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ITERATED TILTED AND TILTED STABLY HEREDITARY ALGEBRAS

BY

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Abstract. We prove that a stably hereditary bound quiver algebra $A = KQ/I$ is iterated tilted if and only if (Q, I) satisfies the clock condition, and that in this case it is of type Q. Furthermore, A is tilted if and only if (Q, I) does not contain any double-zero.

Introduction. Two algebras A and B over a field K are called *stably equivalent* if there is a K-linear equivalence between the quotient categories $\underline{\mathrm{mod}}\, A$ and $\underline{\mathrm{mod}}\, B$ deduced from the categories of modules $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$ by annihilating the projective modules. An algebra A is called *stably hereditary* if it is stably equivalent to a hereditary algebra H. Stably hereditary algebras have been studied from many points of view (see, for instance, [5, 7, 17, 18]), but not from the tilting point of view. Tilted and iterated tilted algebras have been one of the main objects of study in representation theory since their introduction (see, for instance, [2, 9, 11]). Thus, it is natural to ask whether a stably hereditary algebra is iterated tilted or not. For instance, it is shown in [3] that an iterated tilted algebra of type $\widetilde{\mathbb{A}}_n$ satisfies the clock condition, that is, on the unique cycle of its bound quiver, the number of clockwise oriented relations equals the number of counterclockwise oriented relations. Furthermore, it is shown in [15, 20] that if such an algebra is tilted, then its bound quiver cannot contain a double-zero, that is, two consecutive monomial relations pointing in the same direction. In this paper, we prove the following result:

THEOREM. *Let $A = KQ/I$ be a stably hereditary algebra. Then*

(a) *A is iterated tilted if and only if (Q, I) satisfies the clock condition. In this case the type of A is Q.*

(b) *A is tilted if and only if (Q, I) satisfies the clock condition and does not contain any double-zero.*

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It is worthwhile to note that a hereditary algebra that is tilting-cotilting equivalent to A is connected when A is connected, which is not the case in general for the hereditary algebra H that is stably equivalent to A.

This paper consists of two sections. The first is devoted to preliminaries and the second to the proof of the main result.

1. Preliminaries

1.1. *Notation.* All algebras in this paper are basic, connected, associative, finite-dimensional algebras with identities over a fixed algebraically closed field K, and all modules are finitely generated right modules. For an algebra A, we denote by $\operatorname{mod} A$ its module category, by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable objects in $\operatorname{mod} A$, and by $\operatorname{proj} A$ the full subcategory of $\operatorname{ind} A$ consisting of the projective objects. Given an A-module M, we denote by $\operatorname{pd} M$ its projective dimension and by $\operatorname{id} M$ its injective dimension.

We recall that a *quiver* Q is defined by a set of points Q_0 and a set of arrows Q_1. A *relation* from $x \in Q_0$ to $y \in Q_0$ is a linear combination of paths from x to y of length at least two. Let I denote an ideal of KQ generated by a set of relations; then the pair (Q, I) is called a *bound quiver*. A relation $\varrho = \sum_{i=1}^{m} \lambda_i w_i$ in I (where the λ_i are non-zero scalars and the w_i are paths) with $m \geq 2$ is called *minimal* if there is no proper non-empty subset $J \subset \{1, \ldots, m\}$ such that $\sum_{i \in J} \lambda_i w_i$ is also a relation in I, and is called *monomial* if it equals a path ($m = 1$). It is well known that, if A is a basic and connected finite-dimensional K-algebra, then there exists a connected bound quiver (Q_A, I) such that $A \cong KQ_A/I$ (see [6]). For a point a in the quiver of A, we denote by $P(a)$ the corresponding indecomposable projective A-module, and by $I(a)$ the corresponding indecomposable injective A-module. Given an A-module M, we denote by $\operatorname{Supp} M$ the full bound subquiver of Q_A generated by the points a such that $\operatorname{Hom}_A(P(a), M) \neq 0$. We say that A is *triangular* whenever its quiver Q_A has no oriented cycles.

For an arrow α of Q, we denote by $s(\alpha)$ its source, by $t(\alpha)$ its target and by α^{-1} its formal inverse of source $s(\alpha^{-1}) = t(\alpha)$ and of target $t(\alpha^{-1}) = s(\alpha)$. A *walk* in Q is a sequence $w = c_1 \ldots c_n$ with c_i an arrow or the inverse of an arrow such that $t(c_i) = s(c_{i+1})$ for all i such that $1 \leq i < n$. A walk w in Q is called *reduced* if $w = c_1 \ldots c_n$ with $c_i \neq c_{i+1}^{-1}$ for all i such that $1 \leq i < n$. It is called a *non-zero walk* if it does not contain any zero-relation. Finally, a reduced walk is called a *double-zero* if it contains exactly two zero-relations that point in the same direction in w. The double-zero has been used for the classification of tilted and quasi-tilted special biserial algebras, string algebras and gentle algebras [1, 12–15].

For general properties of the category $\operatorname{mod} A$ of finitely generated right A-modules, we refer the reader to [6, 19]. For tilted and iterated tilted algebras, we refer the reader to [2, 9, 11, 19].

1.2. *The bound quiver of a stably hereditary algebra.* In our main results, we use some properties of the bound quiver of a stably hereditary algebra that are easy to identify. This subsection is therefore devoted to the bound quiver of a stably hereditary algebra.

