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TAME TENSOR PRODUCTS OF ALGEBRAS

BY

ZBIGNIEW LESZCZYŃSKI and ANDRZEJ SKOWROŃSKI (Toruń)

Abstract. With the help of Galois coverings, we describe the tame tensor products $A \otimes_K B$ of basic, connected, nonsimple, finite-dimensional algebras A and B over an algebraically closed field K. In particular, the description of all tame group algebras AG of finite groups G over finite-dimensional algebras A is completed.

Introduction. Throughout the paper K will denote a fixed algebraically closed field. By an *algebra* we mean a finite-dimensional K-algebra (associative, with an identity) which we moreover assume to be basic and connected. An algebra A can be written as a bound quiver algebra $A \cong KQ/I$, where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q.

By Drozd's Tame and Wild Theorem [9] the class of algebras may be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite-dimensional algebras over K. Accordingly, we may realistically hope to classify the indecomposable finite-dimensional modules only for the tame algebras. The representation theory of arbitrary tame algebras is still only emerging.

We are concerned with the problem of describing when the tensor product $A \otimes_K B$ of two nonsimple algebras A and B is tame. The class of tensor product algebras contains several important classes of algebras, including:

(1) the group algebras $AG \cong A \otimes_K KG$ of finite groups G with coefficients in algebras A;

(2) the upper triangular $n \times n$ matrix algebras $T_n(A) \cong A \otimes_K T_n(K)$ with coefficients in algebras A;

(3) the enveloping algebras $A^e \cong A \otimes_K A^{\mathrm{op}}$ of algebras A.

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There is a long record of papers devoted to the tameness of the above classes of algebras (see [2], [6], [7], [17], [21]–[25], [29]–[31]). We also note that the tensor product $A \otimes_K B \otimes_K C$ of three nonsimple algebras A, B, C is tame if and only if A, B, C are isomorphic to $T_2(K)$ (see [22]). Hence the tensor product $A \otimes_K B \otimes_K C \otimes_K D$ of any four nonsimple algebras A, B, C, D is always wild.

The paper is organized as follows. In Section 1 we present the main theorem and related background. Section 2 is devoted to basic results applied in the proof of our main theorem. In Section 3 we prove the tameness of the tensor products of some particular Nakayama algebras. Section 4 is devoted to the proof of the main theorem. In the final Sections 5, 6 and 7 we give respectively descriptions of tame group algebras, tame triangular matrix algebras and tame enveloping algebras.

1. The main theorem and related background. Given a locally finite quiver Q (each vertex is the source and end of only finitely many vertices) the path category KQ of Q has as objects the vertices of Q, and as morphisms between two objects x and y the space $KQ(x, y)$ of K-linear combinations of paths from x to y. For $n \geq 1$ and objects x and y of KQ, we denote by $KQ(x, y)_n$ the subspace of $KQ(x, y)$ generated by all paths in Q of length $\geq n$. An ideal I of the path category KQ is called *admissible* if the following conditions are satisfied:

(a) $I(x, y) \subseteq KQ(x, y)_2$ for all objects x, y of KQ,

(b) for every object x of KQ there exists a positive integer n_x such that $KQ(x, y)_{n_x} \subseteq I(x, y)$ and $KQ(y, x)_{n_x} \subseteq I(y, x)$ for all objects y of KQ.

In that case, (Q, I) is called a *bounded quiver* and the residue category $R = KQ/I$ a *locally bounded K-category* [5]. If R is *bounded* (Q has only finitely many vertices) then R may be identified with the algebra $\oplus R$ of all matrices $(a_{yx})_{x,y\in R}$ with $a_{yx} \in R(x, y)$. A locally bounded K-category $R = KQ/I$ with Q having no oriented cycles is called *triangular*. Following [1] a locally bounded K-category R is said to be *simply connected* if, for any presentation $R = KQ/I$ of R as a bound quiver category, the fundamental group $\Pi_1(Q, I)$ of (Q, I) is trivial. Finally, a full subcategory Λ of a locally bounded K-category $R = KQ/I$ is said to be *convex* if any path in Q with source and end in Λ lies entirely in the quiver of Λ.

Assume that R is a locally bounded K-category and G a group of K-linear automorphisms of R acting freely on the objects of R. According to [13] the quotient category R/G exists. Its objects are the G-orbits of the objects of R. Moreover, we have

$$(R/G)(a, b) = \Big\{ (f_{yx}) \in \prod_{(x,y)\in a\times b} R(x, y) \;\Big|\; g f_{yx} = f_{g(y)g(x)} \quad \forall_{g\in G,\, x\in a,\, y\in b} \Big\}$$

and the composition of $e \in (R/G)(b,c)$ with $f \in (R/G)(a,b)$ is given by

$$(ef)_{zx} = \sum_{y \in b} e_{zy} f_{yx}.$$

The canonical functor $F : R \to R/G$, which assigns to each object x of R its G-orbit Gx and to each morphism $\xi \in R(x,y)$ the family $F(\xi)$ given by $F(\xi)_{h(y)g(x)} = \delta_{gh} g\xi$ is called a *Galois covering* of R/G with Galois group G. For G having only finitely many orbits of objects in R, R/G is a bounded K-category and hence may be identified with the associated algebra $\oplus(R/G)$.

Let $R = KQ/I$ and $R' = KQ'/I'$ be two locally bounded K-categories, and let Q_0 and Q'_0 (respectively, Q_1 and Q'_1) be the sets of vertices (respectively, arrows) of Q and Q'. Then we may define the tensor product $R \otimes_K R'$ of R and R' as the locally bounded K-category $K(Q \otimes Q')/I \,\square\, I'$, where $(Q \otimes Q')_0 = Q_0 \times Q'_0$ is the set of vertices of the quiver $Q \otimes Q'$, $(Q \otimes Q')_1 = (Q_0 \times Q'_1) \cup (Q_1 \times Q'_0)$ is the set of arrows of $Q \otimes Q'$, and $I \,\square\, I'$ is the ideal in the path algebra $K(Q \otimes Q')$ generated by $Q_0 \times I'$, $I \times Q'_0$, and the elements of the form $(\alpha, t)(p, \beta) - (r, \beta)(\alpha, s)$ for all arrows $p \xrightarrow{\alpha} r$ in Q_1 and $s \xrightarrow{\beta} t$ in Q'_1. If R and R' are algebras (bounded K-categories) then $K(Q \otimes Q')/I \,\square\, I'$ is a bound quiver presentation of the tensor product algebra $R \otimes_K R'$ (see [22, Lemma 1.3]). We note that $I \,\square\, I' \neq 0$ even if $I = 0$ and $I' = 0$. Moreover, it is easy to see that $R \otimes_K R'$ is simply connected if both R and R' are simply connected.

Following Drozd [9], an algebra Λ is said to be *tame* if, for any dimension d, there exist a finite number of Λ-$K[X]$-bimodules M_i, $1 \le i \le n_d$, which are finitely generated and free as right $K[X]$-modules, and all but finitely many isoclasses of indecomposable left Λ-modules of dimension d are of the form $M_i \otimes_{K[X]} K[X]/(X - \lambda)$ for some $\lambda \in K$ and some i. More generally, a locally bounded K-category R is said to be tame if every full bounded subcategory of R is tame [11].

