COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 2

HEAT KERNEL OF FRACTIONAL LAPLACIAN IN CONES

BY

KRZYSZTOF BOGDAN and TOMASZ GRZYWNY (Wrocław)

Krzysztof Bogdan dedicates this work to Professor Andrzej Hulanicki and Mrs. Barbara Hulanicka, with fond memories of a semester spent together in West Lafayette

Abstract. We give sharp estimates for the transition density of the isotropic stable Lévy process killed when leaving a right circular cone.

1. Introduction. Explicit sharp estimates for the Green function of the Laplacian in $C^{1,1}$ domains were given in 1986 by Zhao [64] (see also [38, 31]). Sharp estimates of the Green function of Lipschitz domains were given in 2000 by Bogdan [11]. Explicit qualitatively sharp estimates for the classical heat kernel in $C^{1,1}$ domains were established in 2002 by Zhang [62] (see also [63, 32], and [30, 54] for further extensions). Qualitatively sharp heat kernel estimates in Lipschitz domains were given in 2003 by Varopoulos [59].

The development of the boundary potential theory of the fractional Laplacian follows an analogous path. Green function estimates were obtained in 1997 and 1998 by Kulczycki and Chen and Song for $C^{1,1}$ domains [46, 29] (see [26, Corollary 1.8] for the case of dimension one, see also [14]), and in 2002 by Jakubowski for Lipschitz domains [45] (see also [51, 17]). In 2008 Chen, Kim and Song gave a sharp and explicit estimate for the heat kernel of the fractional Laplacian on $C^{1,1}$ domains [26] (see (9) below).

In this note we give an extension of the estimate to the right circular cones. We also conjecture, in agreement with the results of [59], a likely form of the estimate for a more general class of domains:

$$p_t^D(x,y) \approx P_x(\tau_D > t)P_y(\tau_D > t)p_t(x,y). \tag{1}$$

Here $p_t(x,y)$ is the heat kernel of the fractional Laplacian on the whole space $\mathbb{R}^d$, and $P_x(\tau_D > t) = \int_{\mathbb{R}^d} p_t^D(x,y)\,dy$ is the *survival probability* of the corresponding isotropic α-stable Lévy process.

2010 *Mathematics Subject Classification*: Primary 60J35, 60J50; Secondary 60J75, 31B25.
Key words and phrases: cone, fractional Laplacian, killed stable Lévy process, transition density, heat kernel.

DOI: 10.4064/cm118-2-1

The main result of this paper, Theorem 1, asserts that (1) holds indeed for the right circular cones for all $t > 0$, $x, y \in \mathbb{R}^d$ (see also (23) for a more explicit statement). It is noteworthy that all the above-mentioned estimates for bounded $C^{1,1}$ domains have the same form as for the ball (in this connection compare [26, Corollary 1.2] with [14, Corollary 3]; see also (9) below). We also remark that the right circular cones are merely special Lipschitz domains, but a number of techniques and explicit formulas make them an interesting and important test case (see [5, 33, 34, 23, 22]). We hope to encourage a further study of Lipschitz and more general domains for stable and other jump-type processes [25, 41, 36, 12]. We should emphasize that generally the estimates for Lipschitz domains cannot be as explicit as those for $C^{1,1}$ domains. For instance, the decay rate of harmonic and parabolic functions in the vertex of a cone delicately depends on the aperture of the cone (see [2, 51], and also [11]). Nevertheless, Lipschitz domains offer a natural setting for studying the boundary behavior of the Green function and the heat kernel of the Brownian motion and α-stable Lévy processes $(0 < \alpha < 2)$. This is so because of *scaling*, the rich range of asymptotic behaviors depending on the local geometry of the domain's boundary, connections to the boundary Harnack principle, approximate factorization of the Green function, and applications in the perturbation theory of generators, in particular via the 3G Theorem [11, 2, 64, 1, 19, 42, 43, 13, 18, 16], and 3P Theorem [18]. It is noteworthy that (1) is an approximate factorization of the heat kernel (see [11, 19] in this connection).

Cones are also examples of unbounded domains, which are only partially resolved by the results of [26, 27] (note that (9) is valid only for bounded times). We should note that the upper bound in (9) was proved in 2006 by Siudeja for semibounded convex domains [57, Theorem 1.6] (stated for general convex domains in [57, Remark 1.7]). It appears that the impulse for the proof of (9) was given by Siudeja and Kulczycki in [48, Theorem 4.2]; see also [4, Proposition 2.9] by Kulczycki and Bañuelos. A similar but weaker upper bound was earlier given in [2, (26)] (see also [50, 49, 52]). We also remark that [40, Theorem 4.4] gives a sharp explicit estimate for the survival probability of the relativistic process on a half-line. Generally, the subject is far from exhausted—and it seems manageable with the existing techniques.

