COLLOQUIUM MATHEMATICUM

VOL. 136 2014 NO. 1

QCH KÄHLER MANIFOLDS WITH $\kappa = 0$

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Abstract. The aim of this paper is to describe all Kähler manifolds with quasi-constant holomorphic sectional curvature with $\kappa = 0$.

1. Introduction. The aim of the present paper is to describe all connected Kähler manifolds (M, g, J) admitting a global, 2-dimensional, J-invariant distribution $\mathcal{D}$ having the following property: The holomorphic curvature $K(\pi) = R(X, JX, JX, X)$ of any J-invariant 2-plane $\pi \subset T_xM$, where $X \in \pi$ and $g(X, X) = 1$, depends only on the point x and the number $|X_{\mathcal{D}}| = \sqrt{g(X_{\mathcal{D}}, X_{\mathcal{D}})}$, where $X_{\mathcal{D}}$ is the orthogonal projection of X on $\mathcal{D}$. In this case we have

$$R(X, JX, JX, X) = \phi(x, |X_{\mathcal{D}}|)$$

where $\phi(x, t) = a(x) + b(x)t^2 + c(x)t^4$ and a, b, c are smooth functions on M. Also $R = a\Pi + b\Phi + c\Psi$ for certain curvature tensors $\Pi, \Phi, \Psi \in \bigotimes^4 \mathfrak{X}^*(M)$ of Kähler type. The investigation of such manifolds, called *QCH Kähler manifolds*, was started by G. Ganchev and V. Mihova [G-M-1], [G-M-2]. In [J-1] we partially classify QCH manifolds with non-vanishing invariant κ of the distribution $\mathcal{D}$ (see also [G-M-2]). In that case the distribution $\mathcal{D}$ is integrable and the foliation induced by $\mathcal{D}$ turns out to be a holomorphic, homothetic foliation by curves.

In the present paper we shall investigate QCH Kähler manifolds satisfying the condition $\kappa = 0$. It turns out that from the results in [G-M-1] it easily follows that in that case a, b are constant, and if $b \neq 0$ and M is complete and simply connected with $\dim M \geq 6$ then $M = M_a \times \Sigma$ where M_a is a Kähler manifold of constant holomorphic curvature $a \neq 0$ and Σ is a Riemannian surface.

If $b = 0$ then the situation is much more complicated and we shall mainly investigate the case where the distribution $\mathcal{D}$ is integrable or M is complete. If $\mathcal{D}$ is integrable then M is locally a manifold with constant holomorphic sectional curvature or a product $M_0 \times \Sigma$ where M_0 has holomorphic sectional

2010 *Mathematics Subject Classification*: Primary 53C55; Secondary 53C25.
Key words and phrases: Kähler manifold, holomorphically pseudosymmetric Kähler manifold, QCH Kähler manifold.

DOI: 10.4064/cm136-1-3

curvature 0 and Σ is a Riemannian surface. If $a = b = 0$, $c \neq 0$ on M and (M, g, J) is complete and simply connected then M is a product $\mathbb{C}^{n-1} \times \Sigma$ where Σ has non-vanishing scalar curvature. If $a > 0$, $b = 0$ and (M, g) is complete then (M, g, J) has constant holomorphic curvature. If $a = 0$, $b = 0$ and M is complete then M is the union of a manifold of constant holomorphic sectional curvature 0 and a manifold which is a local product of a $(2n-2)$-manifold of constant holomorphic sectional curvature 0 and a Riemannian surface Σ.

2. The invariant κ. Let (M, g, J) be a $2n$-dimensional Kähler manifold with a 2-dimensional J-invariant distribution $\mathcal{D}$. Let $\mathfrak{X}(M)$ denote the algebra of all differentiable vector fields on M, and $\Gamma(\mathcal{D})$ the set of local sections of the distribution $\mathcal{D}$. If $X \in \mathfrak{X}(M)$ then we shall denote by $X^\flat$ the 1-form $\phi \in \mathfrak{X}^*(M)$ dual to X with respect to g, i.e. $\phi(Y) = X^\flat(Y) = g(X, Y)$. By Ω we shall denote the *Kähler form* of (M, g, J), i.e. $\Omega(X, Y) = g(JX, Y)$. Let $\mathcal{E}$ denote the distribution $\mathcal{D}^\perp$, which is a $2(n-1)$-dimensional, J-invariant distribution.

