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# A CHARACTERIZATION OF SEQUENCES WITH THE MINIMUM NUMBER OF $k$-SUMS MODULO $k$

BY

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**Abstract.** Let $G$ be an additive abelian group of order $k$, and $S$ be a sequence over $G$ of length $k+r$, where $1 \leq r \leq k-1$. We call the sum of $k$ terms of $S$ a $k$-sum. We show that if 0 is not a $k$-sum, then the number of $k$-sums is at least $r+2$ except for $S$ containing only two distinct elements, in which case the number of $k$-sums equals $r+1$. This result improves the Bollobás–Leader theorem, which states that there are at least $r+1$ $k$-sums if 0 is not a $k$-sum.

**1. Introduction.** Let $k \geq 2$ and $r$ be integers with $1 \leq r \leq k-1$, and let $G$ be an additive abelian group of order $k$. For any given sequence $S$ of elements of $G$ of length $k+r$, we call the sum of $k$ terms of the sequence $S$ a *$k$-sum*. Then the renowned Erdős–Ginzburg–Ziv theorem [3] can be stated as follows: If $G$ is a cyclic group of order $k$ and $r = k-1$, then some $k$-sum is 0. The study of $k$-sums has received a lot of attention from several authors: see, for example, [1, 4, 5, 7, 8, 9]. For detailed background information about $k$-sums, we refer the readers to [2] and [6].

For convenience, we use the following notation and terminology, which are consistent with [5] and [7]. Let $\mathcal{F}(G)$ denote the free abelian monoid with basis $G$; its elements are called *sequences* over $G$. An element $S \in \mathcal{F}(G)$ will be written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{i=1}^{l} g_i = \prod_{g \in G} g^{v_g(S)} \in \mathcal{F}(G),$$

where $v_g(S) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is called the *multiplicity* of $g$ in $S$. We say that $S$ contains some $g \in G$ if $v_g(S) \geq 1$. A sequence $T \in \mathcal{F}(G)$ is called a *subsequence* of $S$ if $v_g(T) \leq v_g(S)$ for every $g \in G$, denoted by $T \mid S$. Whenever $T \mid S$, the element $R = ST^{-1} \in \mathcal{F}(G)$ denotes the sequence with $T$ deleted from $S$. Clearly, $RT = S$.

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We define

$$\begin{aligned}
|S| = l = \sum_{g\in G} v_g(S) \in \mathbb{N}_0, &\qquad \text{the } \textit{length} \text{ of } S,\\
\sigma(S) = \sum_{i=1}^{k} g_i = \sum_{g\in G} v_g(S)g \in G, &\qquad \text{the } \textit{sum} \text{ of } S,\\
\mathrm{supp}(S) = \{g\in G : v_g(S) > 0\}, &\qquad \text{the } \textit{support} \text{ of } S,\\
\textstyle\sum_k(S) = \Big\{\sum_{i\in I} g_i : I \subset [1,l] \text{ with } |I| = k\Big\},&\\
&\qquad \text{the } \textit{set of } k\textit{-sums} \text{ of } S, \text{ for all } k\in\mathbb{N}.
\end{aligned}$$

For sets $A$ and $B$ in an abelian group $G$, we write $A + B$ for the set $\{a+b : a \in A, b \in B\}$. Similarly, for $b \in G$, we write $b-A$ for $\{b-a : a \in A\}$. Moreover, we denote by $|A|$ the cardinality of $A$.

In 1999, Bollobás and Leader [2] posed the interesting problem of estimating the number of $k$-sums, and obtained the following result.

THEOREM A. *Let $k \geq 2$ and $r$ be integers with $1 \leq r \leq k-1$, and let $G$ be an additive abelian group of order $k$. Let $S \in \mathcal{F}(G)$ with $|S| = k + r$. If $0 \notin \sum_k(S)$, then $|\sum_k(S)| \geq r+1$.*

In the same paper, Bollobás and Leader [2] also raised a conjecture related to the problem of minimizing the number of sums from a sequence of given length in $G$ and the problem of minimizing the number of $k$-sums, which was solved by Gao and Leader [6]. In 2003, Yu [11] gave a simple proof of Theorem A.

In this paper, we mainly focus on the estimate for the number of $k$-sums. Using the natural bijection between $\sum_k(S)$ and $\sum_r(S)$, it is enough to estimate $|\sum_r(S)|$. By counting the number of $r$-sums, we get our main result.

