \langle m_0 \rangle$. Put $\varepsilon^* = \Phi(\varepsilon)$. Then $\varepsilon^*(N)$ is a direct summand of N with annihilator I. By our assumption on R, we see that $\varepsilon^*(N)$ has a cyclic summand $\langle n_0 \rangle$ with annihilator I. But if $\langle n_0 \rangle$ had a nonzero complement in $\varepsilon^*(N)$, then there would be a nonzero map of $\langle n_0 \rangle$ into that complement, hence $\operatorname{End}_R\ \varepsilon^*(N)$ would be noncommutative. This is contrary to $\operatorname{End}_R\ \varepsilon^*(N) \cong \operatorname{End}_R\ \varepsilon(M) = R/I$, thus $\varepsilon^*(N) = \langle n_0 \rangle$.

We now define $\phi : M \to N$ in the usual fashion. Given $m \in M$, we may choose $\alpha_m \in \operatorname{End}_R M$ with $\alpha_m(m_0) = m$. If we put $\phi(m) = \Phi(\alpha_m)(n_0)$, then ϕ is well defined, and is an R-homomorphism since we may take $r\alpha_{m_1} + \alpha_{m_2}$ for $\alpha_{rm_1+m_2}$. Moreover, ϕ is an isomorphism since Φ^{-1}, n_0 and m_0 can be used to construct an inverse.

Finally, if $\alpha \in \operatorname{End}_R M$, then $\Phi(\alpha)\phi(m) = \Phi(\alpha)\Phi(\alpha_m)(n_0) = \Phi(\alpha\alpha_m)(n_0) = \phi(\alpha(m))$ since $\alpha\alpha_m(m_0) = \alpha(m)$. Thus, $\Phi(\alpha) = \phi\alpha\phi^{-1}$. ∎

To give an example of a ring in $\mathcal{SUM}$, we mimic the proof that a pure cyclic p-subgroup of an abelian group is a direct summand.

Lemma 2. *Let R be a local ring with a maximal ideal which is principal and nilpotent. Then $R \in \mathcal{SUM}$, thus $R \in \mathcal{ISO}$.*

Proof. Let M be an R-module. Passing to $R/\operatorname{Ann}_R M$, we may assume that M is faithful. Let $P = \langle p \rangle$ be the maximal ideal of R. Since the result is clear for fields, we may assume that $P \neq 0$. Thus, there exists $k \geq 2$ such that the distinct ideals of R are $R \supset \langle p \rangle \supset \cdots \supset \langle p^k \rangle = 0$. Since M is faithful, we may choose $m_0 \in M$ such that $\operatorname{Ann}_R(m_0) = 0$. It suffices to show that $\langle m_0 \rangle$ is a direct summand of M. Choose a submodule N maximal with respect to $\langle m_0 \rangle \cap N = 0$. To show that $M = \langle m_0 \rangle \oplus N$, for the sake of contradiction we may assume there exists $m \in M \setminus (\langle m_0 \rangle \oplus N)$. We may further assume that $pm \in \langle m_0 \rangle \oplus N$, say $pm = rm_0 + n$ ($r \in R$, $n \in N$). Then $0 = p^k m = p^{k-1} r m_0 + p^{k-1} n$, thus $p^{k-1} r m_0 = 0$ and we conclude that $r = pr_1$ for some $r_1 \in R$. Hence $p(m - r_1 m_0) = n$. But $m - r_1 m_0 \notin N$, thus $\langle m_0 \rangle \cap (N + \langle m - r_1 m_0 \rangle) \neq 0$ by maximality, say $r_2 m_0 = n_0 + r_3(m - r_1 m_0) \neq 0$. We cannot have $p \mid r_3$ since $\langle m_0 \rangle \cap N = 0$. Thus r_3 is a unit, giving the contradiction $m \in \langle m_0 \rangle \oplus N$. ∎

We take note of some relevant behavior under the formation of quotient ring. Let I be an ideal of R. Regarding (R/I)-modules as R-modules via the natural map identifies the category of (R/I)-modules with a full subcategory of the category of R-modules. Thus we see

Lemma 3. *In the setting just mentioned, if an (R/I)-module shows that R/I does not belong to $\mathcal{INN}$ or to $\mathcal{ISO}$, then it has the same property for R.*

By using standard results on split realization of R-algebras, we prove that any R in $\mathcal{INN}$ must have dimension 0.