By h, m we shall denote the tensors $h = g \circ (p_{\mathcal{D}} \times p_{\mathcal{D}}), m = g \circ (p_{\mathcal{E}} \times p_{\mathcal{E}})$, where $p_{\mathcal{D}}, p_{\mathcal{E}}$ are the orthogonal projections on $\mathcal{D}, \mathcal{E}$ respectively. It follows that $g = h + m$. By ω we shall denote the Kähler form of $\mathcal{D}$, i.e. $\omega(X, Y) = h(JX, Y)$, and by Ω_m the Kähler form of $\mathcal{E}$, i.e. $\Omega_m(X, Y) = m(JX, Y)$.

For any local section $X \in \Gamma(\mathcal{D})$ we define $\operatorname{div}_{\mathcal{E}} X = \operatorname{tr}_m \nabla X^\flat = m^{ij} \nabla_{e_i} X^\flat(e_j)$ where $\{e_1, \ldots, e_{2(n-1)}\}$ is any basis of $\mathcal{E}$ and $[m^{ij}]$ is a matrix inverse to $[m_{ij}]$, where $m_{ij} = m(e_i, e_j)$. Note that if $f \in C^\infty(M)$ then $\operatorname{div}_{\mathcal{E}}(fX) = f \operatorname{div}_{\mathcal{E}} X$ for $X \in \Gamma(\mathcal{D})$.

Let $\xi \in \Gamma(\mathcal{D})$ be a unit local section of $\mathcal{D}$. Then $\{\xi, J\xi\}$ is an orthonormal basis of $\mathcal{D}$. Let $\eta(X) = g(\xi, X)$ and $J\eta = -\eta \circ J$, which means that $J\eta(X) = g(J\xi, X)$. Let us denote by κ the function

$$\kappa = \sqrt{(\operatorname{div}_{\mathcal{E}} \xi)^2 + (\operatorname{div}_{\mathcal{E}} J\xi)^2}.$$

Then κ does not depend on the choice of the section ξ. Note that $\kappa = 0$ if and only if $\operatorname{div}_{\mathcal{E}} \xi = 0$ for all $\xi \in \Gamma(\mathcal{D})$.

3. Curvature tensor of a QCH Kähler manifold. We shall recall some results from [G-M-1]. Let $R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$ and write

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

If R is the curvature tensor of a QCH Kähler manifold (M, g, J), then there exist functions $a, b, c \in C^\infty(M)$ such that

$$R = a\Pi + b\Phi + c\Psi, \tag{1}$$

where Π is the standard Kähler tensor of constant holomorphic curvature, i.e.

$$\begin{aligned}\Pi(X,Y,Z,U) = \tfrac{1}{4}\big(&g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\\ &+ g(JY,Z)g(JX,U) - g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)\big),\end{aligned}$$

the tensor Φ is defined by

$$\begin{aligned}\Phi(X,Y,Z,U) = \tfrac{1}{8}\big(&g(Y,Z)h(X,U) - g(X,Z)h(Y,U)\\ &+ g(X,U)h(Y,Z) - g(Y,U)h(X,Z) + g(JY,Z)\omega(X,U)\\ &- g(JX,Z)\omega(Y,U) + g(JX,U)\omega(Y,Z) - g(JY,U)\omega(X,Z)\\ &- 2g(JX,Y)\omega(Z,U) - 2g(JZ,U)\omega(X,Y)\big),\end{aligned}$$

and finally

$$\Psi(X,Y,Z,U) = -\omega(X,Y)\omega(Z,U) = -(\omega\otimes\omega)(X,Y,Z,U).$$

If (M,g,J) is a QCH Kähler manifold then one can show that the Ricci tensor r of (M,g,J) satisfies the equation

$$r(X,Y) = \lambda m(X,Y) + \mu h(X,Y)$$

where $\lambda = \frac{n+1}{2}a + \frac{b}{4}, \mu = \frac{n+1}{2}a + \frac{n+3}{4}b + c$ are eigenvalues of the Ricci tensor (see [G-M-1, Corollary 2.1 and Remark 2.1]). In particular the distributions $\mathcal{E},\mathcal{D}$ are eigendistributions of the Ricci tensor corresponding to the eigenvalues λ,μ.