Let $A = KQ/I$ be a stably hereditary algebra. Then, by [7], we have $I = I_{\Sigma_A}$ with

$$\Sigma_A = \{x \in Q_0 \mid S(x) \text{ is a non-projective submodule of } A\},$$
$$I_{\Sigma_A} = \langle \alpha\beta \mid t(\alpha) \in \Sigma_A,\ s(\beta) \in \Sigma_A \rangle.$$

That is, I_{Σ_A} is the ideal generated by all paths $\alpha\beta$ with $t(\alpha) = s(\beta) \in \Sigma_A$. In particular, A is a monomial algebra (that is, I_{Σ_A} is generated by monomial relations).

Definition. A cycle C in (Q, I_{Σ_A}) satisfies the *clock condition* if the number of clockwise oriented relations on C equals the number of counterclockwise oriented relations. We say that (Q, I_{Σ_A}) satisfies the *clock condition* if all cycles in (Q, I_{Σ_A}) satisfy the clock condition.

The following theorem, due to Skowroński, allows us to characterize the bound quiver of A using the clock condition.

Theorem ([21]). *For an algebra A tilting equivalent to a hereditary or canonical algebra, and for any idempotent e of A and special cycle C in G_{eAe}, C satisfies the clock condition. In particular, Q_A has no oriented cycles.* ∎

Actually, in our context of stably hereditary algebras, it is easily seen to be equivalent to say that if A is tilting equivalent to a hereditary algebra, then any cycle of (Q, I_{Σ_A}) satisfies the clock condition. Therefore, if (Q, I_{Σ_A}) does not satisfy the clock condition, A is not iterated tilted. Hence, from now on, we suppose that (Q, I_{Σ_A}) satisfies the clock condition. In particular, A is a triangular algebra.

We want to decompose Q into maximal subquivers which do not contain any relation, that is, the ordinary quivers of the algebras $A_1, \ldots, A_n$ such that A is stably equivalent to $A_1 \times \ldots \times A_n$ (see [7]).

Let $\alpha \in Q_1$, and let Q_α be the subquiver of Q such that

$$(Q_\alpha)_1 = \left\{ \beta \in Q_1 \;\middle|\; \begin{array}{l} \text{there exists a non-zero walk } w \text{ such that} \\ w = \alpha^* w' \beta^* \text{ where } \alpha^* \in \{\alpha, \alpha^{-1}\},\ \beta^* \in \{\beta, \beta^{-1}\} \end{array} \right\}.$$

Remarks. (1) Since A is a stably hereditary algebra, we easily see that Q_α is a full subquiver of Q. Moreover, since (Q, I_{Σ_A}) satisfies the clock condition, Q_α is convex in Q.

(2) Since (Q, I_{Σ_A}) satisfies the clock condition, every walk w containing zero-relations that point in the same direction in w:

$$w = \bullet \longrightarrow p_1 \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow p_2 \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow p_s \longrightarrow \bullet$$

is such that $p_1 \neq p_2 \neq \ldots \neq p_s$. In particular, every double-zero in (Q, I_{Σ_A}) contains two distinct vertices of Σ_A.

(3) Let $\alpha \in Q_1$ and $\beta \in (Q_\alpha)_1$. Then $Q_\alpha = Q_\beta$.

(4) If $Q_\alpha \neq Q_\beta$, then $(Q_\alpha)_1 \cap (Q_\beta)_1 = \emptyset$ and $(Q_\alpha)_0 \cap (Q_\beta)_0 \subseteq \Sigma_A$.

(5) For all $\alpha \in Q_1$ and $x \in (Q_\alpha)_0 \setminus \Sigma_A$, we have $I(x) \in \operatorname{ind} KQ_\alpha$ and $P(x) \in \operatorname{ind} KQ_\alpha$.

Let $\alpha_1, \ldots, \alpha_t \in Q_1$ be such that $Q_{\alpha_i} \neq Q_{\alpha_j}$ when $i \neq j$, and such that for all $\beta \in Q_1$, there exists $i \in \{1, \ldots, t\}$ such that $\beta \in (Q_{\alpha_i})_1$. Let $I_\alpha = \{\alpha_1, \ldots, \alpha_t\}$.

Then $Q_1 = \bigcup_{i=1}^t (Q_{\alpha_i})_1$ and $Q_0 = \bigcup_{i=1}^t (Q_{\alpha_i})_0$, and it follows from the remarks above and from [7] that A is stably equivalent to $KQ_{\alpha_1} \times \ldots \times KQ_{\alpha_t}$.

Example. Let (Q, I) be the following bound quiver:

$$\begin{tikzcd}
11 \arrow[dr, "\beta_{11}"'] & & & 6 \arrow[r, "\beta_5"] & 5 \arrow[dr, "\beta_4"] & & & 1 \\
& 10 \arrow[r, "\beta_{10}"] & 9 \arrow[ur, "\beta_6"] \arrow[dr, "\beta_9"'] & & & 4 \arrow[r, "\beta_3"] & 3 \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"'] & \\
12 \arrow[ur, "\beta_{12}"] & & & 8 \arrow[r, "\beta_8"'] & 7 \arrow[ur, "\beta_7"'] & & & 2
\end{tikzcd}$$