Proposition 1. *If R has dimension greater than 0, then there exist arbitrarily large R-modules M such that $\operatorname{End}_R M$ has outer automorphisms.*

Proof. Passing to R modulo a nonmaximal prime ideal, by Lemma 3 we may assume that R is an integral domain of dimension > 0. We may choose a nonzero nonunit $p \in R$ and put $I = \bigcap_{i<\omega} p^i R$. It suffices to find arbitrarily large (R/I)-modules. It is easy to verify that the image of p in R/I is a nonunit which is not a zero divisor, thus we may assume that $I = 0$ and that p is a nonunit which is not a zero divisor. Put $S = \{p^i \mid i < \omega\}$. Then R is S-separable and S-torsion-free.

We may apply results of [2] or [1] to obtain an arbitrarily large R-module M with an S-torsion submodule T such that T is a direct sum of cyclic modules of form $R/\langle p^k\rangle$ for unbounded k, M is S-separable, M/T is S-divisible, and $\operatorname{End}_R M$ has the form which we now describe. The restriction of endomorphisms from M to T gives an algebra homomorphism $\operatorname{End}_R M \to \operatorname{End}_R T$ which is injective since M/T is S-divisible and M is S-separable. Regarding $\operatorname{End}_R M$ as embedded in $\operatorname{End}_R T$, if $\operatorname{Bd}(T)$ denotes the ideal in $\operatorname{End}_R T$ of S-bounded endomorphisms, then $\operatorname{End}_R M = R \oplus \operatorname{Bd}(T)$.

We claim that there exists an automorphism γ of T such that for every $k < \omega$, there exists a cyclic summand of T which is not invariant under $p^k\gamma$. Since T is unbounded, we may choose a summand of T of form $\bigoplus_{i<\omega}\langle m_i\rangle$, where $\langle m_i\rangle \cong R/\langle p^{k_i}\rangle$ $(i < \omega)$, and $\{k_i\}$ is an unbounded increasing sequence of positive integers. Then there exists an automorphism γ of T such that $\gamma(m_0) = m_0$ and $\gamma(m_i) = m_i + m_{i-1}$ for $i \geq 1$. If $k < k_{i-1}$, then $p^k\gamma$ does not map $\langle m_i\rangle$ into itself.

Now define $\Phi : \operatorname{End}_R M \to \operatorname{End}_R T$ by $\Phi(\alpha) = \gamma\alpha\gamma^{-1}$ for every $\alpha \in \operatorname{End}_R M$. Since $\operatorname{Bd}(T)$ is an ideal of $\operatorname{End}_R T$, we see that Φ is an algebra automorphism of $\operatorname{End}_R M$. To show that Φ is outer, we shall assume that Φ is conjugation by a unit $\theta \in \operatorname{End}_R M$ and derive a contradiction. The automorphism $\theta^{-1}\gamma$ must centralize $\operatorname{End}_R M$. Since $\langle m_i\rangle$ is a direct summand, $\langle m_i\rangle$ is invariant under $\theta^{-1}\gamma$ for every i. But $\theta^{-1} = u + \beta$ for some unit $u \in R$ and $\beta \in \operatorname{Bd}(T)$. Let k be such that $p^k\beta = 0$. Then $p^k\theta^{-1}\gamma = p^k u\gamma$ maps every $\langle m_i\rangle$ into itself, contrary to the choice of γ. ■

Corollary. *If R belongs to $\mathcal{INN}$, then R has dimension 0.*

Now we show that certain local rings do not belong to $\mathcal{INN}$.