Now let us assume that (M,g,J) is a QCH Kähler manifold of dimension $2n \geq 6$ and let ξ be a local unit section of $\mathcal{D}$ and $\eta(Z) = g(\xi,Z)$. Let us define two 1-forms ϵ,ϵ^* by

$$\begin{aligned}\epsilon(Z) &= g(p_{\mathcal{E}}(\nabla_\xi\xi),Z) = g(\nabla_\xi\xi,Z) - pJ\eta(Z),\\ \epsilon^*(Z) &= g(p_{\mathcal{E}}(\nabla_{J\xi}J\xi),Z) = g(\nabla_{J\xi}J\xi,Z) - p^*\eta(Z),\end{aligned}$$

where $p = g(\nabla_\xi\xi,J\xi)$,$p^* = g(\nabla_{J\xi}J\xi,\xi)$ and $p_{\mathcal{E}}$ denotes the orthogonal projection on $\mathcal{E}$. Note that the distribution $\mathcal{D}$ is integrable if and only if $\epsilon+\epsilon^* = 0$ (see [G-M-1, Lemma 3.3]). In fact for $Z \in \Gamma(\mathcal{E})$ we have

$$\begin{aligned}g([\xi,J\xi],Z) &= g(\nabla_\xi J\xi - \nabla_{J\xi}\xi,Z) = g(J\nabla_\xi J\xi - J\nabla_{J\xi}\xi,JZ)\\ &= -g(\nabla_\xi\xi + \nabla_{J\xi}J\xi,JZ) = -(\epsilon(JZ)+\epsilon^*(JZ)).\end{aligned}$$

Let $\{Z_\lambda\}$ be any complex basis of the complex subbundle $\mathcal{E}^{1,0}$ of the complex tangent bundle $T^cM = \mathbb{C}\otimes TM$. We also write $Z_{\bar\lambda} = \bar Z_\lambda$. Then the Bianchi identity for the tensor R of the form (1) gives in the case $\kappa = 0$ and

$\dim M \geq 6$ the following relations (see [G-M-1, Theorem 3.5]):

$$\nabla a = 0, \quad \nabla b = 0, \tag{2}$$

$$b\nabla_{Z_\lambda}\eta(Z_\mu) = 0, \quad c\nabla_{Z_\lambda}\eta(Z_\mu) = 0, \tag{3}$$

$$b\nabla_{Z_\lambda}\eta(Z_{\bar\mu}) = 0, \quad c\nabla_{Z_\lambda}\eta(Z_{\bar\mu}) = 0, \tag{4}$$

$$b\epsilon(Z_\lambda) = 0, \quad b\epsilon^*(Z_\lambda) = 0, \tag{5}$$

$$c(\epsilon(Z_\lambda) + \epsilon^*(Z_\lambda)) = dc(Z_\lambda). \tag{6}$$

Hence we will obtain

THEOREM 1. *Let (M, g, J) be a QCH Kähler manifold with $\kappa = 0$ and $\dim M \geq 6$. If $b \neq 0$ and M is complete and simply connected then $M = M_a \times \Sigma$ where M_a is a Kähler manifold of constant holomorphic sectional curvature $a \neq 0$ and Σ is a Riemannian surface. Conversely, every such product is a QCH manifold with $\kappa = 0$ with respect to the distribution $T\Sigma \subset TM$.*

Proof. From (2) we obtain $a = \text{const}$, $b = \text{const}_1$. If $b \neq 0$ then $\nabla_{Z_\lambda}\eta(Z_\mu) = 0$, $\nabla_{Z_\lambda}\eta(Z_{\bar\mu}) = 0$, $\epsilon = 0$, $\epsilon^* = 0$. It follows that the distribution $\mathcal{D}$ is integrable and totally geodesic. In fact if $\epsilon = 0$, $\epsilon^* = 0$ then $\nabla_\xi \xi = pJ\xi$, $\nabla_{J\xi}J\xi = p^*\xi$ and consequently $\nabla_\xi J\xi = -p\xi$, $\nabla_{J\xi}\xi = -p^*J\xi$. Thus if $X, Y \in \Gamma(\mathcal{D})$ then $h(X, Y) = p_{\mathcal{E}}(\nabla_X Y) = 0$ and $\mathcal{D}$ is totally geodesic.

On the other hand since $\nabla_{Z_\lambda}\eta(Z_{\bar\mu}) = 0$ and $\nabla_{Z_\lambda}\eta(Z_\mu) = 0$ we obtain $g(\nabla_X \xi, Y) = 0$ for $X, Y \in \Gamma(\mathcal{E})$. Analogously $g(\nabla_X J\xi, Y) = 0$ for $X, Y \in \Gamma(\mathcal{E})$. Hence $g(\xi, \nabla_X Y) = g(J\xi, \nabla_X Y) = 0$ and the foliation $\mathcal{E}$ is totally geodesic with leaves of constant holomorphic curvature a. Now the result follows from the de Rham theorem (see [K-N]). Note that for $M = M_a \times \Sigma$ we have $R = a\Pi - 2a\Phi + (\frac{1}{2}\tau + a)\Psi$ where τ is the scalar curvature of Σ (see [J-2]). ■

Now we consider the case $\dim M \geq 6$ and $b = 0$. Let $M_0 = \{x \in M : c(x) = 0\}$. Then M_0 is closed in M and let $U = M - M_0 = \{x \in M : c(x) \neq 0\}$. Note that $\mu - \lambda = c$ where λ, μ are eigenvalues of the Ricci tensor of (M, g, J) corresponding respectively to eigensubbundles $\mathcal{E}, \mathcal{D}$.