A possible choice for I_α is $I_\alpha = \{\beta_1, \beta_3, \beta_6, \beta_{11}\}$, and then we have

$$Q_{\beta_{11}} = \begin{tikzcd}
11 \arrow[dr, "\beta_{11}"] & \\
& 10 \\
12 \arrow[ur, "\beta_{12}"'] &
\end{tikzcd}
\qquad
Q_{\beta_6} = \begin{tikzcd}
& & 6 & \\
10 \arrow[r, "\beta_{10}"] & 9 \arrow[ur, "\beta_6"] \arrow[dr, "\beta_9"'] & & \\
& & 8 \arrow[r, "\beta_8"'] & 7
\end{tikzcd}$$

$$Q_{\beta_3} = \begin{tikzcd}
6 \arrow[r, "\beta_5"] & 5 \arrow[dr, "\beta_4"] & & \\
& & 4 \arrow[r, "\beta_3"] & 3 \\
& 7 \arrow[ur, "\beta_7"'] & &
\end{tikzcd}
\qquad
Q_{\beta_1} = \begin{tikzcd}
& 1 \\
3 \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"'] & \\
& 2
\end{tikzcd}$$

Now, we want to characterize (Q, I_{Σ_A}) in such a way as to show that there always exists $\alpha_i \in I_\alpha$ such that any indecomposable KQ_{α_i}-module M is of projective dimension at most one when considered as an A-module, and that, dually, there always exists $\alpha_j \in I_\alpha$ such that any indecomposable KQ_{α_j}-module N is of injective dimension at most one when considered as an A-module. This characterization will also allow us to state a sufficient and necessary condition to see if (Q, I_{Σ_A}) contains a double-zero.

So, first, let

$$V_1 = \{\alpha_i \in I_\alpha \mid \text{there exists } \beta \in (Q_{\alpha_i})_1 \text{ such that } t(\beta) \in \Sigma_A\},$$
$$V_2 = \{\alpha_i \in I_\alpha \mid \text{there exists } \beta \in (Q_{\alpha_i})_1 \text{ such that } s(\beta) \in \Sigma_A\}.$$

In the previous example, we have $V_1 = \{\beta_3, \beta_6, \beta_{11}\}$ and $V_2 = \{\beta_1, \beta_3, \beta_6\}$.

1.3. LEMMA. (a) *There exists $\alpha_i \in I_\alpha \cap (V_2 \setminus V_1)$.*

(b) *There exists $\alpha_i \in I_\alpha \cap (V_1 \setminus V_2)$.*

Proof. (a) Let $\alpha_{i1} \in I_\alpha$. If $\alpha_{i1} \in V_2 \setminus V_1$, we are done. Otherwise, $Q_{\alpha_{i1}}$ contains an arrow β_{i1} such that $t(\beta_{i1}) \in \Sigma_A$:

$$\begin{tikzcd} \bullet \arrow[r, "\beta_{i1}"'] \arrow[rr, bend left, dash, dashed] & p_{i1} \arrow[r, "\beta'_{i1}"'] & \bullet \end{tikzcd}$$

Let $\alpha_{i2} \in I_\alpha \setminus \{\alpha_{i1}\}$ be such that $\beta'_{i1} \in (Q_{\alpha_{i2}})_1$. If $\alpha_{i2} \in V_2 \setminus V_1$, we are done. Otherwise, $Q_{\alpha_{i2}}$ contains an arrow β_{i2} such that $t(\beta_{i2}) \in \Sigma_A$, and so there is a double-zero of the form

$$\begin{tikzcd} \bullet \arrow[r, "\beta_{i1}"'] \arrow[rr, bend left, dash, dashed] & p_{i1} \arrow[r, "\beta'_{i1}"'] & \bullet \arrow[r, dotted, dash] & \bullet \arrow[r, "\beta_{i2}"'] \arrow[rr, bend left, dash, dashed] & p_{i2} \arrow[r, "\beta' i2"'] & \bullet \end{tikzcd} \qquad (p_{i1} \neq p_{i2}).$$

We then consider $\alpha_{i3} \in I_\alpha \setminus \{\alpha_{i1}, \alpha_{i2}\}$ such that $\beta'_{i2} \in (Q_{\alpha_{i3}})_1$. We repeat the argument, and since $|I_\alpha|$ and $|\Sigma_A|$ are finite, the statement follows from induction.

(b) Dual proof. ■

1.4. LEMMA. *The bound quiver (Q, I_{Σ_A}) does not contain any double-zero if and only if $V_1 \cap V_2 = \emptyset$.*

Proof. If $V_1 \cap V_2 \neq \emptyset$, there exists $\alpha_i \in I_\alpha$ such that Q_{α_i} contains an arrow whose target is in Σ_A, and an arrow whose source is in Σ_A. This implies the existence of a double-zero in (Q, I_{Σ_A}).

On the other hand, if (Q, I_{Σ_A}) contains a double-zero:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left, dash, dashed] & p_1 \arrow[r, "\beta_1"'] & \bullet \arrow[r, dotted, dash] & \bullet \arrow[r, "\beta_2"'] \arrow[rr, bend left, dash, dashed] & p_2 \arrow[r] & \bullet \end{tikzcd}$$

then we see that there exists a non-zero walk containing the arrows β_1 and β_2. Hence there exists $\alpha_i \in I_\alpha$ such that $\beta_1, \beta_2 \in (Q_{\alpha_i})_1$, and so $\alpha_i \in V_1 \cap V_2$. ■

1.5. LEMMA. (a) *Let $\alpha_i \in I_\alpha \cap (V_2 \setminus V_1)$ and let $M \in \operatorname{ind} KQ_{\alpha_i}$. Then $\operatorname{pd} M_A \leq 1$ and $\tau_A M \in \operatorname{ind} KQ_{\alpha_i}$.*

(b) *Let $\alpha_i \in I_\alpha \cap (V_1 \setminus V_2)$ and let $M \in \operatorname{ind} KQ_{\alpha_i}$. Then $\operatorname{id} M_A \leq 1$ and $\tau_A^{-1} M \in \operatorname{ind} KQ_{\alpha_i}$.*

Proof. (a) Let P be the projective cover of M and $f : P \to M$ be the canonical projection. Since $M \in \operatorname{ind} KQ_{\alpha_i}$ and $\alpha_i \in V_2 \setminus V_1$, we have $P \in \operatorname{proj} KQ_{\alpha_i}$.

