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A NOTE ON CONFORMAL VECTOR FIELDS ON A RIEMANNIAN MANIFOLD

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Abstract. We consider an n-dimensional compact Riemannian manifold (M, g) and show that the presence of a non-Killing conformal vector field ξ on M that is also an eigenvector of the Laplacian operator acting on smooth vector fields with eigenvalue $\lambda > 0$, together with an upper bound on the energy of the vector field ξ, implies that M is isometric to the n-sphere $S^n(\lambda)$. We also introduce the notion of φ-analytic conformal vector fields, study their properties, and obtain a characterization of n-spheres using these vector fields.

1. Introduction. The use of differential equations in studying the geometry of a Riemannian manifold was initiated by Obata (cf. [O1], [O2]). His work is about characterizing specific Riemannian manifolds by second order differential equations. According to his main result, a necessary and sufficient condition for an n-dimensional complete and connected Riemannian manifold (M, g) to be isometric to the n-sphere $S^n(c)$ is that there exists a non-constant smooth function f on M that satisfies the differential equation $H_f = -cfg$, where H_f is the Hessian of f. Then Tashiro [TA] showed that the Euclidean spaces R^n are characterized by the differential equation $H_f = cg$, and Tanno [T] obtained a similar characterization of spheres. Recently García-Río et. al. [EGKU], [GKU] have considered the Laplacian operator Δ acting on smooth vector fields on a Riemannian manifold (M, g) and generalized the result of Obata using a differential equation satisfied by a vector field to characterize the n-sphere $S^n(c)$ (cf. [GKU, Theorem 3.5]). These authors have also proved that the differential equation

$$\Delta Z = -cZ, \quad c = \frac{S}{n(n-1)},$$

where Z is a non-trivial smooth vector field on an n-dimensional compact Einstein manifold (M, g) of constant scalar curvature $S > 0$ (that is, Z is an eigenvector of the Laplacian operator Δ), is a necessary and sufficient condition for M to be isometric to the n-sphere $S^n(c)$ (cf. [EGKU, Theorem 6]).

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A smooth vector field ξ on a Riemannian manifold (M, g) is said to be a *conformal vector field* if there exists a smooth function f on M that satisfies $\pounds_\xi g = 2fg$, where $\pounds_\xi g$ is the Lie derivative of g with respect to ξ. If in addition ξ is a closed vector field, then ξ is said to be a *closed conformal vector field*. Riemannian manifolds admitting closed conformal vector fields or conformal gradient vector fields have been investigated in [DA], [MP], [O3], [TW], [TA] and it has been observed that there is a close relationship between the potential functions of conformal vector fields and Obata's differential equation. In [D2], conformal vector fields which are also eigenvectors of the Laplacian operator have been studied on a compact Riemannian manifold of constant scalar curvature and under a suitable restriction on the Ricci curvature of this manifold, and it is shown there that the Riemannian manifold must be isometric to a sphere. Note that there are several examples of non-trivial conformal vector fields which are also eigenvectors of the Laplacian operator acting on smooth vector fields (cf. [D2]). A natural question arises whether we could prove the result in [D2] without these curvature assumptions or by replacing the curvature assumptions with a suitable analytic condition. In the present paper, we answer this question as well as initiate the study of *φ-analytic conformal vector fields*, i.e. conformal vector fields whose flow leaves invariant a certain tensor field associated to the conformal vector field. We also obtain a characterization of spheres using φ-analytic conformal vector fields.

2. Preliminaries. Let (M, g) be an n-dimensional Riemannian manifold with the Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. Recently García-Río et. al. [GKU] have studied the Laplacian operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\Delta X = \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X),$$

where ∇ is the Riemannian connection and $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M. This operator is elliptic and self-adjoint with respect to the inner product $\langle\, ,\rangle$ on $\mathfrak{X}^C(M)$, the space of compactly supported vector fields in $\mathfrak{X}(M)$, defined by