In what follows we assume that the distribution $\mathcal{D}$ is integrable. From (6) this means that $\nabla c \in \Gamma(\mathcal{D})$. We will show that in U the distribution $\mathcal{D}$ induces a holomorphic, homothetic foliation by curves (see [Ch-N]), and moreover $L_V g = 0$ on $T\mathcal{D}^\perp = T\mathcal{E}$ and $L_V J(TM) \subset \mathcal{D}$ for every $V \in \Gamma(\mathcal{D})$.

From (3)–(4) it follows that $g(\nabla_X \xi, Y) = 0$ for every $X, Y \in \Gamma(\mathcal{E})$ and therefore $L_\xi g = 0$ on $T\mathcal{E}$. Analogously $L_{J\xi} g = 0$ on $T\mathcal{E}$ and consequently $L_V g = 0$ on $T\mathcal{E}$ for $V \in \Gamma(\mathcal{D})$. Note that $L_V J = J \circ \nabla V - \nabla V \circ J = [J, T]$ where $TX = \nabla_X V$. It is enough to show that $L_\xi JTM \subset \mathcal{D}$. From (2.5) it is clear that $\nabla_X \xi \in \mathcal{D}$ for $X \in \mathcal{E}$ and $L_\xi J(\mathcal{E}) \subset \mathcal{D}$. We also have $L_\xi J\xi = J\nabla_\xi \xi - \nabla_{J\xi}\xi = [\xi, J\xi] \in \mathcal{D}$ if $\mathcal{D}$ is integrable. Hence the result follows.

Let us define an almost complex structure J' on M by $J'X = JX$ if $X \in \mathcal{E}$ and $J'X = -JX$ if $X \in \mathcal{D}$. It is clear that J commutes with J'.

PROPOSITION 1. *The manifold (U, g, J') is an almost Kähler manifold belonging to the class $\mathcal{AK}_2$, i.e.*

$$R(X,Y,Z,W) - R(J'X,J'Y,Z,W) = R(J'X,Y,J'Z,W) + R(J'X,Y,Z,J'W) \tag{7}$$

for all $X, Y, Z, W \in TU$. The distribution $\mathcal{E}$ is included in the Kähler nullity of J'. If $\nabla J' \neq 0$ then (U, g, J') is a normal almost Kähler manifold.

Proof. The Ricci form ρ of (U, g, J) equals $\rho = \lambda\Omega_m + \mu\omega$ where $\lambda = \frac{n+1}{2}a$ is constant and $\mu = \lambda + c$. From $d\rho = 0$ we get (note that $d\omega + d\Omega_m = 0$)

$$d\omega = -d\ln|\lambda - \mu| \wedge \omega = -d\ln|c| \wedge \omega.$$

Hence if $\mathcal{D}$ is integrable we obtain $d\omega = d\Omega_m = 0$ and (U, g, J') is an almost Kähler manifold. Since J commutes with J' it is easy to prove (7).

Now note that $\nabla_X J' = 0$ for every $X \in \mathcal{E}$. Since $J = J'$ on $\mathcal{E}$ and the foliation induced by $\mathcal{E}$ is totally geodesic we get $\nabla_X J'Y = 0$ for every $Y \in \mathcal{E}$. If $\xi \in \mathcal{D}$ then

$$(\nabla_X J')\xi = \nabla_X(J'\xi) - J'(\nabla_X \xi) = -\nabla_X(J\xi) + J(\nabla_X \xi) = 0$$

since $\nabla_X \xi \in \mathcal{D}$ if $X \in \mathcal{E}$.