For completeness we mention recent estimates [28, 21, 53, 60, 56, 37, 6] for transition density and potential kernel of jump-type processes. We need to point out that generally these are estimates for processes without killing. Killing is a dramatic "perturbation" analogous to Schrödinger perturbations with singular negative potentials [14, 16, 10, 12], and it strongly influences the asymptotics of the transition density and Green function. The asymptotics is crucial for solving the Dirichlet problem for the corresponding operators (see also [39, 40]). As we shall see, the heat kernel of the fractional

Laplacian in the right circular cones has a power-type asymptotics at infinity, and it decays like the distance to the boundary to the power $\alpha/2$ except at the vertex, where it decays with the rate of $\beta \in (0, \alpha)$.

The paper is composed as follows. Below in this section we recall basic facts about the transition density of the α-stable Lévy processes killed when first leaving a domain. In Section 2 we give a sharp explicit estimate for the survival probability $P_x(\tau_D > t)$ for $C^{1,1}$ domains D. In Section 3 we prove our main estimates, Theorem 1 and (23), by using the ideas and results of [26] and [2]. Our general references on the boundary potential theory of the fractional Laplacian are [13] and [19]. We also refer the reader to [15] for a broad non-technical overview of the goals and methods of the theory.

In what follows, $\mathbb{R}^d$ denotes the Euclidean space of dimension $d \ge 1$, dx is the Lebesgue measure on $\mathbb{R}^d$, and $0 < \alpha < 2$. For $t > 0$ we let p_t be the smooth real-valued function on $\mathbb{R}^d$ with the following Fourier transform:

$$\int_{\mathbb{R}^d} p_t(x) e^{ix\cdot\xi}\, dx = e^{-t|\xi|^\alpha}, \qquad \xi \in \mathbb{R}^d. \tag{2}$$

For instance, $\alpha = 1$ yields

$$p_t(x) = \Gamma((d+1)/2)\pi^{-(d+1)/2} \frac{t}{(|x|^2 + t^2)^{(d+1)/2}},$$

the Cauchy convolution semigroup of functions [58]. We generally have

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x), \qquad x \in \mathbb{R}^d,\ t > 0. \tag{3}$$

This follows from (2). The semigroup $P_t f(x) = \int_{\mathbb{R}^d} f(y) p_t(y-x)\, dy$ has $\Delta^{\alpha/2}$ as infinitesimal generator [7, 61, 13, 44], where

$$\Delta^{\alpha/2}\varphi(x) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \lim_{\varepsilon\downarrow 0} \int_{\{|y|>\varepsilon\}} \frac{\varphi(x+y) - \varphi(x)}{|y|^{d+\alpha}}\, dy, \qquad x \in \mathbb{R}^d.$$

Here $\phi \in C_c^\infty(\mathbb{R}^d)$, i.e. $\phi : \mathbb{R}^d \to \mathbb{R}$ is smooth and compactly supported on $\mathbb{R}^d$. Put differently,

$$\int_s^\infty \int_{\mathbb{R}^d} p_{u-s}(z-x)[\partial_u \phi(u,z) + \Delta_z^{\alpha/2}\phi(u,z)]\, dz\, du = -\phi(s,x),$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ ([16]). We denote by

$$\nu(y) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} |y|^{-d-\alpha}$$

the density function of the Lévy measure of the semigroup $\{P_t\}$ [55, 20, 15].

There is a constant c such that (see [20] or [9])

$$c^{-1}\left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}\right) \le p_t(x) \le c\left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}\right), \qquad x \in \mathbb{R}^d,\ t > 0. \tag{4}$$

Inequality (4) and similar *sharp estimates* (i.e. such that the lower and upper bounds are comparable) will be abbreviated as follows:

$$p_t(x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \qquad x \in \mathbb{R}^d,\ t > 0. \tag{5}$$

The standard isotropic α-stable Lévy process (X_t, P_x) on $\mathbb{R}^d$ may be constructed by specifying the following time-homogeneous transition probability:

$$P_t(x, A) = \int_A p_t(y - x)\, dy, \qquad t > 0,\ x \in \mathbb{R}^d,\ A \subset \mathbb{R}^d,$$

and stipulating that $P_x(X(0) = x) = 1$. Thus, P_x, E_x denote the distribution and expectation for the process starting from x. The distribution of the process is concentrated on right continuous functions $[0, \infty) \to \mathbb{R}^d$ with left limits, and for all $s \ge 0$ and $x \in \mathbb{R}^d$ we have $P_x(X_s = X_{s-}) = 1$. It is well-known that (X_t, P_x) is strong Markov with respect to the so-called standard filtration [8, 10]. The Lévy system (see [35, VII.68], [28, Appendix A], also [57, Theorem 2.4], [48, Corollary 2.8] and [3, Lemma 1]) for (X_t, P^x) amounts to the equality

$$E_x\Big[\sum_{s \le T} f(s, X_{s-}, X_s)\Big] = E_x\Big[\int_0^T \Big(\int_{\mathbb{R}^d} f(s, X_s, y)\nu(w - X_s)\, dw\Big)\, ds\Big], \tag{6}$$

where $x \in \mathbb{R}^d$, $f \ge 0$ is a Borel function on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ such that $f(s, z, w) = 0$ if $z = w$, and T is a stopping time with respect to the filtration of X.

