COLLOQUIUM MATHEMATICUM

VOL. 98 2003 NO. 1

# PRODUCTS OF DISJOINT BLOCKS OF CONSECUTIVE INTEGERS WHICH ARE POWERS

BY

MARIUSZ SKAŁBA (Warszawa)

**Abstract.** The product of consecutive integers cannot be a power (after Erdős and Selfridge), but products of disjoint blocks of consecutive integers can be powers. Even if the blocks have a fixed length $l \geq 4$ there are many solutions. We give the bound for the smallest solution and an estimate for the number of solutions below $x$.

Our starting point is the celebrated theorem of Erdős and Selfridge stating that the product of consecutive integers is never a power ([3], [4]).

On the other hand, the product of disjoint blocks of consecutive integers can be a power ([4]). Let us first consider the case of three consecutive integers. Let $f(x) = x^3 - x$, fix $k \geq 2$ and consider the diophantine equation

$$\prod_{i=1}^{k} f(x_i) = y^2. \tag{1}$$

We look for solutions in natural numbers $x_1, \ldots, x_k, y$ which satisfy

$$2 \leq x_1 < \ldots < x_k, \quad x_{j+1} \geq x_j + 3. \tag{2}$$

For $k = 2$, K. R. S. Sastry takes

$$x_1 = n, \quad x_2 = 2n - 1,$$

where $n, m \in \mathbb{N}$ satisfy the equation

$$(n+1)(2n-1) = m^2$$

of the Pellian type with infinitely many solutions ([4]). For $k = 3$ one can take

$$x_1 = F_{2u-1}, \quad x_2 = F_{2u+1}, \quad x_3 = F_{2u}^2, \quad u \geq 2,$$

where $F_u$ is the $u$th term of Fibonacci sequence.

Since each $k \geq 2$ is of the form $3a + 2b$ with nonnegative $a, b$ and we can combine solutions we obtain

THEOREM 1. *For any fixed $k \geq 2$ the equation* (1) *has infinitely many natural solutions satisfying* (2).

---

2000 *Mathematics Subject Classification*: 11A05, 11D57.

From now on, we will consider a more general problem. Let $l \geq 2$ be a fixed natural number and

$$f(x) = (x+1)(x+2)\cdot\ldots\cdot(x+l). \tag{3}$$

Moreover let $q \geq 2$ be a fixed prime and consider the equation

$$\prod_{i=1}^{k} f(x_i) = y^q. \tag{4}$$

We are interested in nonnegative integral solutions which satisfy

$$x_1 < \ldots < x_k, \quad x_{j+1} \geq x_j + l. \tag{5}$$

Erdős and Graham [2, p. 67] suggest that if $l \geq 4$, $q = 2$, then (4) has only finitely many solutions satisfying (5) (cf. [4, Problem D.17]).

If we allow $k$ to vary, the above suggestion is not true, because we have

THEOREM 2. (a) *There exist nonnegative integers* $x_1, \ldots, x_k, y$ *satisfying* (4), (5) *and*

$$x_k + l < e^{cql},$$

*where*

$$c = \sup_{n \geq 1} \frac{\pi(n)\log n}{n} = \frac{30\log 113}{113} < 1.25506.$$

(b) *Let* $N(x)$ *denote the number of solutions of* (4) *satisfying* (5) *and* $x_k + l \leq x$. *Then*

$$N(x) \gg 2^{\frac{x}{l}(1+o(1))} \quad \text{as } x \to \infty.$$

*Proof.* (a) Let $G = \mathbb{Q}_+/\mathbb{Q}_+^q$ and $G(x) = \langle \overline{p}_1, \ldots, \overline{p}_{\pi(x)} \rangle$, where $\overline{m}$ denotes the image of $m \in \mathbb{Q}_+$ in $G$, $x \geq 1$. Obviously $G(x) \simeq C_q^{\pi(x)}$, where $C_q$ stands for a cyclic group of order $q$. Now define

$$g_j = \overline{f((j-1)l)} \quad \text{for } j = 1, \ldots, [x/l]. \tag{6}$$

The elements $g_j$ belong to $G(x)$. We recall the definition of the Davenport constant of a finite Abelian group $H$ ([1]). It is the smallest natural number $D(H)$ such that for any sequence $g_1, \ldots, g_k$ of $k$ elements of $H$ with $k \geq D(H)$ one can choose a subsequence $g_{j_1}, \ldots, g_{j_u}$ with

$$g_{j_1} \cdot \ldots \cdot g_{j_u} = 1.$$

It can be proved ([6], [5]) that

$$D(C_q^t) = (q-1)t + 1. \tag{7}$$

By the above definition, in order to prove part (a) it is sufficient to show that

$$\left[\frac{x}{l}\right] \geq (q-1)\pi(x) + 1 \tag{8}$$

for $x = e^{cql}$. For $x \geq 2$, we have $\pi(x) \geq 1$, hence (8) will follow from

$$\frac{x}{l} \geq q \cdot \frac{cx}{\log x},$$

which is equality for $x = e^{cql}$.

(b) We apply the following theorem of J. E. Olson [6]:

*For each sequence* $(g_1, \ldots, g_k)$, $g_i \in H$, *let* $N(g_1, \ldots, g_k)$ *be the number of solutions* $(e_1, \ldots, e_k)$, $e_i = 0$ *or* $1$, *of the equation*

$$g_1^{e_1} \cdot \ldots \cdot g_k^{e_k} = 1.$$

*Let* $N(H, k)$ *be the minimum value of* $N(g_1, \ldots, g_k)$. *Then*

$$N(H, k) = \max(1, 2^{k+1-D(H)}).$$

Our result follows now immediately:

$$N(x) \gg 2^{\frac{x}{l} - (q-1)\frac{x}{\log x}} \gg 2^{\frac{x}{l}(1+o(1))} \quad \text{as } x \to \infty.$$

Remark. The numerical value of $c$ is taken from [7].

**Acknowledgments.** I would like to thank Prof. J. Browkin for very stimulating examples of nontrivial solutions of (4) for $l \geq 4$. I thank also Prof. A. Schinzel for his kind interest, remarks improving presentation and bibliographical hints.

## *REFERENCES*

[1] H. Davenport, in: Proc. Midwestern Conf. on Group Theory, Ohio State Univ., 1966.

[2] P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, Monograph. Enseign. Math. 28, Genève, 1980.

[3] P. Erdős and J. L. Selfridge, *The product of consecutive integers is never a power*, Illinois J. Math. 19 (1975), 292–301.

[4] R. K. Guy, *Unsolved Problems in Number Theory*, Springer, 1994.

[5] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, PWN, Warszawa, 1990.

[6] J. E. Olson, *A combinatorial problem on finite Abelian groups, I, II*, J. Number Theory 1 (1969), 8–10, 195–199.

[7] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some function of prime numbers*, Illinois J. Math. 6 (1962), 64–89.

Department of Mathematics, Computer Science and Mechanics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: skalba@mimuw.edu.pl

*Received 19 December 2002* (4295)