Now we show that (U, g, J') is a normal almost Kähler manifold, i.e. $\nabla J' \circ J'(\mathcal{D}, \mathcal{D}) \subset \mathcal{E}$ (see [N]) assuming that $\nabla J' \neq 0$. It is enough to show that $\nabla_\xi J'\eta \in \mathcal{E}$ if $\xi, \eta \in \mathcal{D}$. Indeed,

$$\begin{aligned}\nabla_\xi J'\eta &= -\nabla_\xi(J\eta) - J'(\nabla_\xi \eta_{|\mathcal{D}} + \nabla_\xi \eta_{|\mathcal{E}}) \\ &= -J(\nabla_\xi \eta) + J(\nabla_\xi \eta_{|\mathcal{D}}) - J(\nabla_\xi \eta_{|\mathcal{E}}) = -2J(\nabla_\xi \eta_{|\mathcal{E}}) \in \mathcal{E}. \ \blacksquare\end{aligned}$$

THEOREM 2. *Let (M, g, J) be a QCH manifold with $\dim M \geq 6$, $\kappa = 0$, $b = 0$ and integrable distribution $\mathcal{D}$. If $a = 0$ then M is the union of a manifold of constant holomorphic sectional curvature 0 and a manifold which is a local product of a $(2n-2)$-manifold of constant holomorphic sectional curvature 0 and a Riemannian surface Σ. If $a \neq 0$ then (M, g, J) has constant holomorphic sectional curvature a.*

Proof. Write $U = U_1 \cup U_2$ where $U_1 = \{x \in U : \nabla J'(x) = 0\}$ and $U_2 = \{x \in U : \nabla J' \neq 0\}$. If $\operatorname{int} U_1 \neq \emptyset$ then in $\operatorname{int} U_1$, g is locally a product metric and consequently $a = -b/2 = 0$. Hence if $a \neq 0$ then $\operatorname{int} U_1 = \emptyset$. The manifold (U_2, g, J') is a normal almost Kähler non-Kähler manifold whose opposite almost Kähler structure is Kähler. Such manifolds are described in [N]. In particular $\mathcal{D}$ is spanned locally by two holomorphic Killing vector fields ξ, η which commute, $[\xi, \eta] = 0$. It follows (see [Bes] for the proof in the compact case, but the result also holds locally) that $\rho(\xi, \eta) = \mu\omega(\xi, \eta) =$

$-\frac{1}{2}\delta(J[\xi,\eta]) = 0$ and consequently $\mu = 0$, which means that $c = -\frac{n+1}{2}a$ and $c \neq 0$ is constant. In particular $a \neq 0$.

The metric of the manifold (U_2, g, J') can locally be described as follows (see [N]). Let (M_a, g_a, J_a) be a space of constant holomorphic curvature $a \neq 0$ and $f : M_a \to \mathbb{C}$ be a holomorphic function $f = u + iv$ such that $u > 0$. Then $M = M_a \times \mathbb{R}^2$ with the metric

$$g = g_a + u dx^{\otimes 2} + \frac{1}{u}(dy + v dx)^{\otimes 2}$$

where x, y are standard coordinates on $\mathbb{R}^2$, and the complex structure J which coincides with the complex structure J_a of M_a on M_a and which is given by $J(dx) = \frac{1}{u}(dy + v dx)$ on $\mathbb{R}^2$. The Kähler form is $\Omega = \omega_a + dx \wedge dy$ where ω_a is the Kähler form of (M_a, g_a, J_a). From Proposition 7.2 in [N] it follows that a normal $\mathcal{AK}_2$ manifold is locally a product of a Kähler manifold and a strictly normal $\mathcal{AK}_2$ manifold. It follows that (M, g, J') is strictly normal. On the other hand it follows from [N] that (M, g, J') cannot be strictly normal, since $\dim M_a \geq 4$ and no holomorphic function $f : M_a \to \mathbb{C}$ can be immersive. ∎

Corollary. *Let (M, g, J) be a QCH manifold with* $\dim M \geq 6$, $\kappa = 0$ *and constant scalar curvature. Then two cases are possible:*

(a) (M, g, J) *has constant holomorphic sectional curvature,*
(b) (M, g, J) *is locally a product $M_a \times \Sigma$ where M_a is a Kähler manifold of constant holomorphic sectional curvature a and Σ is a Riemannian surface of constant scalar curvature.*

Proof. The scalar curvature is constant if and only if c is constant. From (6) it follows that if $c \neq 0$ is constant then $\mathcal{D}$ is integrable. ∎

Remark. If we take Σ with non-zero scalar curvature τ on $\Sigma - V$ and which vanishes on some closed set $V \subset \Sigma$ with non-empty interior then $\mathbb{C}^{n-1} \times \Sigma$ is a QCH manifold with $\kappa = 0, b = 0, a = 0$ for which we have $M_0 = \mathbb{C}^{n-1} \times V$ and $U_1 = \mathbb{C}^{n-1} \times (\Sigma - V)$ where on $\mathbb{C}^{n-1}$ there is the standard Euclidean metric. The curvature tensor is $R = \frac{1}{2}\tau\Psi$. Note that QCH manifolds with $\kappa = 0$ and constant scalar curvature are locally symmetric (if $a \geq 0$ and M is compact this also follows from [O]).