$$\langle X, Y\rangle = \int_M g(X, Y), \qquad X, Y \in \mathfrak{X}^C(M).$$

A non-trivial vector field X is said to be an *eigenvector* of the Laplacian operator Δ if there is a constant μ such that $\Delta X = -\mu X$. For a compact Riemannian manifold (M, g), using the properties of Δ with respect to the inner product $\langle\, ,\rangle$, it is easy to conclude that the eigenvalue satisfies $\mu \geq 0$. For example consider the n-sphere $S^n(c)$ of constant curvature c (that is, of radius $\sqrt{1/c}$) as a hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ with unit normal vector field N and take a constant vector field Z on $\mathbb{R}^{n+1}$, which can be expressed as $Z = \xi + fN$, where ξ is the tangential component of Z to $S^n(c)$ and $f = \langle Z, N\rangle$ is treated as a smooth function on $S^n(c)$, $\langle\, ,\rangle$ being the Euclidean metric on $\mathbb{R}^{n+1}$. Then it is easy to show that ξ is a conformal vector field on $S^n(c)$ and that $\Delta\xi = -c\xi$.

We shall denote by Δ both Laplacian operators, the one acting on smooth functions on M as well as that acting on smooth vector fields. The *Ricci operator* Q is a symmetric $(1,1)$-tensor field that is defined by $g(QX, Y) = \mathrm{Ric}(X, Y)$, $X, Y \in \mathfrak{X}(M)$, where Ric is the Ricci tensor of the Riemannian manifold.

A vector field $\xi \in \mathfrak{X}(M)$ is said to be a *conformal vector field* if

$$\pounds_\xi g = 2fg \tag{2.1}$$

for a smooth function $f \in C^\infty(M)$ called the *potential function*, where $\pounds_\xi$ is the Lie derivative with respect to ξ. Using Koszul's formula (cf. [D1], [DD]), we immediately obtain the following for a vector field ξ on M:

$$2g(\nabla_X\xi, Y) = (\pounds_\xi g)(X, Y) + d\eta(X, Y), \qquad X, Y \in \mathfrak{X}(M), \tag{2.2}$$

where η is the 1-form dual to ξ, that is, $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$. Define a skew-symmetric tensor field φ of type $(1,1)$ on M by

$$d\eta(X, Y) = 2g(\varphi X, Y), \qquad X, Y \in \mathfrak{X}(M). \tag{2.3}$$

Then using equations (2.1)–(2.3), we immediately get

$$\nabla_X\xi = fX + \varphi X, \qquad X \in \mathfrak{X}(M), \tag{2.4}$$

and we say that φ is the tensor field *associated* to the conformal vector field ξ.

LEMMA 2.1. *Let ξ be a conformal vector field on a Riemannian manifold (M, g) with potential function f. Then*

$$(\nabla\varphi)(X, Y) = R(X, \xi)Y + Y(f)X - g(X, Y)\nabla f, \qquad X, Y \in \mathfrak{X}(M),$$

where $(\nabla\varphi)(X, Y) = \nabla_X(\varphi Y) - \varphi(\nabla_X Y)$, R is the curvature tensor field and ∇f is the gradient of the function f.

Proof. Equation (2.4) gives

$$R(X, Y)\xi = X(f)Y - Y(f)X + (\nabla\varphi)(X, Y) - (\nabla\varphi)(Y, X). \tag{2.5}$$

Note that the 2-form given by $g(\varphi X, Y)$ in (2.3) is closed, whence

$$g((\nabla\varphi)(X, Y), Z) + g((\nabla\varphi)(Y, Z), X) + g((\nabla\varphi)(Z, X), Y) = 0,$$

which together with the skew-symmetry of φ and with (2.5) gives

$$g(R(X, Y)\xi + Y(f)X - X(f)Y, Z) + g((\nabla\varphi)(Z, X), Y) = 0;$$

this proves the lemma. ∎

LEMMA 2.2 ([D1]). *Let ξ be a conformal vector field on an n-dimensional compact Riemannian manifold (M,g) with potential function f. Then*

$$\int_M f = 0, \qquad \int_M g(\nabla f, \xi) = -n \int_M f^2,$$

where ∇f is the gradient of the function f.