Assume that $\operatorname{pd} M_A > 1$. Then $\operatorname{Ker} f$ has a non-projective direct summand S. Since A is stably hereditary, $S = S(p)$ with $p \in \Sigma_A$.

Hence $S(p)$ is a direct summand of $\operatorname{soc} P$, and therefore Q_{α_i} contains an arrow whose target is in Σ_A, which contradicts the fact that $\alpha_i \in V_2 \setminus V_1$. Hence $\operatorname{pd} M \leq 1$.

Therefore $0 \to \operatorname{Ker} f \to P \to M \to 0$ is a minimal projective resolution of M. If there exists $p \in (\operatorname{Supp} P)_0 \cap \Sigma_A$, then $S(p)$ is a direct summand of $\operatorname{top} M$ since $\alpha_i \in V_2 \setminus V_1$. Hence $p \notin (\operatorname{Supp} \operatorname{Ker} f)_0$.

Therefore, there exists $V \subseteq (Q_{\alpha_i})_0 \setminus \Sigma_A$ such that $\operatorname{Ker} f = \bigoplus_{v \in V} P(v)$.

Applying the Nakayama functor ν_A yields the exact sequence

$$0 \to \tau_A M \to \nu_A \operatorname{Ker} f \to \nu_A P$$

and $\nu_A \operatorname{Ker} f = \nu_A(\bigoplus_{v \in V} P(v)) \cong \bigoplus_{v \in V} I(v)$.

Since $V \subseteq (Q_{\alpha_i})_0 \setminus \Sigma_A$, we have $\bigoplus_{v \in V} I(v) \in \operatorname{mod} KQ_{\alpha_i}$. Therefore $\tau_A M \in \operatorname{ind} KQ_{\alpha_i}$.

(b) Dual proof. ■

2. Main results. To show that A is iterated tilted of type Q, we need a particular tilting (or cotilting) A-module T, and we give its construction in Lemma 2.1. We see in Lemma 2.3 that $\operatorname{End} T$ is still a stably hereditary algebra, with the same ordinary quiver as A. Moreover, this new stably hereditary algebra has less non-projective simple submodules. This is the key to the proof that A is iterated tilted.

2.1. Lemma. (a) *Let $J \subseteq \{1, \ldots, t\}$ be such that for all $j \in J$, $\alpha_j \in V_2 \setminus V_1$. Let $Q_{J_0} = \bigcup_{j \in J} (Q_{\alpha_j})_0$. Then*

$$T = \Big(\bigoplus_{i \in Q_0 \setminus Q_{J_0}} P(i) \Big) \oplus \Big(\bigoplus_{p \in \Sigma_A \cap Q_{J_0}} S(p) \Big) \oplus \Big(\bigoplus_{k \in Q_{J_0} \setminus \Sigma_A} I(k) \Big)$$

is a tilting A-module.

(b) *Let $J \subseteq \{1, \ldots, t\}$ be such that for all $j \in J$, $\alpha_j \in V_1 \setminus V_2$. Let $Q_{J_0} = \bigcup_{j \in J} (Q_{\alpha_j})_0$. Then*

$$T = \Big(\bigoplus_{i \in Q_{J_0} \setminus \Sigma_A} P(i) \Big) \oplus \Big(\bigoplus_{p \in \Sigma_A \cap Q_{J_0}} S(p) \Big) \oplus \Big(\bigoplus_{k \in Q_0 \setminus Q_{J_0}} I(k) \Big)$$

is a cotilting A-module.

Proof. (a) It is clear that T has $|Q_0|$ pairwise non-isomorphic indecomposable summands.

By Remark (5) and Lemma 1.5, we have $\operatorname{pd} T_A \leq 1$.

Let

$$P = \bigoplus_{i \in Q_0 \setminus Q_{J_0}} P(i), \qquad S = \bigoplus_{p \in \Sigma_A \cap Q_{J_0}} S(p), \qquad I = \bigoplus_{k \in Q_{J_0} \setminus \Sigma_A} I(k).$$

Then $\operatorname{Ext}^1_A(T,T) = \operatorname{Ext}^1_A(I \oplus S, P \oplus S) \cong \operatorname{DHom}_A(P \oplus S, \tau_A I \oplus \tau_A S)$. It follows from the proof of Lemma 1.5 that the socles of $\tau_A I$ and $\tau_A S$ have no non-projective simple summand which is a submodule of A (that is, has the form $S(p)$ with $p \in \Sigma_A$). This and Lemma 1.5 imply that $\operatorname{Hom}_A(P, \tau_A I) = 0$, $\operatorname{Hom}_A(P, \tau_A S) = 0$, $\operatorname{Hom}_A(S, \tau_A I) = 0$ and $\operatorname{Hom}_A(S, \tau_A S) = 0$.

Hence $\operatorname{Ext}^1_A(T,T) = 0$, and so T is a tilting A-module.

