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CONVERGENCE TO STATIONARY SOLUTIONS IN A MODEL OF SELF-GRAVITATING SYSTEMS

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Abstract. We study convergence of solutions to stationary states in an astrophysical model of evolution of clouds of self-gravitating particles.

1. Introduction. In this paper we study asymptotic properties of solutions of the system introduced in [8], [7] for describing the temporal evolution of the density $u(x,t) \ge 0$ and the uniform-in-space temperature $\vartheta(t) > 0$ of a cloud of self-gravitating particles confined to a bounded subdomain $\Omega \subset \mathbb{R}^d$, $d = 2,3$.

This system consists of the continuity equation

$$u_t(x,t) = \operatorname{div}\{\vartheta(t)\nabla u(x,t) + u(x,t)\nabla\varphi(x,t)\} \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1}$$

coupled with the Poisson equation

$$\Delta\varphi(x,t) = u(x,t) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{2}$$

which gives the relation between the gravitational potential $\varphi(x,t)$ and the distribution of mass $u(x,t)$.

The equations (1)–(2) are supplemented with the no-flux boundary condition

$$(\vartheta(t)\nabla u + u\nabla\varphi) \cdot \vec{\nu} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{3}$$

and the initial data

$$u(x,0) = u_0(x) \ge 0 \quad \text{in } \Omega. \tag{4}$$

Here $\vec{\nu}$ denotes the exterior normal vector to $\partial\Omega$.

Without loss of generality, we assume that the total mass of the particles is equal to one:

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx = 1. \tag{5}$$

The potential φ satisfies either the Dirichlet condition

$$\varphi(x,t) = 0 \quad \text{for } x \in \partial\Omega \tag{6}$$

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or the physically acceptable "free" condition

$$\varphi = E_d \star u, \tag{7}$$

where E_d is the fundamental solution of the Laplacian in $\mathbb{R}^d$.

The total energy $\mathcal{E}$ is the sum of the thermal energy $\int_\Omega \vartheta(t)u(x,t)\,dx$ and the potential energy $\frac{1}{2}\int_\Omega u(x,t)\varphi(x,t)\,dx$. For simplicity, we set all the physical constants to be one. In our case $\int_\Omega u(x,t)\,dx = 1$, hence the energy $\mathcal{E}$ takes the form

$$\mathcal{E} = \vartheta(t) + \frac{1}{2}\int_\Omega u(x,t)\varphi(x,t)\,dx. \tag{8}$$

Its conservation permits one to determine the temperature $\vartheta(t)$, uniform in Ω.

For a given energy level $\mathcal{E}$, (1)–(8) is *problem* $\mathcal{P}_\mathcal{E}$ for the unknown quantities u, φ, ϑ. Below we consider $\mathcal{P}_\mathcal{E}$ in the ball; in this case there is no qualitative difference between the conditions (6) and (7).

The problem of existence and uniqueness of solutions of the problem $\mathcal{P}_\mathcal{E}$ for $d = 2, 3$ was studied in [6] and [9]. For $u_0 \in L^2(\Omega)$ the local existence and uniqueness of solution was proved. The existence of global-in-time solutions was obtained in [6] for $d = 2$, and in [9] for the three-dimensional radially symmetric case under some assumptions on the initial density and temperature. The solutions of the model under consideration may exhibit finite time blow-up for large initial data [6], [9]. The structure of the set of stationary solutions of the problem $\mathcal{P}_\mathcal{E}$ was investigated in [1] and [5].

Our aim is to prove that for some initial distribution of mass u_0 and initial temperature ϑ_0 (or fixed energy $\mathcal{E}$), the solution converges to the unique stationary state.