Proposition 2. *Let R be a local ring with maximal ideal P. Assume that the dimension of P/P^2 over R/P is ≥ 2. Then there exist arbitrarily large R-modules M such that* $\mathrm{End}_R M$ *has outer automorphisms.*

Proof. We may choose an ideal I with $P \supseteq I \supseteq P^2$ such that the dimension of P/I over R/P is 2. Thus, by Lemma 3 we may assume that $P^2 = 0$ and that P has dimension 2 over R/P. Choosing a basis $\{x, y\}$ for P, we find that x and y are annihilated by P and $P = Rx \oplus Ry$.

To begin, we shall construct a small R-module M. Let C be the cyclic submodule of $R \oplus R$ generated by (x, y) and put $M = (R \oplus R)/C$. Next, we construct an algebra of 2×2 matrices over R. Let $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ denote a generic 2×2 matrix over R. Define A to be the set of all σ such that $a - d, b, c \in P$. Then A is an R-algebra. Let J consist of all σ such that $a, b, c, d \in P$. Then J is an ideal of A such that $A = R + J$. Moreover, A is commutative since $J^2 = 0$. Regarding $R \oplus R$ as a left module over the R-algebra A, we see that C is an A-submodule since $JC = 0$, hence we obtain an R-algebra homomorphism $\Psi : A \to \mathrm{End}_R M$.

We claim that Ψ is onto. Any element of $\mathrm{End}_R M$ is induced by a matrix σ acting on $R \oplus R$ and mapping C into itself. Consequently, there exists $r \in R$ such that $ax + by = rx$ and $cx + dy = ry$. Thus $a - r, b, c, d - r \in P$, and hence $\sigma \in A$. Since Ψ is onto, $\mathrm{End}_R M$ is commutative, so any nontrivial algebra automorphism that we construct will be outer, as desired. Let K denote the kernel of Ψ. Since $A = R + J$ and $J^2 = 0$, any R-module automorphism $\Phi : A \to A$ which is the identity on R and satisfies $\Phi(J) \subseteq J$ will be an R-algebra automorphism of A. If, in addition, $\Phi(K) = K$, then Φ will induce an R-algebra automorphism of $\mathrm{End}_R M$.

Before defining Φ, we compute K. Clearly, $\sigma \in K$ if and only if $\sigma(R \oplus R) \subseteq C$, thus $\sigma = \left(\begin{smallmatrix} ux & vx \\ uy & vy \end{smallmatrix}\right)$, $(u, v \in R)$. To define Φ on A, the entries b and c of σ may be written as $b = vx + sy$, $c = tx + uy$. Now put

$$\Phi\begin{pmatrix} a & vx + sy \\ tx + uy & d \end{pmatrix} = \begin{pmatrix} a & vx + ty \\ sx + uy & d \end{pmatrix}.$$

Then Φ is an R-module automorphism of A which is the identity on R and K, and maps J into itself. The algebra automorphism induced on $\mathrm{End}_R M$ is not the identity since if σ is given by $a = c = d = 0$, $b = y$, then $\Phi(\sigma) - \sigma \notin K$.

To obtain arbitrarily large modules, let λ be a cardinal number and put $N = \bigoplus_{i<\lambda} M$. Since M is finitely generated, $\mathrm{End}_R N$ can be identified with the column-finite matrices (α_{ij}) $(i, j < \lambda$, $\alpha_{ij} \in \mathrm{End}_R M)$. If ϕ denotes the outer automorphism of $\mathrm{End}_R M$ that we have just obtained, then $(\alpha_{ij}) \mapsto (\phi(\alpha_{ij}))$ is an algebra automorphism of $\mathrm{End}_R N$. Suppose that it is inner, say $(\beta_{ij}) \in \mathrm{End}_R N$ is a unit such that $(\beta_{ij})(\alpha_{ij}) = (\phi(\alpha_{ij}))(\beta_{ij})$ for all (α_{ij}). Taking (α_{ij}) with all zero entries except for a single off-diagonal entry of 1, we see that (β_{ij}) must commute with all such (α_{ij}), hence must be diagonal. Therefore β_{00} is a unit, giving the contradiction $\beta_{00}\alpha_{00} = \phi(\alpha_{00})\beta_{00}$. ∎

The case for noetherian R can now be settled.