LEMMA 2.3. *Let ξ be a conformal vector field on a compact Riemannian manifold (M,g) with potential function f. Then*

$$\int_M \big(\mathrm{Ric}(\xi,\xi) - n(n-1)f^2 - \|\varphi\|^2\big) = 0.$$

Proof. Since $Q(\xi) = \sum R(\xi, e_i)e_i$, Lemma 2.1 gives

$$\sum (\nabla\varphi)(e_i, e_i) = -Q(\xi) - (n-1)\nabla f. \tag{2.6}$$

Using (2.4), (2.6) to evaluate $\operatorname{div}\varphi(\xi)$, we obtain

$$\begin{aligned}\operatorname{div}\varphi(\xi) &= -\sum g(fe_i + \varphi e_i, \varphi e_i) + \mathrm{Ric}(\xi,\xi) + (n-1)g(\nabla f, \xi)\\ &= -\|\varphi\|^2 + \mathrm{Ric}(\xi,\xi) + (n-1)g(\nabla f,\xi).\end{aligned}$$

Integrating the above equation and using Lemma 2.2, we get the result. ∎

3. A characterization of spheres. We observe that for a non-Killing conformal vector field ξ on a Riemannian manifold (M,g), the length of ξ cannot be a constant. Indeed, if the length $\|\xi\|$ is a constant, then equation (2.4) shows that $\varphi(\xi) = f\xi$, and so

$$f\|\xi\|^2 = 0,$$

that is, either $f = 0$ or $\xi = 0$, which again by (2.4) implies that ξ is a Killing vector field, a contradiction. Recall that the *energy* of the conformal vector field ξ on a compact Riemannian manifold (M,g) is given by

$$e(\xi) = \int_M \|\xi\|^2.$$

Consider the conformal vector field ξ on $S^n(c)$ induced by a constant vector field Z on $\mathbb{R}^{n+1}$ which satisfies $\nabla_X\xi = -\sqrt{c}\,\rho X$ and $\nabla\rho = \sqrt{c}\,\xi$, where the restriction of Z to $S^n(c)$ is expressed as $Z = \xi + \rho N$ and N is the unit normal to $S^n(c)$. Thus the potential function satisfies $f = -\sqrt{c}\,\rho$, and we have $\nabla f = -c\xi$ and $\Delta f = -ncf$, where Δ is the Laplacian operator acting on the smooth functions on $S^n(c)$. Hence the energy of the conformal vector field ξ is given by

$$e(\xi) = \frac{1}{c^2}\int_M \|\nabla f\|^2 = nc^{-1}\int_M f^2.$$

Moreover, ξ satisfies $\Delta\xi = -c\xi$. This raises a question: is a compact Riemannian manifold (M, g) that admits a non-Killing conformal vector field with $\Delta\xi = -c\xi$, and having energy satisfying the above equality for a constant c, necessarily isometric to the sphere $S^n(c)$? In this section, we show that the answer is affirmative and prove the following:

THEOREM 3.1. *An n-dimensional compact connected Riemannian manifold (M, g) admits a non-Killing conformal vector field ξ with potential function f that satisfies $\Delta\xi = -\lambda\xi$, $\lambda > 0$, with energy*

$$e(\xi) \le n\lambda^{-1} \int_M f^2,$$

if and only if M is isometric to the n-sphere $S^n(\lambda)$.

Proof. Choose a pointwise constant local orthonormal frame $\{e_1, \ldots, e_n\}$ on M and use (2.4) to get

$$\Delta\xi = \sum \nabla_{e_i}(fe_i + \varphi(e_i)) = \nabla f + \sum (\nabla\varphi)(e_i, e_i). \tag{3.1}$$