2. Radially symmetric solutions. We consider radially symmetric solutions of the system (1)–(8) in the unit ball $\Omega = \{x \in \mathbb{R}^d : |x| \le 1\}$, $d = 2, 3$. Hence, we may assume

$$\varphi(x,t) = 0 \quad \text{for } |x| = 1. \tag{9}$$

Following [2] we write the problem $\mathcal{P}_\mathcal{E}$ in terms of the integrated density

$$Q(r,t) := \int_{B_r(0)} u(x,t)\,dx \quad \text{for } r \in (0,1] \text{ and } t \in [0,T),\ T \le \infty.$$

Let σ_d denote the area of the unit sphere in $\mathbb{R}^d$. Rescaling $t := (d/\sigma_d)t$ and $\vartheta := d\sigma_d\vartheta$, we obtain as in [9] (cf. also [2]), for $Q(y,t) := Q(r,t)$ with $y = r^d$, the equation

$$Q_t = y^{2-2/d}\vartheta(t)Q_{yy} + QQ_y \tag{10}$$

for $(y,t) \in D_T = \{(y,t) : y \in (0,1),\ t \in (0,T)\}$.

Using the variable Q we transform the energy relation (8) into the form

$$\mathcal{E} = \vartheta(t) - \frac{1}{2}\int_0^1 Q^2(y,t) y^{2/d-2}\, dy, \tag{11}$$

where $\mathcal{E} := d\sigma_d \mathcal{E}$.

The equation (10) is supplemented with the boundary conditions

$$Q(0,t) = 0, \qquad Q(1,t) = 1, \qquad \text{for } t \in [0,T), \tag{12}$$

and the initial data

$$Q(y,0) = Q_0(y) := \int_{B_r(0)} u_0(x)\, dx. \tag{13}$$

The equation (10), boundary conditions (12), initial data (13) and a given total energy (11) define the *problem* $\mathcal{Q}_{\mathcal{E}}$.

Formally, the transformation of $\mathcal{P}_{\mathcal{E}}$ to $\mathcal{Q}_{\mathcal{E}}$ allows us to consider densities u from L^1, which was not possible in the framework of the L^2 theory used in [6], [9]. In our case, we stress that the problem $\mathcal{Q}_{\mathcal{E}}$ plays only an auxiliary role, i.e. each solution Q we take into account comes from a density u. Here, remember that $Q_y = (\sigma_d/d)u$.

We prove our main result:

THEOREM 2.1. *Assume that the initial data Q_0 and the energy $\mathcal{E}$ are chosen so that*

(a) *the stationary solution Q^s, ϑ^s of the problem $\mathcal{Q}_{\mathcal{E}}$ is unique*,

(b) *the problem $\mathcal{Q}_{\mathcal{E}}$ has a global solution $Q(y,t), \vartheta(t)$ with the uniformly bounded derivative Q_y*,

(c) *the temperature $\vartheta(t)$ satisfies $0 < c \le \vartheta(t) \le C < \infty$.*

Then $Q(y,t)$ tends to Q^s uniformly on $[0,1]$ and $\vartheta(t)$ converges to ϑ^s as $t \to \infty$.

Proof. The idea of the proof comes from [11], where a simpler case of electrically repulsing particles has been considered.

We introduce the entropy functional W for the problem $\mathcal{Q}_{\mathcal{E}}$ by

$$W(t) := \int_0^1 Q_y \log Q_y\, dy - \log \vartheta. \tag{14}$$

Note that $W(t)$ is well defined and bounded from below for the solutions satisfying the conditions (b) and (c).

Observing that

$$W'(t) = \int_0^1 (Q_t)_y (\log Q_y + 1)\, dy - \frac{\vartheta_t}{\vartheta}$$

and integrating by parts we get

$$W'(t) = -\int_0^1 Q_t \frac{Q_{yy}}{Q_y}\,dy - \frac{\vartheta_t}{\vartheta} = -\int_0^1 Q_t\left(\frac{Q_{yy}}{Q_y} + \frac{1}{\vartheta}\,Qy^{2/d-2}\right)dy \tag{15}$$

$$= -\int_0^1 \frac{Q_t^2}{Q_y\vartheta}\,y^{2/d-2}\,dy \le 0.$$

Hence W is the Lyapunov functional for the problem $\mathcal{Q}_{\mathcal{E}}$.