THEOREM. *Let R be noetherian. Then membership in the classes $\mathcal{SUM}$, $\mathcal{ISO}$ and $\mathcal{INN}$ are all equivalent. This occurs if and only if $R \cong R_1 \times \cdots \times R_m$ for some m, where each R_i is a local ring with a principal nilpotent maximal ideal; equivalently, R is a principal ideal ring of dimension 0. If $R \notin \mathcal{INN}$, then there exist arbitrarily large R-modules M such that $\operatorname{End}_R M$ has outer automorphisms.*

Proof. Assume R is noetherian. If $\dim R > 0$, then Proposition 1 implies that $R \notin \mathcal{INN}$, and there exist arbitrarily large M such that $\operatorname{End}_R M$ has outer automorphisms. Now assume $\dim R = 0$. Then $R \cong R_1 \times \cdots \times R_m$, where each R_i is indecomposable and of dimension 0, hence each R_i is local with finitely generated nilpotent maximal ideal. If R is not a principal ideal ring, then some R_i is not, hence by Proposition 2 and Lemma 3, $R \notin \mathcal{INN}$ and there exist arbitrarily large M such that $\operatorname{End}_R M$ has outer automorphisms. If R is a principal ideal ring, then so is each R_i, and Lemma 2 implies that each $R_i \in \mathcal{SUM}$, thus clearly $R \in \mathcal{SUM}$, therefore $R \in \mathcal{INN}$. ∎

As mentioned at the beginning, we do not know an example of a nonnoetherian ring R belonging to $\mathcal{INN}$; if such exists, it must of course have dimension 0. We can say something about $\mathcal{SUM}$ and $\mathcal{ISO}$. The case of the class $\mathcal{SUM}$ is easily taken care of.

PROPOSITION 3. *If $R \in \mathcal{SUM}$, then R is noetherian and thus described by the Theorem.*

Proof. If R is not noetherian, then there exists a strictly descending chain $I_0 \supset I_1 \supset \cdots$ of ideals. Put $M = \bigoplus_{i<\omega} R/I_i$. Then $\operatorname{Ann}_R M = \bigcap_{i<\omega} I_i$, while for every $x \in M$, $\operatorname{Ann}_R(x) \supseteq I_k$ for some k, thus $\operatorname{Ann}_R M \neq \operatorname{Ann}_R(x)$. ∎

Motivated by the result for the noetherian case, a reasonable nonnoetherian ring to consider might be a local ring of dimension 0 in which every finitely generated ideal is principal, in particular, a dimension 0 valuation ring. In the final proposition, we consider this for the class $\mathcal{ISO}$, and also the case of a direct product of infinitely many fields.

PROPOSITION 4.

(1) *Let $R = \prod_{i\in I} F_i$ be a direct product of an infinite family of fields F_i. Then $R \notin \mathcal{ISO}$.*
(2) *Let R be a nonnoetherian valuation ring. Then $R \notin \mathcal{ISO}$.*

Proof. (1) Let N be the ideal $\bigoplus_{i\in I} F_i$ in R. The natural map $R \to \operatorname{End}_R N$ is injective. It is surjective since $\operatorname{End}_R N = \prod_{i\in I} \operatorname{End}_{F_i} F_i$, hence $\operatorname{End}_R R \cong \operatorname{End}_R N$. Clearly R and N are not isomorphic since N is not cyclic.

(2) Let R be a nonnoetherian valuation ring with maximal ideal P. By Proposition 2, we may assume that the dimension of P/P^2 over R/P is < 2. If the dimension is 1, then modulo an appropriate principal ideal, and taking Lemma 3 into account, we may assume that $P = P^2$, and that $P \neq 0$ since R is nonnoetherian. Moreover, we claim we may assume that P is a faithful ideal. Let $I = \operatorname{Ann}_R P$ and put $\overline{R} = R/I$ and $\overline{P} = P/I$. If $x \in R$ is such that $\bar{x}\overline{P} = 0$, then $xP \subseteq I$, hence $xP = xP^2 = 0$. Thus $\bar{x} = 0$, and we see that $\overline{P}$ is faithful for $\overline{R}$. By Lemma 3, we may thus assume that $P = P^2$ is faithful. By the Corollary to Proposition 1, we may assume that R has dimension 0.