(b) Dual proof. ∎

When (Q, I_{Σ_A}) does not contain any double-zero, the tilting module and the cotilting module of the previous lemma coincide. This is the following corollary:

2.2. Corollary. *Assume that (Q, I_{Σ_A}) does not contain any double-zero, and let $J_{V_1} = \{x \in (Q_{\alpha_i})_0 \setminus \Sigma_A \mid \alpha_i \in V_1\}$ and $J_{V_2} = \{x \in (Q_{\alpha_i})_0 \setminus \Sigma_A \mid \alpha_i \in V_2\}$. Then*

$$T = \Big(\bigoplus_{i \in J_{V_1}} P(i)\Big) \oplus \Big(\bigoplus_{p \in \Sigma_A} S(p)\Big) \oplus \Big(\bigoplus_{k \in J_{V_2}} I(k)\Big)$$

is a tilting and cotilting A-module. ∎

2.3. Lemma. (a) *Let $J \subseteq \{1, \ldots, t\}$ be such that for all $j \in J$, $\alpha_j \in V_2 \setminus V_1$. Let $Q_{J_0} = \bigcup_{j \in J} (Q_{\alpha_j})_0$, and*

$$T = \Big(\bigoplus_{i \in Q_0 \setminus Q_{J_0}} P(i)\Big) \oplus \Big(\bigoplus_{p \in \Sigma_A \cap Q_{J_0}} S(p)\Big) \oplus \Big(\bigoplus_{k \in Q_{J_0} \setminus \Sigma_A} I(k)\Big).$$

Then $\operatorname{End} T \cong KQ/I_{\Sigma_{\operatorname{End} T}}$ *with $\Sigma_{\operatorname{End} T} = \Sigma_A \setminus (\Sigma_A \cap Q_{J_0})$, and $(Q, I_{\Sigma_{\operatorname{End} T}})$ respects the clock condition.*

(b) *Let $J \subseteq \{1, \ldots, t\}$ be such that for all $j \in J$, $\alpha_j \in V_1 \setminus V_2$. Let $Q_{J_0} = \bigcup_{j \in J} (Q_{\alpha_j})_0$, and*

$$T = \Big(\bigoplus_{i \in Q_{J_0} \setminus \Sigma_A} P(i)\Big) \oplus \Big(\bigoplus_{p \in \Sigma_A \cap Q_{J_0}} S(p)\Big) \oplus \Big(\bigoplus_{k \in Q_0 \setminus Q_{J_0}} I(k)\Big).$$

Then $\operatorname{End} T \cong KQ/I_{\Sigma_{\operatorname{End} T}}$ *with $\Sigma_{\operatorname{End} T} = \Sigma_A \setminus (\Sigma_A \cap Q_{J_0})$, and $(Q, I_{\Sigma_{\operatorname{End} T}})$ respects the clock condition.*

Proof. We start by showing that $Q_{\operatorname{End} T} = Q$.

We already know that $|Q_0| = |(Q_{\operatorname{End} T})_0|$. We identify $i \in Q_0$ with the corresponding direct summand $T(i)$ of T. We have $T(i) = P(i)$ if $i \in Q_0 \setminus Q_{J_0}$, $T(i) = S(i)$ if $i \in \Sigma_A \cap Q_{J_0}$, and $T(i) = I(i)$ if $i \in Q_{J_0} \setminus \Sigma_A$.

First, let us show that $Q_1 \subseteq (Q_{\operatorname{End} T})_1$. We have $Q_1 = \bigcup_{i=1}^t (Q_{\alpha_i})_1$. Let $\alpha : k \to l \in Q_1$. There exists $\alpha_i \in I_\alpha$ such that $\alpha \in (Q_{\alpha_i})_1$.

If $i \in J$, we have $T(l) = I(l)$, and $T(k) = S(k)$ if $k \in \Sigma_A \cap (Q_{\alpha_i})_0$, otherwise $T(k) = I(k)$. In both cases, $T(k)$ is a direct summand of $I(l)/\operatorname{soc} I(l)$.

Hence there exists an irreducible morphism from $I(l)$ to $I(k)$ in $\operatorname{mod} A$. Therefore we have an arrow $\alpha' : T(k) \to I(l)$ in $Q_{\operatorname{End} T}$.

If $i \notin J$, we have $T(k) = P(k)$, and $T(l) = P(l)$ if $l \in Q_0 \setminus Q_{J_0}$, otherwise $l \in \Sigma_A \cap Q_{J_0}$ and $T(l) = S(l)$.

When $T(l) = P(l)$, we can have $l \in \Sigma_A$ or not. If not, then $P(l)$ is a direct summand of $\operatorname{rad} P(k)$, and so we have an arrow $\alpha' : P(k) \to P(l)$ in $Q_{\operatorname{End} T}$.

If $l \in \Sigma_A$, then $S(l)$ is a direct summand of $\operatorname{rad} P(k)$ (but remember, here, $T(l) = P(l)$). Since $S(l) = P(l)/\operatorname{rad} P(l)$, we have a morphism $g : P(l) \to S(l)$ in $\operatorname{mod} A$. It is clear that g does not factorize through any other indecomposable projective direct summand of T. Moreover, since $l \in \Sigma_A \setminus Q_{J_0}$, $S(l)$ cannot be a direct summand of the top of an injective direct summand of T. Therefore g cannot factorize through an injective direct summand of T, and of course cannot factorize through a simple direct summand of T. Hence, since we have an irreducible morphism from $S(l)$ to $P(k)$ and since g does not factorize through any other direct summand of T, we have an arrow $\alpha' : P(k) \to P(l)$ in $Q_{\operatorname{End} T}$.