Since W is bounded from below, there exists a sequence $t_m \to \infty$ such that $W'(t_m) \to 0$ as $m \to \infty$. We prove that $Q(y, t_m)$ tends to the stationary solution. Set

$$A(y, t_m) := \int_0^y Q_t(v, t_m)\,dv = \int_0^y (v^{2-2/d}\vartheta(t)Q_{yy}(v, t) + Q(v, t)Q_y(v, t))\,dv. \tag{16}$$

Integrating by parts we have

$$A(y, t_m) = y^{2-2/d}\vartheta(t_m)Q_y(y, t_m) - \left(2 - \frac{2}{d}\right)y^{1-2/d}\vartheta(t_m)Q(y, t_m)$$
$$+ \left(2 - \frac{2}{d}\right)\left(1 - \frac{2}{d}\right)\int_0^y v^{-2/d}\vartheta(t_m)Q(v, t_m)\,dv + \frac{1}{2}\,Q^2(y, t_m).$$

It follows from our assumptions imposed on Q_y and ϑ that

$$\int_0^1 \frac{Q_t^2}{Q_y\vartheta}\,y^{2/d-2}\,dy \ge C\int_0^y |Q_t|\,dy$$

for some $C > 0$. Hence

$$W'(t_m) \le -C|A(y, t_m)|. \tag{17}$$

Thus $A(y, t_m)$ tends to 0 as $m \to \infty$. The family $Q(\cdot, t_m)$ is compact in C^0 topology and $\vartheta(t_m)$ is bounded, so we may assume that $Q(\cdot, t_m) \to \overline{Q}(\cdot)$ uniformly on $[0, 1]$ and $\vartheta(t_m)$ converges to $\overline{\vartheta}$. Again, from $A(y, t_m) \to 0$, we conclude that $Q_y(\cdot, t_m)$ converges almost uniformly on $(0, 1]$ to $\overline{Q}_y$, and $\overline{Q}$ satisfies

$$y^{2-2/d}\overline{\vartheta}\,\overline{Q}_y - \left(2 - \frac{2}{d}\right)y^{1-2/d}\overline{\vartheta}\,\overline{Q}$$
$$+ \left(2 - \frac{2}{d}\right)\left(1 - \frac{2}{d}\right)\int_0^y v^{-2/d}\overline{\vartheta}\,\overline{Q}(v)\,dv + \frac{1}{2}\,\overline{Q}^2(y) = 0.$$

Differentiating the above formula with respect to y we see that $y^{2-2/d}\overline{\vartheta}\,\overline{Q}_{yy} + \overline{Q}\,\overline{Q}_y = 0$, so $\overline{Q}$, $\overline{\vartheta}$ is the unique stationary solution Q^s, ϑ^s of the problem $\mathcal{Q}_{\mathcal{E}}$.

Now we assume that $\{s_m\}$ is an arbitrary sequence which goes to ∞. Since $W(t)$ is bounded, there exists a sequence $\{t_m\}$ such that $|t_m - s_m| \to 0$, $W'(t_m) \to 0$ and $|W(t_m) - W(s_m)| \to 0$ as $m \to \infty$. We may assume that the whole sequence $Q(\cdot, s_m)$ tends to Q_1, and as we proved above $Q(\cdot, t_m)$ goes to Q^s. We have to show that $Q_1 = Q^s$. From (15) we get

$$|W(t_m) - W(s_m)| = \int_0^1 \int_{s_m}^{t_m} \frac{Q_t^2}{Q_y \vartheta} y^{2/d-2}\, dt\, dy \to 0 \quad \text{as } m \to \infty. \tag{18}$$

We derive from (18) that $\int_0^1 \int_{s_m}^{t_m} |Q_t|\, dt\, dy \to 0$, hence

$$\int_0^1 |Q(y, s_m) - Q(y, t_m)|\, dy \le \int_0^1 \int_{t_m}^{s_m} |Q_t|\, dt\, dy \to 0.$$

Thus, $Q_1 = Q^s$. From the energy equation (11) we conclude that $\vartheta \to \vartheta^s$ as $t \to \infty$. ∎

Now our aim is to show that for some values of the energy $\mathcal{E}$ and the initial data Q_0 the assumptions of Theorem 2.1 are satisfied.