One could fashion a proof utilizing Proposition I.4.11 of [3], but for our special case a direct and self-contained argument may be desirable. The principal ideals of R are linearly ordered, therefore so is $\{\operatorname{Ann}_R(x) \mid x \in P\}$, which defines a topology on R. To see this topology is Hausdorff, note that if $x \neq 0$, then $xP \neq 0$, hence there exists $y \in P$ such that $xy \neq 0$. Thus $x \notin \operatorname{Ann}_R(y)$. Now let $\widehat{R}$ be the completion of R in this topology, regarded as the inverse limit $\widehat{R} = \varprojlim(R/\operatorname{Ann}_R(x))$ taken over all $x \in P$. Then R embeds naturally as a subring of $\widehat{R}$.

We claim that $\operatorname{End}_R P \cong \widehat{R}$. First we show that if $\alpha \in \operatorname{End}_R P$, then $\alpha(x) \in \langle x\rangle$ for every $x \in P$. If not, then for some $x \neq 0$ and some $p \in P$ we have $x = p\alpha(x)$ since R is a valuation ring. Thus, $x = p^k\alpha^k(x)$ for all $k \geq 1$. But P is a nil ideal, hence $p^k = 0$ for some k, giving the contradiction $x = 0$. Consequently, each $\alpha \in \operatorname{End}_R P$ maps each cyclic submodule $\langle x\rangle$ into itself, hence is equivalent to a coherent family of endomorphisms on each $\langle x\rangle$. Since $\operatorname{End}_R\langle x\rangle$ is naturally isomorphic to $R/\operatorname{Ann}_R(x)$, we obtain $\widehat{R}$ as $\operatorname{End}_R P$.

Next we show that every R-endomorphism of $\widehat{R}$ is an $\widehat{R}$-endomorphism. Since R is dense in $\widehat{R}$, it will suffice to show that every R-endomorphism of $\widehat{R}$ is continuous. For this, it is enough to show that the topology on $\widehat{R}$ is given by $\{\operatorname{Ann}_{\widehat{R}}(x) \mid x \in P\}$ since $\operatorname{Ann}_{\widehat{R}}(x)$ is invariant under R-endomorphisms. Regarding $\widehat{R}$ as a submodule of $\prod_{y\in P} R/\operatorname{Ann}_R(y)$, an element $\hat{r} \in \widehat{R}$ has form $\hat{r} = (r_y + \operatorname{Ann}_R(y))_{y\in P}$ such that for all $\langle y\rangle \subseteq \langle z\rangle \subseteq P$, we have $r_y - r_z \in \operatorname{Ann}_R(y)$. Note that if $x, y \in P$, then regardless of containment of the principal ideals, we have $(r_x - r_y)xy = 0$, thus $r_x xy = r_y xy$. Fix $x \in P$ and let $\hat{r} \in \widehat{R}$. Then $\hat{r}x = 0 \Leftrightarrow r_y x \in \operatorname{Ann}_R(y)$ for all $y \in P \Leftrightarrow r_y xy = 0 \Leftrightarrow r_x xy = 0 \Leftrightarrow r_x xP = 0 \Leftrightarrow r_x x = 0$ since P is faithful $\Leftrightarrow r_x \in \operatorname{Ann}_R(x)$. This last condition is that $\hat{r}$ belongs to the kernel of the projection of $\widehat{R}$ to $R/\operatorname{Ann}_R(x)$, thus the topologies agree.

We conclude that $\operatorname{End}_R \widehat{R} = \operatorname{End}_{\widehat{R}} \widehat{R} = \widehat{R} \cong \operatorname{End}_R P$. But $\widehat{R}$ is not isomorphic to P since $\widehat{R} \neq P\widehat{R}$, while $P = P^2$. Therefore, R is not in $\mathcal{ISO}$. ∎

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Received 7 November 2005 (4692)