PRODUCTS OF DISJOINT BLOCKS OF CONSECUTIVE INTEGERS WHICH ARE POWERS

BY

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Abstract. The product of consecutive integers cannot be a power (after Erdős and Selfridge), but products of disjoint blocks of consecutive integers can be powers. Even if the blocks have a fixed length \( l \geq 4 \) there are many solutions. We give the bound for the smallest solution and an estimate for the number of solutions below \( x \).

Our starting point is the celebrated theorem of Erdős and Selfridge stating that the product of consecutive integers is never a power ([3], [4]). On the other hand, the product of disjoint blocks of consecutive integers can be a power ([4]). Let us first consider the case of three consecutive integers. Let \( f(x) = x^3 - x \), fix \( k \geq 2 \) and consider the diophantine equation

\[
\prod_{i=1}^{k} f(x_i) = y^2.
\]

We look for solutions in natural numbers \( x_1, \ldots, x_k, y \) which satisfy

\[
2 \leq x_1 < \ldots < x_k, \quad x_{j+1} \geq x_j + 3.
\]

For \( k = 2 \), K. R. S. Sastry takes

\[ x_1 = n, \quad x_2 = 2n - 1, \]

where \( n, m \in \mathbb{N} \) satisfy the equation

\[
(n + 1)(2n - 1) = m^2
\]

of the Pellian type with infinitely many solutions ([4]). For \( k = 3 \) one can take

\[ x_1 = F_{2u-1}, \quad x_2 = F_{2u+1}, \quad x_3 = F_{2u}^2, \quad u \geq 2, \]

where \( F_u \) is the \( u \)th term of Fibonacci sequence.

Since each \( k \geq 2 \) is of the form \( 3a + 2b \) with nonnegative \( a, b \) and we can combine solutions we obtain

Theorem 1. For any fixed \( k \geq 2 \) the equation (1) has infinitely many natural solutions satisfying (2).

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From now on, we will consider a more general problem. Let \( l \geq 2 \) be a fixed natural number and
\[
f(x) = (x + 1)(x + 2) \cdot \ldots \cdot (x + l).
\]
Moreover let \( q \geq 2 \) be a fixed prime and consider the equation
\[
\prod_{i=1}^{k} f(x_i) = y^q.
\]
We are interested in nonnegative integral solutions which satisfy
\[
x_1 < \ldots < x_k, \quad x_{j+1} \geq x_j + l.
\]
Erdős and Graham [2, p. 67] suggest that if \( l \geq 4, q = 2 \), then (4) has only finitely many solutions satisfying (5) (cf. [4, Problem D.17]).

If we allow \( k \) to vary, the above suggestion is not true, because we have

**Theorem 2.** (a) There exist nonnegative integers \( x_1, \ldots, x_k, y \) satisfying (4), (5) and
\[
x_k + l < e^{q l},
\]
where
\[
c = \sup_{n \geq 1} \frac{\pi(n) \log n}{n} = \frac{30 \log 113}{113} < 1.25506.
\]
(b) Let \( N(x) \) denote the number of solutions of (4) satisfying (5) and \( x_k + l \leq x \). Then
\[
N(x) \gg 2^{\pi(1+o(1))} \quad \text{as } x \to \infty.
\]

**Proof.** (a) Let \( G = \mathbb{Q}_+ \otimes \mathbb{Q}_+^q \) and \( G(x) = \langle \overline{p_1}, \ldots, \overline{p_{\pi(x)}} \rangle \), where \( \overline{m} \) denotes the image of \( m \in \mathbb{Q}_+ \) in \( G, x \geq 1 \). Obviously \( G(x) \simeq C_q^{\pi(x)} \), where \( C_q \) stands for a cyclic group of order \( q \). Now define
\[
g_j = \overline{f((j-1)l)} \quad \text{for } j = 1, \ldots, [x/l].
\]
The elements \( g_j \) belong to \( G(x) \). We recall the definition of the Davenport constant of a finite Abelian group \( H \) ([1]). It is the smallest natural number \( D(H) \) such that for any sequence \( g_1, \ldots, g_k \) of \( k \) elements of \( H \) with \( k \geq D(H) \) one can choose a subsequence \( g_{j_1}, \ldots, g_{j_u} \) with \( g_{j_1} \cdot \ldots \cdot g_{j_u} = 1 \).

It can be proved ([6], [5]) that
\[
D(C_q^{\pi(x)}) = (q - 1) t + 1.
\]
By the above definition, in order to prove part (a) it is sufficient to show that
\[
\left\lfloor \frac{x}{l} \right\rfloor \geq (q - 1) \pi(x) + 1
\]
for $x = e^{cql}$. For $x \geq 2$, we have $\pi(x) \geq 1$, hence (8) will follow from

$$\frac{x}{l} \geq q \cdot \frac{c x}{\log x},$$

which is equality for $x = e^{cql}$.

(b) We apply the following theorem of J. E. Olson [6]:

For each sequence $(g_1, \ldots, g_k)$, $g_i \in H$, let $N(g_1, \ldots, g_k)$ be the number of solutions $(e_1, \ldots, e_k)$, $e_i = 0$ or 1, of the equation

$$g_1^{e_1} \cdot \cdots \cdot g_k^{e_k} = 1.$$

Let $N(H, k)$ be the minimum value of $N(g_1, \ldots, g_k)$. Then

$$N(H, k) = \max(1, 2^{k+1-D(H)}).$$

Our result follows now immediately:

$$N(x) \gg 2^{\frac{c}{l} - (q-1)\frac{\pi}{\log x}} \gg 2^{\frac{c}{l}(1+o(1))} \text{ as } x \to \infty.$$

Remark. The numerical value of $c$ is taken from [7].

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