Next we consider general QCH manifolds with $a = b = 0$.

Theorem 3. *A QCH-manifold with $a = b = 0$ and $c \neq 0$ on M is is a semi-symmetric manifold foliated with $(2n-2)$-dimensional Euclidean manifolds* (see [Sz]). *Conversely, every Kähler manifold foliated with $(2n-2)$-dimensional Euclidean spaces is a QCH manifold with $a = b = 0$ and $c \neq 0$ for which $\mathcal{D} = V^1$, where V^1 is a subbundle of TM irreducible with respect to the primitive holonomy group.*

Proof. A QCH-manifold with $a = b = 0$ satisfies $R.R = \left(a + \frac{b}{2}\right)\Pi.R = 0$ (see [J-2]). If $c \neq 0$ on M then (M, g, J) is a manifold foliated with $(2n-2)$-dimensional Euclidean manifolds (see [Sz]). The primitive holonomy group of (M, g, J) at a point $x \in M$ is $K_x = SO(2) = \{\cos t p_{\mathcal{D}} + \sin t J p_{\mathcal{D}}\}$ and $V^0 = \mathcal{E}$, $V^1 = \mathcal{D}$ since $\mathcal{E} = \{X : R(U, V)X = 0 \text{ for all } U, V \in TM\}$.

Conversely, every Kähler manifold foliated with $(2n-2)$-dimensional Euclidean spaces is a QCH manifold with $a = b = 0$ and $c \neq 0$ for which $\mathcal{D} = V^1$. In fact since $J \circ R(U, V) = R(U, V) \circ J$ it follows that $JV^0 = V^0$ and $JV^1 = V^1$. Set $\mathcal{D} = V^1$ and let $X \in TM$ with $\|X\| = 1, X = X_{\mathcal{E}} + X_{\mathcal{D}}$. Then

$$R(X, JX, JX, X) = R(X_{\mathcal{D}}, JX_{\mathcal{D}}, JX_{\mathcal{D}}, X_{\mathcal{D}}) = \|X_{\mathcal{D}}\|^4 c$$

where c is the sectional curvature of $\mathcal{D}$. It follows that $R = c\Psi$ with respect to $\mathcal{D} = V^1$. ∎

THEOREM 4. *Let (M, g, J) be a complete, simply connected Kähler semi-symmetric manifold foliated with $(2n-2)$-dimensional Euclidean spaces. Then (M, g) is the product of $(2n-2)$-dimensional Euclidean space and a Riemannian surface with non-vanishing scalar curvature.*

Proof. From [Sz] it follows that the space $S = \operatorname{span}\{p_{\mathcal{E}}(\nabla_X Y) : X, Y \in \mathcal{D}\}$ is at most 1-dimensional. We have to show that the hyperbolic and parabolic parts M_h and M_p of M are empty. We shall show that in the Kähler case $JS = S$. In fact, if $p_{\mathcal{E}}(\nabla_X Y) = \xi \in S$ then $p_{\mathcal{E}}(\nabla_X JY) = Jp_{\mathcal{E}}(\nabla_X Y) = J\xi \in S$ and consequently $JS = S$. It follows that S cannot be 1-dimensional and $S = 0$. Thus $\mathcal{D}$ is totally geodesic. Since $V^0 = \mathcal{E}$ is also totally geodesic, it follows from the de Rham theorem that (M, g) is the product of $(2n-2)$-dimensional Euclidean space and a Riemannian surface with non-vanishing scalar curvature. ∎

THEOREM 5. *Let (M, g, J) be a QCH Kähler manifold with $\kappa = 0$, and $\dim M \geq 6$ and $U = \{x : c(x) \neq 0\}$. Let $x_0 \in U$ and $m_{x_0} \in T_{x_0}M \cap \mathcal{E}_{x_0}$ be a unit vector. Then there exists a neighbourhood $V \subset U$ of x_0 and a unit field $m \in \Gamma(\mathcal{E}_{|V})$ such that $\nabla_m m = 0$ and $m(x_0) = m_{x_0}$.*

Proof. Let $\Sigma \subset U$ be a hypersurface perpendicular to m_{x_0} and let m be a unit vector field on Σ with values in $\mathcal{E}$. Let $E \subset TM_{|\Sigma}$ be the 1-dimensional vector bundle spanned by m. Let us consider the map $\phi : E \to M$ defined by $\phi(X) = \exp_{p(X)} X$. There exists a neighbourhood W of 0_{x_0} in E such that $\phi_{|W}$ is a diffeomorphism and $\phi(W) \subset U$. Define $m(\phi(X)) = d_{p(X)}^{\phi(X)} m_{p(X)}$ where d is parallel translation along the curve $d(t) = \exp_{p(X)} tX$. Note that $\nabla_m m = 0$ and $m \in \Gamma(\mathcal{E})$ since $\mathcal{E}_{|U}$ is totally geodesic. ∎