Finally, there is the case $l \in \Sigma_A \cap Q_{J_0}$. In this case, we have $T(l) = S(l)$ and $S(l)$ is a direct summand of $\operatorname{rad} P(k)$. Therefore we have an arrow $\alpha' : P(k) \to S(l)$ in $Q_{\operatorname{End} T}$.

This shows that $Q_1 \subseteq (Q_{\operatorname{End} T})_1$. Let us show the reverse inclusion.

Let $T(k)$ and $T(l)$ be indecomposable direct summands of T such that there exists an arrow $\alpha' : T(k) \to T(l)$ in $Q_{\operatorname{End} T}$. The possible cases are:

(1) $T(k) = P(k)$, $T(l) = P(l)$;
(2) $T(k) = P(k)$, $T(l) = S(l)$;
(3) $T(k) = S(k)$, $T(l) = I(l)$;
(4) $T(k) = I(k)$, $T(l) = I(l)$.

In case (1), we have a morphism from $P(l)$ to $P(k)$ in $\operatorname{mod} A$, and this morphism does not factorize through any other direct summand of T. Assume there is no arrow from k to l in Q. Since we have a morphism from $P(l)$ to $P(k)$, we have a non-zero path $v : k \to \cdots \to l$ of length at least 2 in (Q, I_{Σ_A}). We have $l, k \in Q_0 \setminus Q_{J_0}$, and the same holds for every vertex j lying on v. Therefore $T(j) = P(j)$ for all j lying on v, and all morphisms from $P(l)$ to $P(k)$ must factorize through these modules, a contradiction. Hence there is an arrow $\alpha : k \to l$ in Q.

We prove the other cases similarly. Thus, $(Q_{\operatorname{End} T})_1 \subseteq Q_1$.

Therefore, we have $Q_{\operatorname{End} T} = Q$, and so $\operatorname{End} T \cong KQ/J$ with J an admissible ideal. It remains to show that $J = I_{\Sigma_{\operatorname{End} T}}$.

Let us first show that $I_{\Sigma_{\operatorname{End} T}} \subseteq J$.

Let $\varrho = \alpha\beta \in I_{\Sigma_{\operatorname{End} T}}$ (then $t(\alpha) = s(\beta) = p \in \Sigma_{\operatorname{End} T}$). Let i be the source of α and j be the target of β. Since $p \in \Sigma_{\operatorname{End} T}$, we have $T(i) = P(i)$

and $T(j) = P(j)$ or $T(j) = S(j)$. In both cases, we see that $P(i)$ has no composition factors in common with $T(j)$. Hence $\mathrm{Hom}_A(T(j), P(i)) = 0$, and so $\alpha\beta \in J$ and $I_{\Sigma_{\mathrm{End}\,T}} \subseteq J$.

Now it remains to show the reverse inclusion.

Let $\varrho \in J$ be a minimal relation of source $T(k)$ and of target $T(l)$. In particular, this means that ϱ contains a path from k to l in Q. Hence the possible cases are:

(1) $T(k) = P(k)$, $T(l) = P(l)$;
(2) $T(k) = P(k)$, $T(l) = S(l)$;
(3) $T(k) = P(k)$, $T(l) = I(l)$;
(4) $T(k) = S(k)$, $T(l) = I(l)$;
(5) $T(k) = I(k)$, $T(l) = I(l)$.

Let us identify ϱ to the following full subquiver of Q:

$$\begin{tikzcd}
 & k_{11} \arrow[r] & k_{12} \arrow[r, dotted, no head] & k_{1(s_1-1)} \arrow[r] & k_{1s_1} \arrow[ddr] & \\
 & k_{21} \arrow[r] & k_{22} \arrow[r, dotted, no head] \arrow[dd, dotted, no head] & k_{2(s_2-1)} \arrow[r] \arrow[dd, dotted, no head] & k_{2s_2} \arrow[dr] & \\
k \arrow[uur] \arrow[ur] \arrow[dr] & & & & & l \\
 & k_{j1} \arrow[r] & k_{j2} \arrow[r, dotted, no head] & k_{j(s_j-1)} \arrow[r] & k_{js_j} \arrow[ur] &
\end{tikzcd}$$

In cases (1) and (2), we have $T(k_{im_i}) = P(k_{im_i})$ for all i and m_i such that $1 \leq i \leq j$, $1 \leq m_i \leq s_i$. Since ϱ is minimal, we deduce that all paths from k to l are non-zero in (Q, I_{Σ_A}). Moreover, since A is stably hereditary, we see that for all $i \in \{1, \ldots, j\}$, we have a monomorphism $f_i : T(l) \to P(k)$ such that $\{f_1, \ldots, f_j\}$ are linearly independent in $\mathrm{Hom}_A(T(l), P(k))$. But if ϱ is a non-monomial relation, the set $\{f_1, \ldots, f_j\}$ has to be linearly dependent, a contradiction.

Hence ϱ is monomial, and so $\mathrm{Hom}_A(T(l), P(k)) = 0$. By minimality of ϱ, we have $\varrho \in I_{\Sigma_A}$. But since $k \in Q_0 \setminus Q_{J_0}$ and l is in $Q_0 \setminus Q_{J_0}$ or in $\Sigma_A \cap Q_{J_0}$, we must have $\varrho \in I_{\Sigma_{\mathrm{End}\,T}}$.