Now, by $\Delta\xi = -\lambda\xi$ and equations (2.6), (3.1), we have

$$Q(\xi) = -(n-2)\nabla f + \lambda\xi.$$

Taking the inner product of the above equality with ξ and then integrating, we get

$$\int_M \operatorname{Ric}(\xi, \xi) = \int_M (-(n-2)g(\nabla f, \xi) + \lambda\|\xi\|^2),$$

which together with Lemma 2.2 gives

$$\int_M \operatorname{Ric}(\xi, \xi) = \int_M (n(n-2)f^2 + \lambda\|\xi\|^2).$$

Using Lemma 2.3 in the above equation, we get

$$\int_M \|\varphi\|^2 = \int_M (\lambda\|\xi\|^2 - nf^2) \le \lambda\Big(e(\xi) - n\lambda^{-1} \int_M f^2\Big).$$

Thus if the energy of the vector field ξ satisfies the condition in the statement, the above inequality gives $\varphi = 0$. In this situation, equation (3.1) takes the form

$$\nabla f = -\lambda\xi, \qquad \lambda > 0.$$

Note that f is a non-constant function, for otherwise Lemma 2.2 would yield $f = 0$, which together with equation (2.4) would imply that ξ is a Killing vector field, contradicting the fact that ξ is a non-Killing conformal vector field. Thus the above equation together with (2.4) gives

$$\nabla_X \nabla f = -\lambda f X, \qquad X \in \mathfrak{X}(M),$$

which is the Obata equation and hence M is isometric to the n-sphere $S^n(\lambda)$.

The converse is trivial as the sphere $S^n(\lambda)$ admits a non-Killing conformal vector field satisfying the hypothesis. ■

4. φ-analytic conformal vector fields. In this section, we define φ-analytic vector fields on a Riemannian manifold and study their properties.

DEFINITION 4.1. A conformal vector field ξ on a Riemannian manifold (M, g) with associated tensor field φ is said to be a *φ-analytic conformal vector field* if φ is invariant under the flow of ξ.

It follows from the above definition that a conformal vector field ξ is a φ-analytic conformal vector field if and only if

$$(\pounds_\xi \varphi)(X) = 0, \qquad X \in \mathfrak{X}(M). \tag{4.1}$$

An example of a φ-analytic vector field ξ is given by $\xi = \psi + J\psi \in \mathfrak{X}(\mathcal{C}^n)$, where ψ is the position vector field and J is the complex structure on the complex Euclidean space $\mathcal{C}^n$. It is clear that ξ is the conformal vector field with potential function $f = 1$ and associated tensor field $\varphi = J$ and that it satisfies equation (4.1), that is, ξ is indeed a φ-analytic vector field. Also conformal vector fields on the unit sphere S^n induced by constant vector fields on $\mathbb{R}^{n+1}$ are φ-analytic vector fields. The following theorem provides a characterization of φ-analytic vector fields.

THEOREM 4.2. *A conformal vector field ξ on a Riemannian manifold (M, g) with potential function f is a φ-analytic conformal vector field if and only if there exists a smooth function ρ on M such that $\nabla f = \rho\xi$.*

Proof. Suppose ξ is a φ-analytic vector field with potential function f. Then using equations (2.4) and (4.1), we obtain

$$(\nabla\varphi)(\xi, X) = 0, \qquad X \in \mathfrak{X}(M),$$

which, in view of Lemma 2.1, gives

$$g(X, \xi)\nabla f = g(X, \nabla f)\xi, \qquad X \in \mathfrak{X}(M).$$

Thus, we get $\nabla f \wedge \xi = 0$, and consequently the vector fields ∇f and ξ are parallel. Hence, there exists a smooth function ρ on M such that $\nabla f = \rho\xi$.

Conversely, assume that $\nabla f = \rho\xi$. Then from (2.4) and Lemma 2.1, we have

$$(\pounds_\xi \varphi)(X) = [\xi, \varphi X] - \varphi[\xi, X] = (\nabla\varphi)(\xi, X) = g(X, \nabla f)\xi - g(X, \xi)\nabla f = 0,$$

which proves that ξ is a φ-analytic vector field. ■

If a conformal vector field ξ satisfies $\varphi(\xi) = 0$, we say that ξ is a *null conformal vector field*. Next, we prove the following.