Define on U the operator $B_m(X) = p_{\mathcal{D}}(\nabla_X m)$. Note that $B_m(Y) = 0$ if $Y \in \mathcal{E}$. Let m be as in Theorem 4. Then $(\nabla_m B_m)(Y) = 0$ if $Y \in \mathcal{E}$ and to find B_m it is sufficient to consider $B_{m|\mathcal{D}}$. We get, for $X \in \Gamma(\mathcal{D})$,

$$\begin{aligned}\nabla_m B_m(X) + B_m(\nabla_m X) &= \nabla_m p_{\mathcal{D}}(\nabla_X m) = p_{\mathcal{D}}(\nabla_m \nabla_X m)\\ &= p_{\mathcal{D}}(R(m,X)m + \nabla_X \nabla_m m + \nabla_{[X,m]} m)\\ &\quad - \tfrac{1}{4} aX + p_{\mathcal{D}}(\nabla_{\nabla_m X} m) - p_{\mathcal{D}}(\nabla_{\nabla_X m} m)\\ &\quad - \tfrac{1}{4} aX - B_m^2(X) + B_m(\nabla_m X).\end{aligned}$$

Let X, Y be a parallel basis of unit vectors along $d(t) = \exp_x tm$. Such a basis exists in U. If $m(t) = \dot{d}$ then $B_m(t)$ can be considered as a 2×2 matrix $B(t)$ by the choice of parallel basis. This matrix satisfies the ordinary differential equation

$$B' + B^2 = -\tfrac{1}{4} aI. \tag{8}$$

Now let ξ be a section of the bundle $\mathcal{E}$. Then $R(X,\xi)Y = a\Pi(X,\xi)Y$. Let $X, Y, Z \in TM$. Then for $B(X) = p_{\mathcal{D}}(\nabla_X \xi)$ we get

$$\begin{aligned}&\nabla_Z R(X,\xi)Y + R(X, B(Z))Y + R(X, p_{\mathcal{E}}(\nabla_Z \xi))\\ &\qquad = a\Pi(X, B(Z))Y + a\Pi(X, p_{\mathcal{E}}(\nabla_Z \xi))Y.\end{aligned}$$

Consequently,

$$\nabla_Z R(X,\xi)Y = -c\Psi(X, B(Z))Y.$$

Hence

$$\begin{aligned}\nabla_\xi R(X,Y)Z &= -\nabla_Y R(\xi, X)Z - \nabla_X R(Y,\xi)Z\\ &= c\Psi(B(Y), X)Z + c\Psi(Y, B(X))Z\\ &= -c\Psi(B(X), Y)Z - c\Psi(X, B(Y))Z\\ &= -c \operatorname{tr} B\Psi(X,Y)Z.\end{aligned}$$

Now let $x_0 \in U$ and $m_{x_0} \in \mathcal{E}_{x_0}$ be a unit vector. Let $d(t) = \exp_{x_0}(tm_{x_0})$ and $m(t) = \dot{d}(t)$. Then

$$\frac{dc}{dt} = -c \operatorname{tr} B_{m(t)} \tag{9}$$

and consequently

$$c(d(t)) = c(x_0) \exp\Big(-\int_0^t \operatorname{tr} B_{m(s)}\, ds\Big). \tag{10}$$

It follows that if (M, g, J) is complete and d is a geodesic such that $d(0) \in U$, $\dot{d}(0) \in \mathcal{E}_{x_0}$ then $\operatorname{im} d \subset U$. In particular $\mathcal{E}_{|U}$ is integrable and its leaves Σ are complete and contained in U if M is complete.

Let S be the distribution defined by

$$S = \operatorname{span}\{p_{\mathcal{E}} \nabla_X X, p_{\mathcal{E}} \nabla_X Y, p_{\mathcal{E}} \nabla_Y X, p_{\mathcal{E}} \nabla_Y Y\}$$

where $X, Y = JX$ is an orthonormal local basis of $\mathcal{D}$. The dimension of S cannot be constant. Note that $\dim S \leq 4$. It is easy to see that $p_{\mathcal{E}}(\nabla c) \in S$, $Jp_{\mathcal{E}}(\nabla c) \in S$ and $JS = S$. Note also that $p_{\mathcal{E}}([X, JX]) = Jp_{\mathcal{E}}(\nabla \ln c)$.