In case (3), we have $T(k) = P(k)$ and $T(l) = I(l)$. Therefore, for each $i \in \{1, \ldots, j\}$, there exists exactly one k_{im_i} (with $1 \leq m_i \leq s_i$) such that $T(k_{im_i}) = S(k_{im_i})$, $T(k_{in}) = P(k_{in})$ for all n such that $1 \leq n < m_i$, and $T(k_{in}) = I(k_{in})$ for all n such that $m_i < n \leq s_i$. Therefore, for all $i \in \{1, \ldots, j\}$, $S(k_{im_i})$ is a direct summand of both $\mathrm{soc}\, P(k)$ and $\mathrm{top}\, I(l)$. This yields j linearly independant morphisms from $I(l)$ to $P(k)$, a contradiction which implies that ϱ must be monomial.

Therefore $j = 1$. But, as we already saw, $S(k_{1m_1})$ is a direct summand of both $\mathrm{soc}\, P(k)$ and $\mathrm{top}\, I(l)$, and so $\mathrm{Hom}_A(I(l), P(k)) \neq 0$. Hence there is no minimal relation of source $P(k)$ and of target $I(l)$.

For cases (4) and (5), we show similarly that ϱ must be a monomial relation, and that in fact there is no minimal relation of source $S(k)$ or $I(k)$ and of target $I(l)$.

Therefore $J \subseteq I_{\Sigma_{\operatorname{End} T}}$ and thus $\operatorname{End} T \cong KQ/I_{\Sigma_{\operatorname{End} T}}$.

The proof that $(Q, I_{\Sigma_{\operatorname{End} T}})$ respects the clock condition follows from the construction of (Q, I_{Σ_A}) and from the fact that for all $j \in J$, $\alpha_j \in V_2 \setminus V_1$.

(b) Dual proof. ■

2.4. Corollary. *If (Q, I_{Σ_A}) does not contain any double-zero, then A is tilted of type Q.*

Proof. Follows from Corollary 2.2 and Lemma 2.3. ■

The proof of the following lemma is similar to those of [15, 2.3] and [13, 2.6], which are done in the contexts of gentle and special biserial algebras respectively.

2.5. Lemma. *If (Q, I_{Σ_A}) contains a double-zero, then A is not tilted.*

Proof. Suppose that (Q, I_{Σ_A}) contains a double-zero of the form

$$1 \longrightarrow 2 \longrightarrow 3 \;\text{——}\; \bullet \cdots\cdots \bullet \;\text{——}\; t-2 \longrightarrow t-1 \longrightarrow t$$

with $t \geq 4$.

Since A is monomial, if $t = 4$, then, by [8], $\operatorname{gl.dim} A > 2$, hence A is not tilted.

Thus, suppose that $t \geq 5$, and let M be the indecomposable A-module of support

$$3 \;\text{——}\; \bullet \cdots\cdots \bullet \;\text{——}\; t-2$$

such that $M(x) = K$ for all x with $3 \leq x \leq t-2$ (this indecomposable module exists since A is monomial).

Let s be the source of $\operatorname{Supp} M$ such that there exists a path from s to $t-2$ in $\operatorname{Supp} M$. Since A is monomial, we see that the kernel of the canonical morphism $P(s) \to M$ has a non-projective direct summand and hence $\operatorname{pd} M > 1$.

Similarly, one proves that $\operatorname{id} N > 1$. Thus, by [10, III, 2.3], A is not quasi-tilted, and therefore is not tilted. ■

It is now possible to prove the main result of this paper:

2.6. Theorem. *Let $A = KQ/I$ be a stably hereditary algebra. Then*:

(a) *A is iterated tilted if and only if (Q, I) satisfies the clock condition. In this case the type of A is Q.*

(b) *A is tilted if and only if (Q, I) satisfies the clock condition and does not contain any double-zero.*

Proof. The first statement follows from Lemmata 1.3, 2.1 and 2.3, from [21, Cor. 1] (Theorem of the first section) and from the fact that $|\Sigma_A|$ is finite. The second statement follows from Corollary 2.4 and Lemma 2.5. ∎

We easily obtain the following corollary, which, in particular, answers a conjecture of Dieter Happel saying that an algebra $A = KQ/I$ with Q a tree and such that $\operatorname{rad}^2 A = 0$ is iterated tilted.

2.7. COROLLARY. *Let $A = KQ/I$ be a stably hereditary algebra with Q a tree. Then A is iterated tilted of type Q, and is tilted if and only if (Q, I) does not contain any double-zero.* ∎

Let $A = KQ/I$ be an algebra with Q a tree such that I is generated by paths of length two, and such that (Q, I) does not contain any double-zero. If A is not stably hereditary, then it is not tilted in general, as shown in the following example.

2.8. EXAMPLE. Let (Q, I) be the following quiver:

$$\begin{tikzcd}
4 \arrow[ddr, "\beta_1"] & & \\
5 \arrow[dr, "\beta_2"'] & & 1 \\
 & 3 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & \\
6 \arrow[ur, "\beta_3"] & & 2 \\
7 \arrow[uur, "\beta_4"'] & &
\end{tikzcd}$$

bound by $\beta_1\alpha_1 = 0$, $\beta_1\alpha_2 = 0$, $\beta_4\alpha_1 = 0$ and $\beta_4\alpha_2 = 0$. Then $A = KQ/I$ is isomorphic to

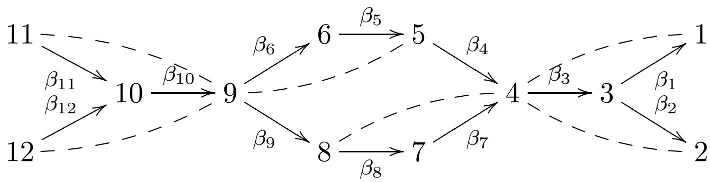
$$A_1 = H[S(3)][S(3)]$$

with H the hereditary algebra with ordinary quiver

$$\begin{tikzcd}
5 \arrow[dr, "\beta_2"'] & & 1 \\
 & 3 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & \\
6 \arrow[ur, "\beta_3"] & & 2
\end{tikzcd}$$

By the proof of [4, 3.2], the component of $\Gamma(\operatorname{mod} A_1)$ containing $S(3)$ is not directed. Hence, it follows from [16, 3.7] that for A_1 to be tilted, this component should be quasi-serial or obtained from a quasi-serial translation quiver by ray or coray insertions, which is not the case. Therefore $A_1 = H[S(3)][S(3)]$ is not tilted.