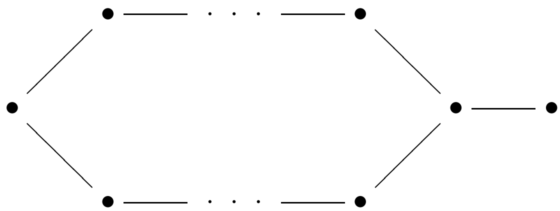
Let $\Lambda = KQ/I$ be a triangular algebra. Denote by Q_0 the set of vertices of Q, by Q_1 the set of arrows of Q, and by $s, e : Q_1 \to Q_0$ the maps which assign to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its end $e(\alpha)$. The *Tits form* q_Λ of Λ is the integral quadratic form on the Grothendieck group $K_0(\Lambda) = \mathbb{Z}^{Q_0}$ of Λ, defined for $\mathbf{x} = (x_i)_{i \in Q_0} \in K_0(A)$ as follows:

$$q_\Lambda(\mathbf{x}) = \sum_{l \in Q_0} x_l^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{e(\alpha)} + \sum_{i,j \in Q_0} r_{ij} x_i x_j$$

where r_{ij} is the cardinality of $\mathcal{R} \cap I(i,j)$ for a minimal set $\mathcal{R} \subset \bigcup_{i,j \in Q_0} I(i,j)$ of K-linear relations generating the ideal I (see [4]). It is well known (see [27]) that if Λ is tame then q_Λ is *weakly nonnegative*, that is, $q_\Lambda(\mathbf{x}) \ge 0$ for any $\mathbf{x}$ in $K_0(A)$ with nonnegative coordinates.

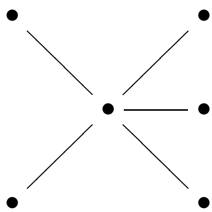
Consider the extended Euclidean graphs

$\widetilde{\widetilde{\mathbb{A}}}_m$:

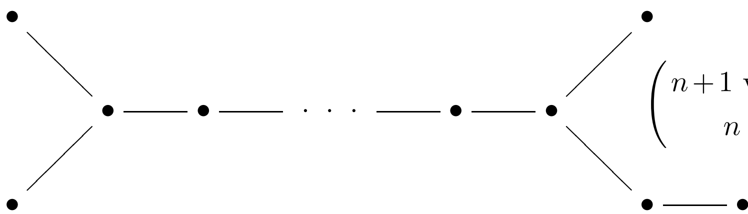


($m+2$ vertices, $m \geq 1$)

T_5 :

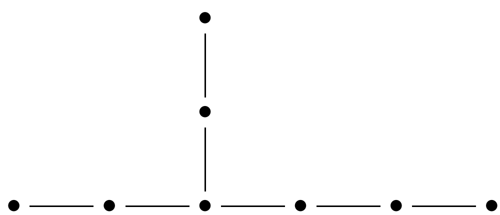


$\widetilde{\widetilde{\mathbb{D}}}_n$:

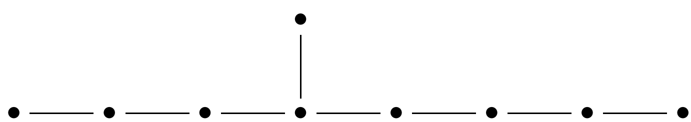


$\left(\begin{array}{c} n+1 \text{ vertices,} \\ n \geq 4 \end{array} \right)$

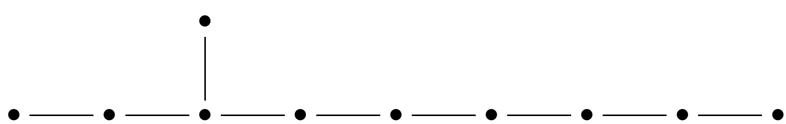
$\widetilde{\widetilde{\mathbb{E}}}_6$:



$\widetilde{\widetilde{\mathbb{E}}}_7$:



$\widetilde{\widetilde{\mathbb{E}}}_8$:



Let $H = K\Delta$ be the path algebra of a quiver Δ (without oriented cycles) whose underlying graph $\overline{\Delta}$ is one of the above extended Euclidean graphs, and T be a preprojective tilting H-module, that is, $\mathrm{Ext}^1_H(T,T) = 0$ and T is a direct sum of $|\Delta_0|$ pairwise nonisomorphic H-modules lying in different

$\operatorname{Tr} D$-orbits of indecomposable projective H-modules. Then $C = \operatorname{End}_H(T)$ is said to be a *concealed algebra* of type $\overline{\Delta}$. It is known that such a C is a wild triangular algebra of global dimension at most 2, and its Tits form q_C is not weakly nonnegative (see [18], [28]). The concealed algebras of types $\widetilde{\widetilde{\mathbb{A}}}_m$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$, $\widetilde{\widetilde{\mathbb{E}}}_8$ have been classified by quivers and relations in [20], [36], [38].

Assume that $A = KQ/I$ and $B = KQ'/I'$ are two nonsimple (basic, connected) algebras such that the tensor product algebra $A \otimes_K B$ is tame. Then, by [22] (see also [30]) A and B are representation-finite and standard algebras, and hence there exist Galois coverings $F^A : \widetilde{A} \to \widetilde{A}/G = A$ and $F^B : \widetilde{B} \to \widetilde{B}/G = B$, where $\widetilde{A} = K\widetilde{Q}/\widetilde{I}$ and $\widetilde{B} = K\widetilde{Q}'/\widetilde{I}'$ are simply connected locally bounded K-categories, and $G = \Pi_1(Q, I)$, $H = \Pi_1(Q', I')$, which are moreover finitely generated free groups. Then we have a Galois covering $F^A \otimes F^B : \widetilde{A} \otimes_K \widetilde{B} \to \widetilde{A} \otimes_K \widetilde{B}/G \times H = A \otimes_K B$, where $\widetilde{A} \otimes_K \widetilde{B} = K(\widetilde{Q}_A \otimes \widetilde{Q}_B)/\widetilde{I}_A \,\square\, \widetilde{I}_B$ is a simply connected locally bounded K-category, and the Galois group $G \times H$ is obviously torsion-free. Therefore, in order to establish criteria for the tameness of the tensor products of nonsimple algebras, we may restrict to the representation-finite standard algebras.

The following theorem is the main result of the paper.

THEOREM 1.1. *Let A and B be two nonsimple representation-finite standard algebras. Then the following statements are equivalent*:

(i) *$A \otimes_K B$ is tame.*

(ii) *The Tits form q_C of any bounded convex subcategory C of $\widetilde{A} \otimes_K \widetilde{B}$ is weakly nonnegative.*

(iii) *$\widetilde{A} \otimes_K \widetilde{B}$ does not contain a bounded convex subcategory which is concealed of type $\widetilde{\widetilde{\mathbb{A}}}_m, m \geq 1$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n, n \geq 4$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$ or $\widetilde{\widetilde{\mathbb{E}}}_8$.*

In the case of 2×2 upper triangular matrix algebras $T_2 \cong A \otimes_K T_2(K)$, the above theorem has been proved in [23]. Hence, our aim is to prove the theorem for algebras A and B which are not isomorphic to K or $T_2(K)$.