THEOREM 1.1. *Let $G$ be an additive abelian group of order $k$, and let $1 \leq r \leq k-1$. Let $S \in \mathcal{F}(G)$ with $|S| = k + r$. If $0 \notin \sum_k(S)$, then $|\sum_k(S)| \geq r+2$ unless $|\mathrm{supp}(S)| = 2$, in which case $|\sum_k(S)| = r+1$.*

Actually, Theorem 1.1 gives us a characterization of sequences $S$ that do not have 0 as a $k$-sum such that $|\sum_k(S)| = r+1$ in Theorem A. In Section 2, we will give the proof of Theorem 1.1, and an application.

**2. Proof of Theorem 1.1, and an application.** We first give the proof of Theorem 1.1, and then give some corollaries and examples. For the proof we need the following result due to Scherk [10].

LEMMA 2.1. *Let $A$ and $B$ be subsets of an abelian group $G$ of order $k$. If $A \cap (-B) = \{0\}$, then $|A+B| \geq \min\{k, |A|+|B|-1\}$.*

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Since $0 \notin \sum_k(S)$ and $|S| = k+r \geq k+1$, we have $|\text{supp}(S)| \geq 2$. From $|G| = k$, one can easily deduce that the $k$-sums do not change when the sequence $S$ is translated. So we may assume that $l = v_0(S) = \max\{v_g(S) : g \in G\}$ (if necessary, we can translate $S$ by $a$ if $v_a(S) = \max\{v_g(S) : g \in G\}$). From $0 \notin \sum_k(S)$, one gets $l \leq k-1$. Since $|S| = k+r$, we have

$$|\textstyle\sum_k(S)| = |\sigma(S) - \sum_r(S)| = |\sum_r(S)|. \tag{2.1}$$

Therefore, to estimate the cardinality of $\sum_k(S)$, it suffices to count the number of distinct elements in $\sum_r(S)$.

Let $U$ be a subsequence of $S0^{-l}$ with maximal length satisfying $\sigma(U) = 0$ and $|U| = u \leq k-1$. Note that $U$ may be an empty sequence. We now have

$$l + u \leq k - 1. \tag{2.2}$$

Otherwise, $0^{k-u}U$ will be a subsequence of $S$ satisfying $\sigma(0^{k-u}U) = 0$ and $|0^{k-u}U| = k$, which is impossible since $0 \notin \sum_k(S)$. Let $W = SU^{-1}0^{-l}$. Then by (2.2),

$$|W| = |SU^{-1}0^{-l}| = k + r - (l+u) \geq r+1. \tag{2.3}$$

We divide the proof of Theorem 1.1 into the following two cases.

CASE 1: $|\text{supp}(W)| = 1$. Let $W = a^h$. Then by (2.3) we obtain $h \geq r+1$. We *claim* that $ja \neq 0$ for any integer $j$ with $1 \leq j \leq r$. Suppose that $j_0 a = 0$ for some integer $j_0 \in [1, r]$. Then from (2.2) we deduce that $\sigma(a^{j_0}U) = 0$ and

$$|U| < |a^{j_0}U| = j_0 + u \leq r + u \leq l + u \leq k - 1$$

since $l \geq h \geq r+1$. This gives us a subsequence $a^{j_0}U$ of $S0^{-l}$ satisfying $\sigma(a^{j_0}U) = 0$ and $|U| < |a^{j_0}U| \leq k-1$, which contradicts the choice of $U$, and the claim is proved.

By the claim, we know that $i_1 a \neq i_2 a$ for any integers $i_1$ and $i_2$ satisfying $0 \leq i_1 < i_2 \leq r$. Hence we have

$$|\{\sigma(0^{r-j}a^j) : 0 \leq j \leq r\}| = \left|\bigcup_{i=0}^{r} \{ia\}\right| = r+1. \tag{2.4}$$

We now consider the following two subcases.

SUBCASE 1.1: $\text{supp}(U) \subseteq \{a\}$. Then $S = 0^l a^{h+u}$. Since $l \geq h \geq r+1$, we have

$$\{S' \mid S : |S'| = r\} = \{0^{r-j}a^j : 0 \leq j \leq r\}.$$

So from (2.4) we deduce that

$$|\sum_r(S)| = |\{\sigma(0^{r-j}a^j) : 0 \le j \le r\}| = r+1.$$

It then follows from (2.1) that $|\sum_k(S)| = r+1$.