Lemma 2.2. *For sufficiently large energy $\mathcal{E}$ there exists a unique stationary solution Q^s, ϑ^s of the problem $\mathcal{Q}_{\mathcal{E}}$.*

Proof. We introduce the new function $\overline{Q} := Q^s/\vartheta^s$ which satisfies the equation

$$y^{2-2/d}\overline{Q}_{yy} + \overline{Q}\,\overline{Q}_y = 0 \quad \text{for } y \in (0,1), \tag{19}$$

and the boundary conditions

$$\overline{Q}(0) = 0, \quad \overline{Q}(1) = 1/\vartheta^s. \tag{20}$$

For $d = 2$ the problem (19)–(20) is integrable, and the unique solution is

$$\overline{Q}(y) = \frac{2Cy}{1 + Cy}, \quad \text{where} \quad C = \frac{1}{2\vartheta^s - 1}, \quad \vartheta^s > 1/2.$$

To obtain the uniqueness of a stationary solution of the problem $\mathcal{Q}_{\mathcal{E}}$ observe that the energy of $\overline{Q}$,

$$\mathcal{E}(\vartheta^s) = \kappa\vartheta^s - \frac{1}{2} \int_0^{1/(2\vartheta^s - 1)} \left(\frac{2v}{1+v}\right)^2 \frac{1}{v}\, dv,$$

is an increasing function of ϑ^s and $\lim_{\vartheta^s \to \infty} \mathcal{E}(\vartheta^s) = \infty$, $\lim_{\vartheta^s \to 1/2} \mathcal{E}(\vartheta^s) = -\infty$.

The three-dimensional case is more complicated. For the proof we introduce the new variables [2]

$$v = 9y^{2/3}\overline{Q}_y, \quad w = 3y^{-1/3}Q, \quad y = e^{3\tau}.$$

A simple computation shows that v, w satisfy the system of equations

$$v' = (2 - w)v, \quad w' = v - w, \tag{21}$$

where the prime denotes $d/d\tau$. The boundary data (20) take the form $w(-\infty) = 0$, $w(0) = 1/\theta^s$. There is a unique trajectory (v, w) of (21) with

$w \geq 0$ which satisfies these boundary conditions (cf. an analogous reasoning in [2]).

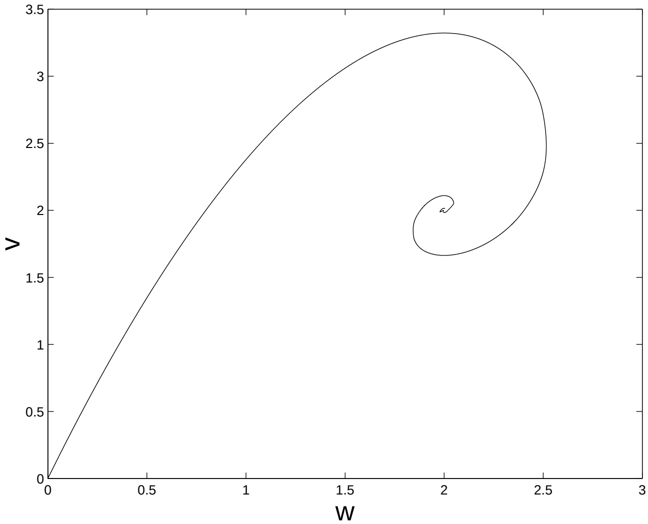

Fig. 1

To finish the proof note that for sufficiently large ϑ^s the energy of the unique solution,

$$\mathcal{E}(\vartheta^s) = \vartheta^s - \int_{-\infty}^{0} w^2(\tau)e^{\tau}\,d\tau,$$