2. Preliminary results. In this section we collect some results applied in the proof of the main theorem. We start with the following proposition proved in [10, Proposition 2] (see also [14, Section 3]).

PROPOSITION 2.1. *Let $F : R \to R/G$ be a Galois covering of locally bounded K-categories, and assume that R/G is tame. Then R is also tame.*

Following [32] a locally bounded K-category R is called *strongly simply connected* if every convex bounded subcategory of R is simply connected. Every strongly simply connected locally bounded K-category is simply connected but the converse is not true. However, all simply connected representation-finite algebras are strongly simply connected. The concealed algebras of types T_5, $\widetilde{\widetilde{\mathbb{D}}}_n, n \geq 4$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$ and $\widetilde{\widetilde{\mathbb{E}}}_8$ are strongly simply connected (see [36]), and are called *hypercritical algebras*. In [26] the class of tame minimal nonpolynomial growth simply connected algebras, called *pg*-critical, has been introduced and classified by quivers and relations. There are only 16 frames of strongly simply connected *pg*-critical algebras. We will need the following criterion for tameness proved in [34, Theorem 2.4] (see also [33, Theorem 4.1] in the simply connected case).

PROPOSITION 2.2. *Let R be a strongly simply connected locally bounded K-category and G a group of K-linear automorphisms of R such that $A = R/G$ is bounded. Assume that R does not contain a convex subcategory which is hypercritical or pg-critical. Then A is tame.*

Following [35] an algebra A is said to be *special biserial* if A is isomorphic to a bound quiver algebra KQ/I, where the bound quiver (Q, I) satisfies the conditions:

(a) each vertex of Q is the source and end of at most two arrows,

(b) for any arrow α of Q there are at most one arrow β and at most one arrow γ with $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

The following fact has been proved in [37] (see also [12]).

PROPOSITION 2.3. *Every special biserial algebra is tame.*

In fact this proposition can be considered as a special case of Proposition 2.2. Indeed, the special biserial algebras admit strongly simply connected Galois coverings whose convex bounded subcategories are all representation-finite (see [12, (5.2)]).

In the proof of our main result we also need a geometric criterion for tameness. For a positive integer d, we denote by $\mathrm{alg}_d(K)$ the affine variety of associative algebra structures with identity on the affine space K^d. Then the general linear group $\mathrm{GL}_d(K)$ acts on $\mathrm{alg}_d(K)$ by transport of structure, and the $\mathrm{GL}_d(K)$-orbits in $\mathrm{alg}_d(K)$ correspond to the isomorphism classes of d-dimensional algebras (we refer to [19] for more details). We shall identify a d-dimensional algebra with the point of $\mathrm{alg}_d(K)$ corresponding to it. For two d-dimensional algebras A and B, we say that B is a *degeneration* of A if B belongs to the closure of the $\mathrm{GL}_d(K)$-orbit of A in the Zariski topology of $\mathrm{alg}_d(K)$. Then we have the following result proved by C. Geiss [15] (see also [8]).

PROPOSITION 2.4. *Let A and B be two d-dimensional algebras, and assume that B is tame and a degeneration of A. Then A is tame.*

We end this section with an example showing that the tensor product of two strongly simply connected algebras is not necessarily strongly simply connected.

Example 2.5. Let A and B be the path algebras KQ of the quiver Q: $\bullet \longleftarrow \bullet \longrightarrow \bullet$. Then $A \otimes_K B$ is the bound quiver algebra $K(Q \otimes Q')/J$, where $Q \otimes Q'$ is the quiver

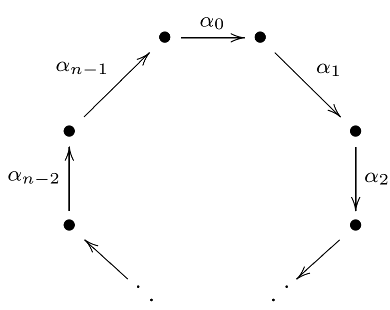
$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "\alpha''"'] \arrow[r, "\beta''"] & \bullet \\
\bullet \arrow[u, "\gamma"] \arrow[d, "\sigma"'] & \bullet \arrow[l, "\alpha'"'] \arrow[r, "\beta'"] \arrow[u, "\gamma'"] \arrow[d, "\sigma'"'] & \bullet \arrow[u, "\gamma''"'] \arrow[d, "\sigma''"'] \\
\bullet & \bullet \arrow[l, "\alpha"'] \arrow[r, "\beta"] & \bullet
\end{tikzcd}$$

and $J = I \,\square\, I'$ (with $I = 0$ and $I' = 0$) is generated by the elements $\alpha\sigma' - \sigma\alpha'$, $\gamma\alpha' - \alpha''\gamma'$, $\gamma''\beta' - \beta''\gamma'$, $\sigma''\beta' - \beta\sigma'$. Then $A \otimes_K B$ is simply connected but not strongly simply connected, because it contains a convex path algebra $H = K\Delta$, where Δ is the quiver of Euclidean type $\widetilde{\mathbb{A}}_8$ formed by the vertices of $Q \otimes Q'$ except the central vertex. In fact, $A \otimes_K B$ is the one-point extension $H[M]$ of H by a simple homogeneous module, and hence $A \otimes_K B$ is a tubular algebra of type $(2, 4, 4)$ in the sense of Ringel [28]. In particular, $A \otimes_K B$ is tame.

3. Tame tensor products of Nakayama algebras. An important role in the proof of our main theorem will be played by the tensor products of some Nakayama algebras. Recall that an algebra A is called a *Nakayama algebra* if all indecomposable projective left and right A-modules have unique composition series. It is well known that a nonsimple algebra A is a Nakayama algebra if and only if the Gabriel quiver Q_A of A is one of the quivers

$$L_n: \quad \bullet \xrightarrow{\alpha_0} \bullet \xrightarrow{\alpha_1} \bullet \longrightarrow \quad \cdots \quad \longrightarrow \bullet \xrightarrow{\alpha_{n-1}} \bullet$$

or



for some $n \geq 1$. We need special families of Nakayama algebras (see [31]).

For $n \geq 2$ and any sequence of positive integers $n_1, \dots, n_s$ satisfying the conditions: $s \geq 1$, $n_s < n-1$ and $n_i + 1 < n_{i+1}$ for $1 \leq i \leq s-1$, we will denote by $A^n_{(n_1,\dots,n_s)}$ the bound quiver algebra $KL_n/I^n_{(n_1,\dots,n_s)}$, where $I^n_{(n_1,\dots,n_s)}$ is the ideal in the path algebra KL_n generated by the paths $\alpha_i\alpha_{i-1}$ for $i \neq n_1, \dots, n_s$, $1 \leq i \leq n-2$. Moreover, for $n \geq 1$, we will denote by A^n the bound quiver algebra KL_n/I^n, where the ideal I^n is generated by all paths $\alpha_i\alpha_{i-1}$, $1 \leq i \leq n-1$.