Subcase 1.2: $\mathrm{supp}(U) \not\subseteq \{a\}$. Then there is an element $b \in \mathrm{supp}(U)$ such that $b \ne a$. Since

$$\{S' \mid S : |S'| = r\} \supseteq \{0^{r-1}b, 0^r, 0^{r-1}a, \dots, a^r\},$$

we have

$$|\sum_r(S)| \ge \Big|\{b\} \cup \bigcup_{i=0}^{r}\{ia\}\Big|.$$

Now it remains to prove that $b \ne ja$ for any integer $j$ with $0 \le j \le r$, from which and (2.4) one can easily deduce that $|\sum_r(S)| \ge r+2$. Clearly, we have $b \ne 0, a$. Suppose that $b = i_0 a$ for some $2 \le i_0 \le r$. Then by (2.2), we get $\sigma(Ub^{-1}a^{i_0}) = 0$ and $|Ub^{-1}a^{i_0}| = u + i_0 - 1 \le u + r \le u + h \le u + l \le k-1$. But $|Ub^{-1}a^{i_0}| \ge |U|+1$. By the maximality of $|U|$, this is impossible. Hence $b \ne ja$ for any integer $j$ with $0 \le j \le r$. So by (2.1), we get $|\sum_k(S)| \ge r+2$.

Case 2: $|\mathrm{supp}(W)| > 1$. By (2.3), we can choose a subsequence $T$ of $W$ such that

$$|T| = r+1 \quad \text{and} \quad |\mathrm{supp}(T)| > 1.$$

Let $h = \max\{v_g(T) : g \in G\}$. Then there exists a decomposition $T = T_1 \cdot \ldots \cdot T_h$ such that $|\mathrm{supp}(T_i)| = |T_i|$ for each integer $i \in [1,h]$, where $T_1, \dots, T_h \in \mathcal{F}(G)$. For each integer $i \in [1,h]$, let $A_i = \mathrm{supp}(T_i) \cup \{0\}$.

Since $h \le l$, we deduce from (2.2) that $h + u \le k-1$. We *claim* that $0 \notin \sum_j(T)$ for any integer $j$ with $1 \le j \le h$. Suppose that there is a subsequence $T'$ of $T$ such that $\sigma(T') = 0$ and $|T'| = j_0$ for some $1 \le j_0 \le h$. It will give us a subsequence $T'U$ of $S0^{-l}$ satisfying $\sigma(T'U) = 0$ and $u = |U| < |T'U| = j_0 + u \le h + u \le k-1$, which is absurd by the choice of $U$. So the claim is true.

It now follows from the claim that

$$\Big(\sum_{i=1}^{j-1} A_i\Big) \cap (-A_j) = \{0\}$$

for each integer $j$ with $2 \le j \le h$. Hence by Lemma 2.1, we obtain

$$\Big|\sum_{i=1}^{h} A_i\Big| \ge \Big|\sum_{i=1}^{h-1} A_i\Big| + |A_h| - 1 \ge \cdots \ge \sum_{i=1}^{h} |A_i| - (h-1) = r+2.$$

Thus for the subsequence $0^hT$ of $S$, we have

$$|\sum_h(0^hT)| = \Big|\sum_{i=1}^{h} A_i\Big| \ge r+2. \tag{2.5}$$

From $|T| = r + 1$, $|\mathrm{supp}(T)| > 1$ and $h = \max\{v_g(T) : g \in G\}$, one can easily deduce that $h \le |T| - 1 = r$. On the other hand, since $r \le k - 1$, we have

$$|ST^{-1}0^{-h}| = k + r - (r + 1 + h) = k - 1 - h \ge r - h \ge 0.$$

Choosing a subsequence $V$ of $ST^{-1}0^{-h}$ with $|V| = r - h$, we have

$$|\textstyle\sum_r(S)| \ge |\sum_r(0^hTV)| \ge |\sum_h(0^hT)| \ge r + 2.$$

Thus by (2.1), we get $|\sum_k(S)| \ge r + 2$. From the above discussion, we can see that $|\mathrm{supp}(S)| = 2$ and $|\sum_k(S)| = r + 1$ in Subcase 1.1, while we have $|\sum_k(S)| \ge r + 2$ and $|\mathrm{supp}(S)| \ge 3$ in Subcase 1.2 and Case 2. Thus we conclude that $|\sum_k(S)| \ge r + 2$ unless $|\mathrm{supp}(S)| = 2$, in which case $|\sum_k(S)| = r + 1$. This completes the proof of Theorem 1.1. ∎

We can immediately get the following consequences of Theorem 1.1.