THEOREM 4.3. *A null conformal vector field ξ with potential function f on a Riemannian manifold (M,g) such that $R(\nabla f,\xi;\xi,\nabla f)\le 0$ is a φ-analytic conformal vector field.*

Proof. Lemma 2.1 gives

$$(\nabla\varphi)(\nabla f,\xi)=R(\nabla f,\xi)\xi+\xi(f)\nabla f-\xi(f)\nabla f,$$

which, together with $\varphi(\xi)=0$, yields

$$-\varphi(f\nabla f+\varphi(\nabla f))=R(\nabla f,\xi)\xi. \tag{4.2}$$

Taking the inner product of the above equality with ∇f, we get

$$R(\nabla f,\xi;\xi,\nabla f)=\|\varphi(\nabla f)\|^2.$$

In view of our hypothesis, $\varphi(\nabla f)=0$, and consequently $[\nabla f,\xi]=f\nabla f-\nabla_\xi\nabla f$, and equation (4.2) gives $R(\nabla f,\xi)\xi=0$. Thus

$$\nabla_{\nabla f}f\xi-\nabla_\xi(f\nabla f)-\nabla_{f\nabla f}\xi+\nabla_{\nabla_\xi\nabla f}\xi=0,$$

which, combined with (2.4), implies that

$$\|\nabla f\|^2\xi-\xi(f)\nabla f+\varphi(\nabla_\xi\nabla f)=0.$$

Taking the inner product of the above equality with ξ, we get

$$g(\nabla f,\xi)^2=\|\nabla f\|^2\|\xi\|^2,$$

that is, $\nabla f=\rho\xi$ for a smooth function ρ on M; by Theorem 4.2, this proves that ξ is a φ-analytic vector field. ∎

Next, we use a specific type of φ-analytic vector field to find the characterization of a sphere. If the function ρ appearing in the characterization of the φ-analytic conformal vector field ξ in Theorem 4.2 is a constant, then we say that ξ is a *φ-analytic conformal vector field of constant type*. Notice that the conformal vector field ξ on $S^n(c)$ induced by a constant vector field Z on $\mathbb{R}^{n+1}$ satisfies $\nabla_X\xi=-\sqrt{c}\,\rho X$ and $\nabla\rho=\sqrt{c}\,\xi$, where the restriction of Z to $S^n(c)$ is expressed as $Z=\xi+\rho N$ and N is the unit normal to $S^n(c)$. Thus, as $\nabla f=-c\xi$, the conformal vector field ξ on $S^n(c)$ is a φ-analytic vector field of constant type. This raises a question: is a compact Riemannian manifold that admits a φ-analytic conformal vector field of constant type necessarily isometric to an n-sphere? We answer this question in the following:

THEOREM 4.4. *Let ξ be a non-Killing φ-analytic conformal vector field of constant type on an n-dimensional compact and connected Riemannian manifold (M,g). Then (M,g) is isometric to the n-sphere $S^n(c)$ for some $c>0$.*

Proof. Note that $\nabla f=\alpha\xi$, where α is a constant. Observe that $\alpha\ne 0$, for otherwise the potential function f would be constant, which by Lemma 2.2 would imply $f=0$, and so ξ would be a Killing vector field. Hence,

$\alpha \neq 0$ and the vector field $\xi = \alpha^{-1}\nabla f$ is closed, which by the equation (2.3) gives $\varphi = 0$. Thus taking the covariant derivatives of both sides of the equation $\nabla f = \alpha\xi$ with respect to $X \in \mathfrak{X}(M)$ and using (2.4), we get

$$\nabla_X \nabla f = \alpha f X, \quad X \in \mathfrak{X}(M). \tag{4.3}$$

We claim that α is a negative constant. To see it, observe that (4.3) gives $\Delta f = n\alpha f$, that is, f is an eigenfunction of the Laplacian operator Δ, which being an elliptic operator on the compact Riemannian manifold has the eigenvalue $n\alpha = 0$ or $n\alpha < 0$. The first option cannot occur as it implies $\Delta f = 0$, that is, f is a constant, which is ruled out as seen above. Hence $\alpha < 0$, which implies that (4.3) is the Obata equation, proving that M is isometric to $S^n(c)$, $c = -\alpha$. ■

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