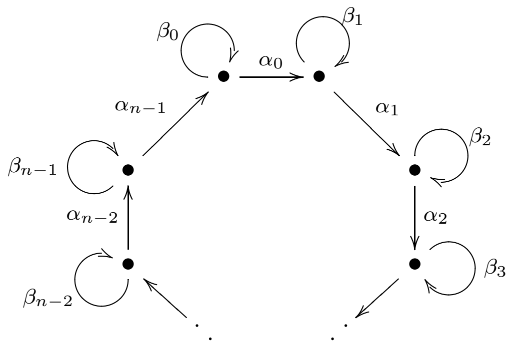
For $n \geq 2$ and any sequence of positive integers $n_1, \dots, n_s$ satisfying the conditions: $s \geq 1$, $n_s < n-1$ and $n_i + 1 < n_{i+1}$ for $1 \leq i \leq s-1$, we denote by $B^n_{(n_1,\dots,n_s)}$ the bound quiver algebra $KC_n/J^n_{(n_1,\dots,n_s)}$, where $J^n_{(n_1,\dots,n_s)}$ is the ideal in the path algebra KC_n generated by the paths $\alpha_i\alpha_{i-1}$ for $i \neq n_1, \dots, n_s$, $1 \leq i \leq n$, and $\alpha_0 = \alpha_n$. Moreover, for $n \geq 1$, we will denote by B^n the bound quiver algebra KC_n/J^n, where the ideal J^n is generated by all paths $\alpha_i\alpha_{i-1}$, $1 \leq i \leq n$.

Hence, A^n and B^n, $n \geq 1$, are representatives of isoclasses of radical square zero nonsimple Nakayama algebras.

The aim of this section is to prove that the tensor product algebras of the forms $A^n_{(n_1,\dots,n_s)} \otimes_K A^m$, $A^n_{(n_1,\dots,n_s)} \otimes_K B^m$, $B^n_{(n_1,\dots,n_s)} \otimes_K A^m$, $B^n_{(n_1,\dots,n_s)} \otimes_K B^m$, $A^n \otimes_K B^m$, $A^n \otimes_K A^m$ and $B^n \otimes_K B^m$ are all tame.

Proposition 3.1. *The algebras* $B^n_{(n_1,\dots,n_s)} \otimes_K B^m$, $n \geq 2$, $m \geq 1$, *are tame.*

Proof. We first prove that the algebras $B^n_{(n_1,\dots,n_s)} \otimes_K B^1$, $n \geq 2$, are tame. The algebra $B^n_{(n_1,\dots,n_s)} \otimes_K B^1$ is isomorphic to the bound quiver algebra $KQ^n/R^n_{(n_1,\dots,n_s)}$, where Q^n is the quiver



and $R^n_{(n_1,\dots,n_s)}$ is the ideal in KQ^n generated by $\alpha_i\alpha_{i-1}$ for $i \neq n_1, \dots, n_s$, $1 \leq i \leq n$, β_i^2, $1 \leq i \leq n-1$, and $\alpha_i\beta_i - \beta_{i+1}\alpha_i$, $1 \leq i \leq n$, where $\alpha_n = \alpha_0$ and $\beta_n = \beta_0$. Put $\Lambda = B^n_{(n_1,\dots,n_s)} \otimes_K B^1$, and let $d = \dim_K \Lambda$. For $a \in K$, consider the bound quiver algebra $\Lambda(a) = KQ^n/R^n_{(n_1,\dots,n_s)}(a)$, where $R^n_{(n_1,\dots,n_s)}(a)$ is the ideal in KQ^n obtained from $R^n_{(n_1,\dots,n_s)}$ by replacing the generators $\alpha_{i-1}\beta_{i-1} - \beta_i\alpha_{i-1}$ and $\alpha_i\beta_i - \beta_{i+1}\alpha_i$, $i = n_1, \dots, n_s$, by $a\alpha_{i-1}\beta_{i-1} - \beta_i\alpha_{i-1}$ and $\alpha_i\beta_i - a\beta_{i+1}\alpha_i$, and keeping the other generators of $R^n_{(n_1,\dots,n_s)}$ unchanged. Then we have an algebraic family $\Lambda(a)$, $a \in K$, of algebras in $\mathrm{alg}_d(K)$. Observe that $\Lambda(0)$ is a special biserial algebra, and hence $\Lambda(0)$ is tame by Proposition 2.3. Moreover, $\Lambda(1) = \Lambda$. In fact, we have $\Lambda(a) \cong \Lambda$ for all $a \in K \setminus \{0\}$. Indeed, for $a \in K \setminus \{0\}$, the automorphism of the path algebra KQ^n mapping $a\beta_{i-1}$ to β_{i-1} and $a\beta_{i+1}$ to β_{i+1}, for $i = n_1, \dots, n_s$, and keeping the other arrows of Q^n unchanged, induces the required isomorphism $\Lambda(a) \xrightarrow{\sim} \Lambda$ of bound quiver algebras. Thus, we proved that $\Lambda(0)$ is a degeneration of Λ. Applying now Proposition 2.4 and invoking the tameness of $\Lambda(0)$, we conclude that Λ is also tame.

For $m \geq 2$, we may prove in a similar way that any algebra $B^n_{(n_1,\dots,n_s)} \otimes_K B^m$ degenerates to a special biserial algebra, and consequently is tame. Alternatively, observe also that there is a canonical Galois covering $B^n_{(n_1,\dots,n_s)} \otimes_K B^m \to B^n_{(n_1,\dots,n_s)} \otimes_K B^1$ with cyclic Galois group of order m, induced by the canonical Galois covering $B^m \to B^1$ of Nakayama algebras. Applying now Proposition 2.1 and invoking the tameness of $B^n_{(n_1,\dots,n_s)} \otimes_K B^1$, we conclude that the algebra $B^n_{(n_1,\dots,n_s)} \otimes_K B^m$ is also tame. ■

Corollary 3.2. *The algebras* $A^n_{(n_1,\dots,n_s)} \otimes_K A^m$, $A^n_{(n_1,\dots,n_s)} \otimes_K B^m$, $B^n_{(n_1,\dots,n_s)} \otimes_K A^m$, $A^n \otimes_K B^m$, $A^n \otimes_K A^m$ *and* $B^n \otimes_K B^m$ *are tame.*

Proof. Observe that there are canonical Galois coverings

$$\widetilde{B}^n_{(n_1,\dots,n_s)} \otimes_K \widetilde{B}^m \to B^n_{(n_1,\dots,n_s)} \otimes_K B^m \quad \text{with group } \mathbb{Z}\times\mathbb{Z},$$
$$\widetilde{B}^n_{(n_1,\dots,n_s)} \otimes_K B^m \to B^n_{(n_1,\dots,n_s)} \otimes_K B^m \quad \text{with group } \mathbb{Z},$$
$$B^n_{(n_1,\dots,n_s)} \otimes_K \widetilde{B}^m \to B^n_{(n_1,\dots,n_s)} \otimes_K B^m \quad \text{with group } \mathbb{Z},$$

and hence by Proposition 2.1, the locally bounded categories $\widetilde{B}^n_{(n_1,\dots,n_s)} \otimes_K \widetilde{B}^m$, $\widetilde{B}^n_{(n_1,\dots,n_s)} \otimes_K B^m$ and $B^n_{(n_1,\dots,n_s)} \otimes_K \widetilde{B}^m$ are tame. Then $A^n_{(n_1,\dots,n_s)} \otimes_K A^m$, $A^n_{(n_1,\dots,n_s)} \otimes_K B^m$ and $B^n_{(n_1,\dots,n_s)} \otimes_K A^m$ are tame, being full finite subcategories of $\widetilde{B}^n_{(n_1,\dots,n_s)} \otimes_K \widetilde{B}^m$, $\widetilde{B}^n_{(n_1,\dots,n_s)} \otimes_K B^m$ and $B^n_{(n_1,\dots,n_s)} \otimes_K \widetilde{B}^m$, respectively. Finally, observe that the algebras $A^n \otimes_K B^m$, $A^n \otimes_K A^m$ and $B^n \otimes_K B^m$ are factor algebras of $A^n_{(n_1,\dots,n_s)} \otimes_K B^m$, $A^n_{(n_1,\dots,n_s)} \otimes_K A^m$ and $B^n_{(n_1,\dots,n_s)} \otimes_K B^m$, and hence are also tame. ■