Corollary 2.2. *Let $G$ be an additive abelian group of order $k \ge 3$, and let $S \in \mathcal{F}(G)$ with $|S| = 2k - 2$. Then either* 0 *is a $k$-sum, or $S = a^{k-1}b^{k-1}$ and $\sum_k(S) = G \setminus \{0\}$.*

Note that Bialostocki and Dierker [1] proved that if $S$ is a sequence over a cyclic group $G$ of order $k$ and $|S| = 2k - 2$, then either 0 is a $k$-sum, or $S = a^{k-1}b^{k-1}$ and $\sum_k(S) = G \setminus \{0\}$. Evidently, if we let $G$ be a cyclic group of order $k$, Corollary 2.2 becomes the Bialostocki–Dierker theorem [1].

Corollary 2.3. *Let $G$ be an additive abelian group of order $k \ge 4$, and let $S \in \mathcal{F}(G)$ with $|S| = 2k - 3$. If $0 \notin \sum_k(S)$, then every non-zero element of $G$ can be expressed as a $k$-sum except for $S = a^{k-1}b^{k-2}$ with $a$ and $b$ being elements of $G$, in which case only one non-zero element of $G$ cannot be expressed as a $k$-sum.*

In [2], Bollobás and Leader pointed out that the lower bound $r + 1$ may not be best possible in the non-cyclic case. Applying Theorem 1.1, we construct a class of sequences such that $|\sum_k(S)| \ge r + 2$.

Proposition 2.4. *Let $n \ge 2$ and $t \ge 2$ be integers, and let $G = \mathbb{Z}_n^t$. Let $S \in \mathcal{F}(G)$ with $|S| = n^t + r$, where $n - 1 \le r \le n^t - 1$. If $0 \notin \sum_{n^t}(S)$, then $|\sum_{n^t}(S)| \ge r + 2$.*

*Proof.* Suppose that $|\sum_{n^t}(S)| = r + 1$. Then by Theorem 1.1, $S$ must be of the form $a^lb^h$, where $n \le h \le l \le n^t - 1$. Take $x = (n^{t-1} - \lfloor h/n \rfloor)n$ and $y = \lfloor h/n \rfloor n$. Clearly, $1 \le x \le l$ and $1 \le y \le h$. But $x + y = n^t$ and $xa + yb = 0$, a contradiction, since $0 \notin \sum_{n^t}(S)$. Thus by Theorem 1.1, we obtain $|\sum_{n^t}(S)| \ge r + 2$, as desired. ∎

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*REFERENCES*

[1] A. Bialostocki and P. Dierker, *On the Erdős–Ginzburg–Ziv theorem and the Ramsey numbers for stars and matchings*, Discrete Math. 110 (1992), 1–8.

[2] B. Bollobás and I. Leader, *The number of $k$-sums modulo $k$*, J. Number Theory 78 (1999), 27–35.

[3] P. Erdős, A. Ginzburg and A. Ziv, *A theorem in additive number theory*, Bull. Res. Council Israel 10F (1961), 41–43.

[4] W. Gao, *Addition theorems for finite abelian groups*, J. Number Theory 53 (1995), 241–246.

[5] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. 24 (2006), 337–369.

[6] W. Gao and I. Leader, *Sums and $k$-sums in abelian groups of order $k$*, J. Number Theory 120 (2006), 26–32.

[7] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure Appl. Math. 278, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[8] Y. O. Hamidoune, O. Ordaz and A. Ortuño, *On a combinatorial theorem of Erdős, Ginzburg and Ziv*, Combin. Probab. Comput. 7 (1998), 403–412.

[9] J. E. Olson, *An addition theorem for finite abelian groups*, J. Number Theory 9 (1977), 63–70.

[10] P. Scherk, *Distinct elements in a set of sums*, Amer. Math. Monthly 62 (1955), 46–47.

[11] H. Yu, *A simple proof of a theorem of Bollobás and Leader*, Proc. Amer. Math. Soc. 131 (2003), 2639–2640.

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