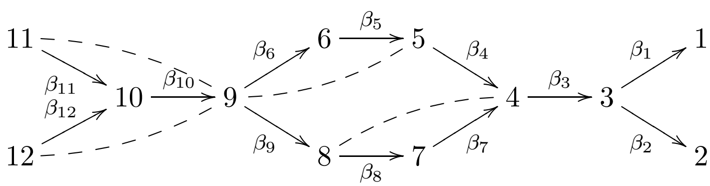
2.9. Example. Let $A = KQ/I_{\Sigma_A}$ be the stably hereditary algebra appearing in the first section:



We have $\Sigma_A = \{3, 6, 7, 10\}$, and $\{\beta_1, \beta_3, \beta_6, \beta_{11}\}$ is a possible choice for I_α. Hence $V_1 = \{\beta_3, \beta_6, \beta_{11}\}$, $V_2 = \{\beta_1, \beta_3, \beta_6\}$ and $V_2 \setminus V_1 = \{\beta_1\}$, $(Q_{\beta_1})_0 = \{1, 2, 3\}$, $\Sigma_A \cap (Q_{\beta_1})_0 = \{3\}$, and thus by Lemma 2.1(a),

$$T_1 = \Big(\bigoplus_{i=4}^{12} P(i)\Big) \oplus S(3) \oplus \Big(\bigoplus_{k=1}^{2} I(k)\Big)$$

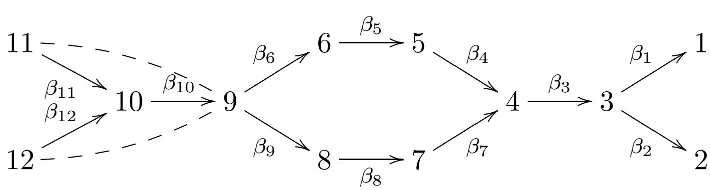
is a tilting A-module. Moreover, by Lemma 2.3(a), $\operatorname{End} T_1 \cong KQ/I_{\Sigma_{\operatorname{End} T_1}}$ with $\Sigma_{\operatorname{End} T_1} = \Sigma_A \setminus (\Sigma_A \cap (Q_{\beta_1})_0) = \{6, 7, 10\}$. Therefore $(Q, I_{\Sigma_{\operatorname{End} T_1}})$ is the following bound quiver:



Hence $\operatorname{End} T_1$ is a stably hereditary algebra with $\Sigma_{\operatorname{End} T_1} = \{6, 7, 10\}$, and $\{\beta_1, \beta_6, \beta_{11}\}$ is a possible choice for I_α; then $V_1 = \{\beta_6, \beta_{11}\}$, $V_2 = \{\beta_1, \beta_6\}$ and $V_2 \setminus V_1 = \{\beta_1\}$. Here we have $(Q_{\beta_1})_0 = \{1, 2, 3, 4, 5, 6, 7\}$ and $\Sigma_{\operatorname{End} T_1} \cap (Q_{\beta_1})_0 = \{6, 7\}$, and thus by Lemma 2.1(a),

$$T_2 = \Big(\bigoplus_{i=8}^{12} P(i)\Big) \oplus S(6) \oplus S(7) \oplus \Big(\bigoplus_{k=1}^{5} I(k)\Big)$$

is a tilting $\operatorname{End} T_1$-module. Moreover, by Lemma 2.3(a), we have $\operatorname{End} T_2 \cong KQ/I_{\Sigma_{\operatorname{End} T_2}}$ with $\Sigma_{\operatorname{End} T_2} = \Sigma_{\operatorname{End} T_1} \setminus (\Sigma_{\operatorname{End} T_1} \cap (Q_{\beta_1})_0) = \{10\}$. Therefore $(Q, I_{\Sigma_{\operatorname{End} T_2}})$ is the following bound quiver:



We see that $\operatorname{End} T_2$ is a stably hereditary algebra with $\Sigma_{\operatorname{End} T_2} = \{10\}$, and $\{\beta_1, \beta_{11}\}$ is a possible choice for I_α; then $V_1 = \{\beta_{11}\}$, $V_2 = \{\beta_1\}$ and thus $V_1 \cap V_2 = \emptyset$. Hence by Lemma 1.4, $(Q, I_{\Sigma_{\operatorname{End} T_2}})$ does not contain any double-zero (which is easy to see here by taking a look at $(Q, I_{\Sigma_{\operatorname{End} T_2}})$), and therefore it follows from Corollary 2.2 that

$$T_3 = \Big(\bigoplus_{i=11}^{12} P(i)\Big) \oplus S(10) \oplus \Big(\bigoplus_{k=1}^{9} I(k)\Big)$$

is a tilting-cotilting $\operatorname{End} T_2$-module. Finally, by Lemma 2.3 we see that $\operatorname{End} T_2$ is a tilted algebra of type Q, and so A is iterated tilted of type Q.

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