For $n \geq 3$, we denote by D^n the bound quiver algebra KL_n/R^n, where R^n is the ideal in the path algebra KL_n generated by all paths $\alpha_{i+1}\alpha_i\alpha_{i-1}$, $1 \leq i \leq n-2$. Moreover, for $n \geq 1$, let E^n be the bound quiver algebra KC_n/T^n, where T^n is the ideal in KC_n generated by all paths $\alpha_{i+1}\alpha_i\alpha_{i-1}$, $1 \leq i \leq n+1$ (with $\alpha_n = \alpha_0$, $\alpha_{n+1} = \alpha_1$).

PROPOSITION 3.3. *The algebras* $E^n \otimes_K A^2$, $n \geq 1$, *are tame.*

Proof. For $n = 1$, this is shown in [31, Section 4]. For $n \geq 2$, we have a canonical Galois covering $E^n \otimes_K A^2 \to E^1 \otimes_K A^2$ with cyclic Galois group of order n, induced by the canonical Galois covering $E^n \to E^1$. Then, invoking Proposition 2.1 and the tameness of $E^1 \otimes_K A^2$, we conclude that all algebras $E^n \otimes_K A^2$ are tame. ■

COROLLARY 3.4. *The algebras* $D^n \otimes_K A^2$, $n \geq 3$, *are tame.*

Proof. Apply Proposition 2.1 and the fact that $D^n \otimes_K A^2$ are bounded convex subcategories of the locally bounded K-category $\widetilde{E}^1 \otimes_K A^2 = \widetilde{E}^n \otimes_K A^2$. ■

We end this section with the following facts.

PROPOSITION 3.5. *The algebras* $KL_3 \otimes_K B^n$, $n \geq 1$, *are tame.*

Proof. For $n = 1$, this has been proved in [31, Section 3]. For $n \geq 2$, it follows from Proposition 2.1, because we have a canonical Galois covering $KL_3 \otimes_K B^n \to KL_3 \otimes_K B^1$ with cyclic Galois group of order n. ■

COROLLARY 3.6. *The algebras* $KL_3 \otimes_K B^n$, $n \geq 2$, *are tame.*

Proof. As above, this follows from Proposition 2.1 and the fact that $KL_3 \otimes_K B^n$ are bounded convex subcategories of the locally bounded K-category $KL_3 \otimes_K \widetilde{B}^1 = KL_3 \otimes_K \widetilde{B}^n$. ■

4. Proof of Theorem 1.1. Let $A = KQ/I$ and $B = KQ'/I'$ be two nonsimple representation-finite standard algebras, and $\widetilde{A} \to \widetilde{A}/G = A$ and $\widetilde{B} \to \widetilde{B}/H = B$, with $\widetilde{A} = K\widetilde{Q}/\widetilde{I}$, $\widetilde{B} = K\widetilde{Q}'/\widetilde{I}'$, $G = \Pi_1(Q, I)$, $H = \Pi_1(Q', I')$, their (strongly) simply connected Galois coverings. Then we have the canonical Galois covering $\widetilde{B} \otimes_K \widetilde{B} \to \widetilde{A} \otimes_K \widetilde{B}/G \times H = A \otimes_K B$ with $\widetilde{A} \otimes_K \widetilde{B} = K(Q \otimes Q')/I_A \,\square\, I_B$ simply connected and $G \times H$ torsion-free.

Assume that $A \otimes_K B$ is a tame algebra. Then it follows from Proposition 2.1 that the locally bounded K-category $\widetilde{A} \otimes_K \widetilde{B}$ is tame. Hence every bounded convex subcategory C of $\widetilde{A} \otimes_K \widetilde{B}$ is tame, and consequently the Tits form q_C of C is weakly nonnegative (see [27]). Therefore, (i) implies (ii).

The implication (ii)⇒(iii) follows from the fact [18, (6.2)] that the Tits form of every concealed algebra of wild type (in particular of type $\widetilde{\widetilde{\mathbb{A}}}_m$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$, $\widetilde{\widetilde{\mathbb{E}}}_8$) is not weakly nonnegative.

The remaining part of this section will be devoted to the proof of the implication (iii)⇒(i). Hence from now on we assume that the locally bounded K-category $\widetilde{A} \otimes_K \widetilde{B}$ does not contain a bounded convex subcategory which is concealed of type $\widetilde{\widetilde{\mathbb{A}}}_m$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$, or $\widetilde{\widetilde{\mathbb{E}}}_8$. Moreover, since for $A = T_2(K)$ or $B = T_2(K)$, the implication (iii)⇒(i) has been proved in [23, Theorem 1], we will assume that A and B are not isomorphic to $T_2(K)$. Clearly, since A and B are basic and nonsimple, they are also not isomorphic to K. Moreover, we may assume that $A \otimes_K B$ is weakly sincere. Recall from [22, (3.1)] that a tensor product algebra $C \otimes_K D$ is called *weakly sincere* if there exists an indecomposable finite-dimensional $C \otimes_K D$-module M whose support is not contained in $\Lambda \otimes_K \Gamma$, for a full subcategory Λ of C and a full subcategory Γ of D with $\Lambda \neq C$ or $\Gamma \neq D$. Hence, every indecomposable finite dimensional $A \otimes_K B$-module is an indecomposable finite-dimensional module over a weakly sincere full subcategory $C \otimes_K D$ of $A \otimes_K B$. Therefore, in order to prove the tameness of $A \otimes_K B$, we may indeed assume that $A \otimes_K B$ is weakly sincere.

The first step in our proof is to reduce it to the case when both A and B are Nakayama algebras. We start with the following fact proved in [22, Theorem 3.2].

Lemma 4.1. *Assume that neither A nor B is a Nakayama algebra. Then A and B are isomorphic to the path algebras of one of the quivers*

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \qquad \textit{or} \qquad \bullet \longrightarrow \bullet \longleftarrow \bullet\ ,$$

and $A \otimes_K B$ is a tame algebra.