THEOREM 6. *Let (M, g, J) be a QCH Kähler manifold with $\kappa = 0$ and $\dim M \geq 6$. If $b = 0$ and $a > 0$ then M cannot be complete unless it is of constant holomorphic curvature. If M is complete and $a = 0$, $b = 0$ then M is the union of a manifold of constant holomorphic sectional curvature 0 and a manifold which is a local product of a $(2n-2)$-manifold of constant holomorphic sectional curvature 0 and a Riemannian surface Σ.*

Proof. Let $x \in U = \{x : c(x) \neq 0\}$. Let $m_0 \in S_x^{\perp} \cap \mathcal{E}_x$. Let m be constructed as in Theorem 5, $m(x) = m_0$ and let $d(t) = \exp_x(tm_0)$. Let $m(t) = \dot{d}(t)$ and $B(t)$ be the 2×2 matrix corresponding to B_m by choosing a parallel orthonormal basis of $\mathcal{D}$ along d. Then $B(0) = 0$ and $B' + B^2 = -\frac{1}{4}aI$. Consequently, $B(t) = -\gamma \tan(\gamma t)I$ if $a = 4\gamma^2 > 0$, $B(t) = 0$ if $a = 0$, and $B(t) = \gamma \tanh(\gamma t)I$ if $a = -4\gamma^2 < 0$ where $\gamma > 0$.

Hence if $\dim M \geq 8$ and $U \neq \emptyset$ then M cannot be complete if $a > 0$ since the solution $B(t) = -\gamma \tan(\gamma t)I$ of equation (8) with initial condition $B(0) = 0$ is defined only on $(-\frac{1}{2\gamma}\pi, \frac{1}{2\gamma}\pi)$.

If $\dim M = 6$ and $a = 4\gamma^2 > 0$ note that the solution of (8) with initial condition $B(0) = B_0$ is

$$B(t) = \gamma(\cos(\gamma t)B_0 - \gamma \sin(\gamma t)I)(\sin(\gamma t)B_0 + \gamma \cos(\gamma t)I)^{-1}.$$

If B_0 has a real eigenvalue different from 0 then $B(t)$ is not defined on the whole of $\mathbb{R}$. On the other hand if $\dim S_x \geq 2$ then as in [Sz] one can prove that there exists $m_0 \in S_x^{\perp} \cap \mathcal{E}_x$ such that $B_{m_0} = B_0$ has a nonzero real eigenvalue. It follows that $S = 0$ and consequently $\mathcal{D}$ is totally geodesic and in particular integrable. Thus (M, g, J) is a complete manifold of constant holomorphic curvature. If $a = 0$ we prove as in Theorem 3 that $S = 0$ in $U = \{x : c(x) \neq 0\}$ and consequently $\mathcal{D}|U$ is totally geodesic, and the theorem follows. ∎

We finish by giving a second proof of Theorem 2. Let $U = \{x : c(x) \neq 0\}$ and let $x \in U$. First we assume that $a \neq 0$. Since $p_{\mathcal{E}}(\nabla c) = 0$ we have $\dim S_x \leq 2$. Let $m_0 \in S_x^{\perp} \cap \mathcal{E}_x$ with $\|m_0\| = 1$. Then $B_{m_0} = 0$ and we deduce from (10) that along the geodesic $d(t) = \exp_x(tm_0)$ we get $c(d(t)) = c(x)/\cos(\gamma t)^2$ if $a = 4\gamma^2$ and $c(d(t)) = c(x)/\cosh(\gamma t)^2$ if $a = -4\gamma^2$ where $\gamma > 0$. On the other hand, since $p_{\mathcal{E}}(\nabla c) = 0$ the function c is constant on the leaves of the foliation $\mathcal{E}$. The geodesic d is contained in a leaf of the foliation $\mathcal{E}$ and c is not constant on d, which gives a contradiction. Hence $U = \emptyset$ and (M, g, J) has constant holomorphic curvature. If $a = 0$, $S \neq 0$ then since $B_{Jm_0} = J \circ B_{m_0}$ using the results of [Sz] one can easily prove that there exists $m_0 \in S$ such that $\operatorname{tr} B_{m(t)} \neq 0$ for t close to 0. Consequently, from (9) we again infer that c is not constant on d, a contradiction. Hence $S = 0$ and $\mathcal{D}$ is totally geodesic in U, which finishes the proof.

Acknowledgements. The author thanks Prof. Derdziński for his remarks which improved the style of the paper. The paper was supported by Narodowe Centrum Nauki grant DEC-2011/01/B/ST1/02643.

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Received 18 October 2013;
revised 23 March 2014 (6047)