is an increasing function of ϑ^s. ■

LEMMA 2.3. *For sufficiently large $\mathcal{E}$ and bounded Q_0' the temperature satisfies*

$$0 < c \leq \vartheta(t) \leq C < \infty \quad \text{for } t > 0. \tag{22}$$

Proof. The estimate from below for ϑ was proved in [9, Proposition 5.4] for the radially symmetric case and in [6, Lemma 2.1] for general domains. The estimate from above valid for any initial data is specific to the system in two-dimensional bounded domains [6, Lemma 2.2]. In the three-dimensional situation, [9, Theorem 5.5] states that for bounded Q_0' and sufficiently large energy $\mathcal{E}$ the inequalities (22) are satisfied. ■

In the next result we provide a class of initial data for the problem $\mathcal{Q}_{\mathcal{E}}$ which gives a bound for Q_y uniform in time.

LEMMA 2.4. *If $Q_0' < Q_0/y$ for $y \in (0,1]$, then the solution Q, ϑ of the problem $\mathcal{Q}_{\mathcal{E}}$ satisfies*

$$Q_y \leq Q/y \quad \text{in } D_T.$$

Proof. Denote by b the auxiliary quantity $b(y,t) := Q(y,t)/y$. It is easy to show that

$$b_t = \vartheta y^{2-2/d} b_{yy} + (2\vartheta y^{1-2/d} + yb) b_y + b^2. \tag{23}$$

Following the ideas of [10], we define $w := yQ_y - Q$, which satisfies

$$w_t = y^{1-2/d}\vartheta w_{yy} + \left(b_y - \frac{2}{d}\vartheta\right) w_y + (yb_y + b)w.$$

To apply the maximum principle [12, Lemma 2.1] we should check that $w(0,t) \le 0$, $w(y,0) \le 0$, $w(1,t) \le 0$ and $yb_y + b$ is a bounded function on $\overline{D}_T$. The first two inequalities follow from the assumptions on Q_0 and Q (recall that Q is the integrated density). To prove $w(1,t) \le 0$, note that $b(y,t) > 1$ for $y < 1$. In fact, $b(1,t) = 1$ and $(b(y,0))' = (Q_0(y)/y)' < 0$. Hence, $b(\cdot,t)$ is a decreasing function for $t \in (0,\delta)$, $0 < \delta < T$. Thus, $1 < b(0,t)$. It is easy to check that the constant function equal to 1 is a subsolution of (23) on $[0,1] \times [0,\delta)$. The strong maximum principle implies that $b(y,\delta) > 1$ for $y < 1$. Thus 1 is a subsolution on D_T.

Applying the Hopf maximum principle we find that $b_y(1,t) = Q_y - Q = w(1,t) < 0$. Since the initial data $(Q_0)' = u_0\sigma_d/d$ is bounded, by the theorem on the regularity of solutions of parabolic systems (cf. [3, Theorem 2]) we get the local bound on $yb_y + b = Q_y = u\sigma_d/d$. ∎

Now we prove the existence of initial data which guarantee the existence of global solutions with bounded Q_y and the temperature ϑ. We begin with the three-dimensional case. It was shown in [9, Theorem 5.5] that if $(Q_0)'$ is bounded, the initial temperature ϑ_0 is sufficiently large and there exists $B > 0$ such that

$$Q_0(y) \le \frac{y(1+B)}{y^{1/3} + B},$$

then there exists a global solution Q, ϑ which satisfies

$$Q(y,t) \le \frac{y(1+B)}{y^{2/3} + B}, \qquad 0 < c < \vartheta < C. \tag{24}$$

Obviously, we can also assume that $(Q_0)' \le Q_0/y$, and if the initial temperature is sufficiently large, we can guarantee that the energy $\mathcal{E}$ is as large as we wish.

For example $Q_0(y) = y$, i.e. $u_0(x) = 3\pi/4$, and $\vartheta \gg 1$ satisfy the assumptions of Theorem 2.1.

In the proof of the existence of Q satisfying (24) the following auxiliary lemmas are used.