Lemma 4.2. *Assume that A and B are simply connected algebras*, *A is not Nakayama but B is Nakayama. Then $A \otimes_K B$ is tame.*

Proof. This follows from the proof of [22, Theorem 3.2]. In fact, in this case $A \otimes_K B$ is a strongly simply connected algebra which does not contain a convex subcategory which is *pg*-critical or hypercritical, and consequently $A \otimes_K B$ is tame by Proposition 2.2 (or [33, Theorem 4.1]). ■

Lemma 4.3. *Assume that A is neither simply connected nor a Nakayama algebra*, *and B is a simply connected Nakayama algebra. Then $A \otimes_K B$ is tame.*

Proof. Since B is a simply connected Nakayama algebra not isomorphic to K and $T_2(K)$, the quiver Q' of B is a linear quiver

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

consisting of at least 2 arrows. We first show that the quiver $\widetilde{Q}$ of $\widetilde{A}$ has no subquivers of the forms

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \qquad \text{or} \qquad \begin{tikzcd} & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \end{tikzcd}$$

or their duals. Indeed, otherwise the Galois covering $\widetilde{A} \otimes_K B = \widetilde{A} \otimes_K \widetilde{B}$ of $A \otimes_K B$ contains a convex hypercritical subcategory which is the path category of

$$\begin{tikzcd}
 & \bullet \arrow[d] & & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \\
 & & \bullet &
\end{tikzcd}
\qquad \text{or} \qquad
\begin{tikzcd}
 & \bullet \arrow[d] & \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[d] \\
 & \bullet \arrow[u] & \bullet
\end{tikzcd}$$

or their duals, a contradiction. We now claim that the bound quiver (Q', I') of B is of the form

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[rr, dash, dashed, bend right] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

where the dashed line means that the composition of these two arrows belongs to I'. Suppose that Q' contains 3 arrows. Since A is not simply connected and not a Nakayama algebra, invoking the above restriction on $\widetilde{Q}$, we conclude that the quiver $\widetilde{Q}$ contains a convex subquiver of the form

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

or its dual. But then $\widetilde{A} \otimes_K \widetilde{B}$ contains a convex hypercritical subcategory of type $\widetilde{\widetilde{\mathbb{E}}}_7$ which is the path category of the quiver

$$\begin{tikzcd}
\bullet \arrow[d] & & & & \\
\bullet & \bullet \arrow[l] \arrow[d] & & \bullet \arrow[r] \arrow[d] & \bullet \\
 & \bullet & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet & \\
 & & \bullet & &
\end{tikzcd}$$

or its dual, a contradiction. Therefore, Q' consists of two arrows. Finally, if the composition of two arrows of Q' is not in I', then $\widetilde{A} \otimes_K \widetilde{B}$ contains a convex hypercritical subcategory of type $\widetilde{\widetilde{\mathbb{E}}}_7$ given by the bound quiver

$$\begin{tikzcd}
\bullet \arrow[d] & & \bullet \arrow[r] \arrow[d] & \bullet \\
\bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \bullet \arrow[d] & \\
\bullet & \bullet \arrow[l] \arrow[r] & \bullet &
\end{tikzcd}$$

where

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, dash, dashed] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

means that the difference of two parallel paths of length 2 belongs to I', again a contradiction. We now claim that the quiver Q of A is a cycle, and consequently the quiver $\widetilde{Q}$ is an infinite line of type ${}_\infty\mathbb{A}_\infty$. Suppose Q is not a cycle. Invoking the restrictions on $\widetilde{Q}$ from the first part of our proof, we infer that then $\widetilde{Q}$ contains a convex subquiver of the form

$$\begin{tikzcd} & \bullet \arrow[d] & & \\ \bullet & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] \end{tikzcd}$$

or its dual. Then $\widetilde{A} \otimes_K \widetilde{B}$ contains a convex hypercritical subcategory of type $\widetilde{\widetilde{\mathbb{E}}}_6$ which is the path category of the quiver

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[d] & & \bullet \arrow[dll] \arrow[d] \\ & \bullet & \bullet \arrow[l] \arrow[d] & \bullet \\ & & \bullet & \bullet \arrow[l] \end{tikzcd}$$

or its dual, a contradiction. Thus, we proved that $\widetilde{A} \otimes_K \widetilde{B} = \widetilde{A} \otimes_K B$ is a strongly simply connected category whose quiver $\widetilde{Q} \otimes \widetilde{Q}' = \widetilde{Q} \otimes Q'$ is a tensor product of an infinite line of type ${}_\infty\mathbb{A}_\infty$ and two consecutive arrows. Applying now our assumption that $\widetilde{A} \otimes_K \widetilde{B}$ does not contain a hypercritical convex subcategory, we easily conclude, invoking [26], that $\widetilde{A} \otimes_K \widetilde{B}$ also does not contain a convex pq-critical subcategory. Hence, applying Proposition 2.2, we conclude that $A \otimes_K B$ is tame. ∎

LEMMA 4.4. *Assume that A is not a Nakayama algebra, and B is a nonsimply connected Nakayama algebra. Then $A \otimes_K B$ is tame.*

Proof. Assume first that A is simply connected. Then $\widetilde{A} = A$, and hence we have a canonical simply connected Galois covering $A \otimes_K \widetilde{B} \to A \otimes_K B$ with infinite cyclic Galois group $H = \Pi_1(Q', I')$. Moreover, every bounded convex subcategory Γ of $\widetilde{B}$ is a simply connected Nakayama algebra, and $A \otimes_K \Gamma$ is a bounded convex subcategory of $A \otimes_K \widetilde{B}$. We consider two cases.

CASE 1. Assume that neither A nor A^{op} is the path algebra of the quiver

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd}$$

Since $\widetilde{B}$ contains convex simply connected Nakayama bounded subcategories with an arbitrarily large numbers of objects, applying [22, Theorem 3.2] again, we conclude that either A or A^{op} is the bound quiver algebra given by the bound quiver

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] \arrow[rr, dash, dashed, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

and B is a radical square zero Nakayama algebra B^n, for some $n \geq 1$. But then $A \otimes_K \widetilde{B}$ is a strongly simply connected locally bounded K-category which does not contain a convex pg-critical or hypercritical bounded subcategory, and invoking Proposition 2.2 we conclude that $A \otimes_K B$ is tame.

Case 2. Assume that A or A^{op} is the path algebra of

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

For $B = B^n$ with $n \geq 1$, the algebra $A \otimes_K B$ is a factor algebra of the algebra considered above, and hence is tame. In fact, the algebra $A \otimes_K B^n$ is even representation-finite (see [25, Lemma 1]). Hence we may assume that B is not a radical square zero algebra. Since $A \otimes_K \widetilde{B}$ contains convex subcategories $A \otimes_K \mathit{\Gamma}$ for convex bounded subcategories $\mathit{\Gamma}$ of $\widetilde{B}$ with large numbers of objects, invoking [22, Theorem 3.2] again, we conclude that B is an algebra $B^n_{(n_1,\dots,n_s)}$ for some $n \geq 2$ and a sequence of positive integers $n_1, \dots, n_s$ satisfying the conditions: $s \geq 1$, $n_s \leq n-1$, and $n_i + 2 < n_{i+1}$ for $1 \leq i \leq s-1$. We note that if $n_i + 2 = n_{i+1}$ for some i with $1 \leq i \leq s-1$, then $A \otimes_K \widetilde{B}$ contains the path algebra of the quiver

$$\begin{tikzcd} & & \bullet \arrow[d] \\ \bullet \arrow[d] & & \bullet \arrow[d] \\ \bullet & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \\ & \bullet \arrow[d] & \\ & \bullet & \end{tikzcd}$$

of type $\widetilde{\widetilde{\mathbb{E}}}_6$ as a convex subcategory, which contradicts our assumption on $\widetilde{A} \otimes_K \widetilde{B}$. Finally, a simple inspection of the families of the pg-critical algebras presented in [26, Theorem 3.2] shows that the strongly simply connected locally bounded category $A \otimes_K \widetilde{B}^n_{(n_1,\dots,n_s)}$, for $n_1, \dots, n_s$ satisfying $n_i + 2 < n_{i+1}$ for all $1 \leq i \leq s-1$, does not contain a convex pg-critical subcategory.

Then applying Proposition 2.2 we infer that $A \otimes_K B = A \otimes_K B^n_{(n_1,\ldots,n_s)}$ is tame.

Now assume that A is arbitrary but not a Nakayama algebra. By the above discussion, every bounded convex subcategory of $\widetilde{A}$ has at most 4 objects, and hence A is in fact simply connected. Therefore $A \otimes_K B$ is tame. ∎

Since $A \otimes_K B \cong B \otimes_K A$, the above lemmas prove the tameness of $A \otimes_K B$ if one of the algebras is not a Nakayama algebra. The next three lemmas reduce our considerations to the Nakayama algebras with radical cube zero.

LEMMA 4.5. *Let A and B be Nakayama algebras. Then all paths of length 4 in Q belong to the ideal I.*

Proof. Suppose that Q contains a path of length 4 which does not belong to I. Then $\widetilde{A} \otimes_K \widetilde{B}$ contains a bounded convex subcategory of the form $T_4(K) \otimes_K \Gamma$ for a Nakayama subcategory Γ of $\widetilde{B}$ with at least three objects. But then $\widetilde{A} \otimes_K \widetilde{B}$ contains a convex subcategory which is concealed of type $\widetilde{\widetilde{\mathbb{E}}}_7$ (see the proof of [22, Theorem 2.5]), a contradiction. ∎

LEMMA 4.6. *Let A and B be Nakayama algebras. Assume that A is not isomorphic to the path algebra KL_3 but its quiver Q contains a path of length 3 which does not belong to the ideal I. Then A is a factor algebra of the bound quiver algebra of one of the bound quivers*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, dash, dashed, bend left] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[rr, dash, dashed, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

or

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[rr, dash, dashed, bend left] & \bullet \arrow[r] \arrow[rr, dash, dashed, bend right] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

and B is the bound quiver algebra of the bound quiver

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, dash, dashed, bend left] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

and $A \otimes_K B$ is a tame algebra.

Proof. If A and B are simply connected, this is proved in [22, Theorem 3.2]. Consequently, A and B must be simply connected. Indeed, if A is not simply connected then $\widetilde{A}$ contains a bounded convex subcategory Λ having at least 9 objects and the path algebra of the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

as a convex subcategory. Then, for any convex subcategory Γ of $\widetilde{B}$ with at least 3 objects, $\Lambda \otimes_K \Gamma$ is a bounded convex subcategory of $\widetilde{A} \otimes_K \widetilde{B}$, a contradiction. Hence $A = \widetilde{A}$ is simply connected. Similarly, we show that $B = \widetilde{B}$ is simply connected. ∎

LEMMA 4.7. *Assume $A = KL_3$ and B is a Nakayama algebra. Then B is isomorphic to one of the algebras A^n, $n \geq 2$, or B^n, $n \geq 1$. In particular, $A \otimes_K B$ is tame.*

Proof. It follows from [22, Theorem 2.5] that every bounded convex subcategory of $\widetilde{B}$ with at least 3 objects is isomorphic to A^n for some $n \geq 2$. Obviously B is then isomorphic to one of the desired algebras A^n or B^n. The tameness of $A \otimes_K B$ follows from Propositions 2.1, 3.5 and Corollary 3.6. ∎

In the next two lemmas we consider the case of Nakayama algebras whose bound quivers contain two consecutive nonzero paths of length 2.

LEMMA 4.8. *Let A and B be radical cube zero Nakayama algebras. Assume that A is not isomorphic to the bound quiver algebra D^3 but its quiver contains a path $\alpha_3\alpha_2\alpha_1$ with $\alpha_3\alpha_2 \notin I$ and $\alpha_2\alpha_1 \notin I$. Then A is a factor algebra of one of the bound quiver algebras D^n, $n \geq 3$, or E^n, $n \geq 1$, and B is the bound quiver algebra A^2. In particular, $A \otimes_K B$ is tame.*

Proof. Assume first that A is simply connected. Since $A \not\cong D^3$, we conclude from [22, Theorem 3.2] that A is a factor algebra of an algebra D^n for some $n \geq 4$, and $B \cong A^2$. If A is not simply connected, then we conclude that every bounded convex subcategory of $\widetilde{A}$ is a factor algebra of an algebra of the form D^n and $B \cong A^2$. But then A is a factor algebra of an algebra E^n for some $n \geq 1$. Hence $A \otimes_K B$ is a factor algebra of an algebra $D^n \otimes_K A^2$, $n \geq 3$, or $E^n \otimes_K A^2$, $n \geq 1$, and consequently is tame, by Proposition 3.3 and Corollary 3.4. ∎

LEMMA 4.9. *Assume that A is isomorphic to D^3 and B is a Nakayama algebra. Then B is isomorphic to one of the algebras A^n, $n \geq 2$, or B^n, $n \geq 1$. In particular, $A \otimes_K B$ is tame.*

Proof. Assume B is simply connected. Then it follows from [22, Theorem 3.2] that B is isomorphic to A^n for some $n \geq 2$. Hence, if B is not simply connected, then every bounded convex category of $\widetilde{B}$ with at least 3 objects is isomorphic to an algebra A^n, and consequently B is isomorphic to an algebra B^n, $n \geq 1$. Observe also that $D^3 \otimes_K A^n$ is a factor algebra of $KL_3 \otimes_K A^n$ and $D^3 \otimes_K B^n$ is a factor algebra of $KL_3 \otimes_K B^n$. Applying Proposition 3.3 and Corollary 3.4, we conclude that $A \otimes_K B$ is tame. ∎

Thus we have reduced our considerations to the Nakayama algebras of the forms $A^n_{(n_1,\ldots,n_s)}$, A^n, $B^n_{(n_1,\ldots,n_s)}$, B^n.

LEMMA 4.10. *Let A and B be Nakayama algebras of one of the forms $A^n_{(n_1,\ldots,n_s)}$ or $B^n_{(n_1,\ldots,n_s)}$. Then one of the following cases holds:*

(i) $A \cong B \cong A^n_{(1)}$ *for some* $n \geq 2$,

(ii) $A \cong B^{\mathrm{op}} \cong A^3_{(1)}$ *or* $A \cong B^{\mathrm{op}} \cong A^3_{(2)}$.

Moreover, $A \otimes_K B$ *is tame.*

Proof. This follows from [22, Theorem 3.2]. In fact, for A and B from (i) and (ii), $A \otimes_K B$ is a strongly simply connected algebra having no *pg*-critical and hypercritical convex subcategories, and hence is tame, by Proposition 2.2. ∎

Observe that this finishes our proof. Indeed, since $A \otimes_K B \cong B \otimes_K A$, we may assume that A is one of the algebras $A^n_{(n_1,\ldots,n_s)}$, A^n, $B^n_{(n_1,\ldots,n_s)}$, B^n, and B is isomorphic to one of the radical square zero Nakayama algebras A^n, B^n. In this case, the tameness of $A \otimes_K B$ follows from Proposition 3.1 and Corollary 3.2.