Lemma 2.5 ([9, Proposition 5.3]). *Suppose Q^i, $i = 1, 2$, is a solution of the problem*

$$(25)\qquad \begin{aligned} &Q^i_t = y^{1-2/d}\vartheta^i(t)Q_{yy} + QQ_y,\\ &Q^i(y,0) = Q^i_0, \quad Q_i(0,t) = 0, \quad Q_i(1,t) = 1, \end{aligned}$$

with a fixed continuous $\vartheta^i(t) > \delta > 0$. If $\vartheta^1(t) \le \vartheta^2(t)$, $Q^1_0 \ge Q^2_0$, and either Q^1_y or Q^2_y is bounded, then $Q^1 \ge Q^2$.

LEMMA 2.6 ([9, Proposition 5.4]). *Let Q, ϑ be a solution of $\mathcal{Q}_{\mathcal{E}}$ with the initial data Q_0, ϑ_0. Then*

$$\vartheta(t) \ge \vartheta_0 \exp\Big(-\int_0^1 Q_0' \log Q_0'\Big).$$

These lemmas together with Lemma 2.4 guarantee the existence of initial data satisfying the assumptions of Theorem 2.1 in the two-dimensional case.

REMARK. In fact, [9, Propositions 5.3 and 5.4] was proved for $d = 3$, but it is easy to check that the arguments used in the proofs work for all $d > 1$.

LEMMA 2.7. *Let $d = 2$. There exist initial data Q_0 and ϑ_0 such that the solution $Q(y,t)$ of $\mathcal{Q}_{\mathcal{E}}$ is global in time and satisfies*

$$(26)\qquad Q(y,t) \le \frac{Ay}{y^2+B} \quad \textit{for some positive constants } A, B.$$

Proof. Consider the auxiliary problem

$$(27)\quad q_t = y\widetilde{\vartheta}q_{yy} + qq_y, \quad q(0,t) = 0, \quad q(1,t) = 1, \quad q(y,0) = q_0(y)$$

with a given constant $\widetilde{\vartheta} > 1/(8\pi)$. Putting $\tau = t\widetilde{\vartheta}$, $q = \widetilde{\vartheta}\bar{q}$, we transform (27) into the problem

$$(28)\qquad \begin{aligned} &\bar{q}_\tau = y\bar{q}_{yy} + \bar{q}\bar{q}_y,\\ &\bar{q}(0,\tau) = 0, \quad \bar{q}(1,\tau) = 1/\widetilde{\vartheta}, \quad \bar{q}(y,0) = q_0(y)/\widetilde{\vartheta} =: \bar{q}_0(y). \end{aligned}$$

It follows from [4, Theorem 1(ii)] that if $\bar{q}_0'(y) \le AB/(y+B)^2$ for some $A < 8\pi$, $B > 0$, $B(8 - A/\pi) \ge 16$, and $\bar{q}_0(y) \ge y^k/\widetilde{\vartheta}$ for some $k \ge 1$, then the problem (28) has a solution $\bar{q}$ such that $\bar{q}_y$ is uniformly bounded and $\bar{q}(y,\tau) \le Cy/(y^2+B)$ (cf. the proof of [4, Theorem 1]). Hence

$$q(y,t) \le \frac{Ay}{y^2+B},$$

where $A = \widetilde{\vartheta}C$.

Now we choose the initial data Q_0, ϑ_0 such that $\vartheta(t) \ge 1/(8\pi)$ (cf. Lemma 2.6). It follows from the comparison principle (Lemma 2.5) that the solution $Q(y,t)$ of (10)–(13) satisfies the estimates

$$Q(y,t) \le q(y,t) \le \frac{Ay}{y^2+B}. \ \blacksquare$$

Using Lemmas 2.7 and 2.4 we are able to construct the initial data which guarantee the existence of global solutions converging to the stationary state, for example for $d = 2$, $Q_0(y) = y$ and $\vartheta_0 > 1/(8\pi)$ will do.

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