5. Tame group algebras. The aim of this section is to complete the results from [31] concerning description of tame group algebras AG of finite groups G over algebras A. For $m \geq 2$, we denote by $\mathbb{Z}_m$ the cyclic group of order m. Moreover, we denote by D_m, $m \geq 1$, S_m, $m \geq 3$, and Q_m, $m \geq 2$, the following 2-groups:

- dihedral groups $D_m = \langle g, h \mid g^2 = h^{2^m} = 1, hg = gh^{-1} \rangle$,
- semidihedral groups $S_m = \langle g, h \mid g^2 = h^{2^m} = 1, hg = gh^{2^{m-1}-1} \rangle$,
- quaternion groups $Q_m = \langle g, h \mid g^2 = h^{2^{m-1}} = 1, g^4 = 1, hg = gh^{-1} \rangle$.

The following characterization of representation-finite group algebras has been established in [25, Theorem].

THEOREM 5.1. *Let G be a finite group*, A *an algebra over* K, *and p be the characteristic of* K. *Then the group algebra AG is representation-finite if and only if one of the following cases holds*:

(i) *p does not divide the order of G and A is representation-finite*,

(ii) *p divides the order of G and one of the following holds*:

(1) $A \cong K$ *and a p-Sylow subgroup of G is cyclic*,

(2) $p = 3$, *a* 3*-Sylow subgroup of G is isomorphic to* $\mathbb{Z}_3$, *and A is isomorphic to* $T_2(K)$,

(3) $p = 2$, *a* 2*-Sylow subgroup of G is isomorphic to* $\mathbb{Z}_2$, *and A is isomorphic to one of the algebras* $A^n_{(n_1)}$, A^n, *the path algebras of the quivers*

$$\bullet \longrightarrow \bullet \qquad \bullet \longrightarrow \bullet \longleftarrow \bullet \qquad \bullet \longleftarrow \bullet \longrightarrow \bullet$$

or the bound quiver algebras given by the bound quivers

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[rr, dash, dashed, bend right] & \bullet \arrow[l] & \bullet \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] \arrow[rr, dash, dashed, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

We note that the characterization of the representation-finite group algebras KG is due to Higman [16].

Now we give a characterization of all representation-infinite tame group algebras.

THEOREM 5.2. *Let G be a finite group, A an algebra over K, and p be the characteristic of K. Then the group algebra AG is representation-infinite and tame if and only if one of the following cases holds*:

(i) *p does not divide the order of G and A is representation-infinite tame*,

(ii) *p divides the order of G and one of the following holds*:

(1) *$p = 3$, a 3-Sylow subgroup of G is isomorphic to $\mathbb{Z}_3$, and A is isomorphic to the bound quiver algebra A^2*,

(2) *$p = 2$, and one of the following holds*:

(a) *$A \cong K$ and a 2-Sylow subgroup of G is isomorphic to one of the groups D_m, S_m, or Q_m*,

(b) *$A \cong T_2(K)$ and a 2-Sylow subgroup of G is isomorphic to $\mathbb{Z}_4$*,

(c) *a 2-Sylow subgroup of G is isomorphic to $\mathbb{Z}_2$ and A is isomorphic to one of the algebras $A^n_{(n_1,\ldots,n_s)}$, A^n, $B^n_{(n_1,\ldots,n_s)}$, or B^n.*

Proof. This is a direct consequence of [31, Theorem 1], Proposition 3.1 and Corollary 3.2. ■

We note that the characterization of the representation-infinite tame group algebras KG is due to Bondarenko and Drozd [3].

6. Tame triangular matrix algebras. In [23] we have described completely all basic connected algebras A for which the algebras $T_2(A)$ of 2×2 upper triangular matrices over A are tame (respectively, representation-finite). Here, we will extend the results of [23] to the algebras $T_n(A)$ of $n \times n$ upper triangular matrices over A with $n \geq 3$.

THEOREM 6.1. *Let $n \geq 3$ and A be an algebra. Then $T_n(A)$ is representation-finite if and only if one of the following cases holds*:

(i) *$A \cong K$*,

(ii) *$n = 3$ and A is a radical square zero Nakayama algebra.*

Proof. Apply [22, Theorem 2.5], [24, Theorem], and their proofs. ■

THEOREM 6.2. *Let $n \geq 3$ and A be an algebra. Then $T_n(A)$ is representation-infinite and tame if and only if one of the following cases holds*:

(i) *$n = 5$ and $A \cong T_2(K)$*,

(ii) *$n = 4$ and A is a radical square zero Nakayama algebra*,

(iii) *$n = 3$ and one of the following holds*:

(a) *A or A^{op} is isomorphic to the path algebra of the quiver* $\bullet \longleftarrow \bullet \longrightarrow \bullet$,

(b) *A is isomorphic to an algebra of one of the forms $A^n_{(n_1,\ldots,n_s)}$ or $B^n_{(n_1,\ldots,n_s)}$.*

Proof. Apply [22, Theorem 3.2], Theorem 1.1, Theorem 5.1, and the fact that the simply connected Galois covering $T_3(K)\otimes_K \widetilde{B}^n_{(n_1,\ldots,n_s)}$ (respectively, $T_3(K)\otimes_K A^n_{(n_1,\ldots,n_s)}$) of $T_3(K)\otimes_K B^n_{(n_1,\ldots,n_s)} \cong T_3(B^n_{(n_1,\ldots,n_s)})$ (respectively, $T_3(K)\otimes_K A^n_{(n_1,\ldots,n_s)} \cong T_3(A^n_{(n_1,\ldots,n_s)})$) does not contain a convex bounded subcategory which is concealed of type $\widetilde{\widetilde{\mathbb{A}}}_m$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$, or $\widetilde{\widetilde{\mathbb{E}}}_8$. ∎

7. Tame enveloping algebras. In this final section we describe the representation-finite and tame enveloping algebras $A^{\mathrm{e}} = A\otimes_K A^{\mathrm{op}}$.

THEOREM 7.1. *Let A be an algebra. Then A^{e} is representation-finite if and only if A is a simply connected radical square zero Nakayama algebra.*

Proof. The necessity is a consequence of our considerations in Section 4. For the sufficiency, we observe that for a simply connected radical square zero Nakayama algebra A, A^{e} is a simply connected special biserial algebra without convex subcategories which are the path algebras of Euclidean quivers of types $\widetilde{\mathbb{A}}_m$, $m \geq 1$, and hence A^{e} is representation-finite (see [35]). ∎

THEOREM 7.2. *Let A be an algebra. Then A^{e} is representation-infinite and tame if and only if one of the following cases holds*:

(i) *A or A^{op} is isomorphic to the bound quiver algebra of one of the bound quivers*

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

(ii) *A is isomorphic to a nonsimply connected radical square zero Nakayama algebra B^n, $n \geq 1$.*

Proof. The necessity is a consequence of our considerations in Section 4. The sufficiency follows from Corollary 3.2, Lemma 4.1, Theorem 6.2, and Theorem 7.1. ∎

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Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: zbles@mat.uni.torun.pl
skowron@mat.uni.torun.pl

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