For open $D \subset \mathbb{R}^d$ we let $\tau_D = \inf\{t > 0 : X_t \notin D\}$, and we define

$$p_t^D(x, y) = p_t(x, y) - E_x[\tau_D < t;\ p_{t-\tau_D}(X_{\tau_D}, y)], \qquad x, y \in \mathbb{R}^d,\ t > 0$$

(see, e.g., [26, 13]). Clearly,

$$0 \le p_t^D(x, y) \le p_t(y - x). \tag{7}$$

By the strong Markov property, p_t^D is the transition density of the isotropic stable process *killed* on leaving D, meaning that p^D satisfies the Chapman–Kolmogorov equation

$$\int_{\mathbb{R}^d} p_s^D(x, z) p_t^D(z, y)\, dz = p_{s+t}^D(x, y), \qquad x, y \in \mathbb{R}^d,\ s, t > 0,$$

and for every $x \in \mathbb{R}^d$, $t > 0$ and bounded Borel function f,

$$\int_{\mathbb{R}^d} f(y) p_t^D(x, y)\, dy = E_x[\tau_D < t;\ f(X_t)].$$

Furthermore, for $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times D)$, we have

$$\int_s^\infty \int_D p_{u-s}^D(x, z)[\partial_u \phi(u, z) + \Delta_z^{\alpha/2} \phi(u, z)]\, dz\, du = -\phi(s, x),$$

which justifies calling p^D the heat kernel of the fractional Laplacian *on* D. In analogy with (3) we have the following scaling property:

$$p_t^D(x,y) = t^{-d/\alpha} p_1^{t^{-1/\alpha}D}(t^{-1/\alpha}x, t^{-1/\alpha}y), \qquad x,y \in \mathbb{R}^d,\ t>0. \tag{8}$$

2. $C^{1,1}$ domains. Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ domain, meaning that D is open and there is $r_0 > 0$ such that for every $z \in \partial D$ there exist balls $B_z(r_0) \subset D$ and $B'_z(r_0) \subset D^c$ of radius r_0, *tangent* at z. Set $\delta_D(x) = \operatorname{dist}(x, D^c)$, the distance to D^c. The transition density of the stable Lévy process killed off D satisfies ([26])

$$p_t^D(x,y) \approx \left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D^{\alpha/2}(y)}{\sqrt{t}}\right) p_t(x,y), \qquad 0 < t \le 1,\ x,y \in \mathbb{R}^d. \tag{9}$$

COROLLARY 1. *If D is a $C^{1,1}$ domain then*

$$P_x(\tau_D > t) \approx 1 \wedge \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}, \qquad 0 < t \le 1,\ x,y \in \mathbb{R}^d. \tag{10}$$

Proof. We have

$$P_x(\tau_D > t) = \int_{\mathbb{R}^d} p_t^D(x,y)\,dy.$$

By (9),

$$P_x(\tau_D > t) = \int_D p_t^D(x,y)\,dy \approx \left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}\right) I_t(x), \qquad 0 < t \le 1,\ x,y \in \mathbb{R}^d,$$

where

$$I_t(x) = \int_D \left(1 \wedge \frac{\delta_D^{\alpha/2}(y)}{\sqrt{t}}\right) p_t(x,y)\,dy.$$

Clearly, $I_t(x) \le \int_{\mathbb{R}^d} p_t(x,y)\,dy = 1$. This yields the upper bound in (10). To prove the lower bound we consider $0 < t \le 1$ and we will first assume that $\delta_D(x) > t^{1/\alpha}$. If $|y-x| < t^{1/\alpha}/2$, then $p_t(x,y) \approx t^{-d/\alpha}$, and we get

$$I_t(x) \ge c \int_{|y-x|<t^{1/\alpha}/2} t^{-d/\alpha}\,dy = c > 0.$$

If $\delta_D(x) \le t^{1/\alpha}$, then let $z \in \partial D$ be such that $|x-z| = \delta_D(x)$, and consider the inner tangent ball $B_z(t^{1/\alpha} \wedge r_0)$ for D at z, with center at, say, w. We have

$$I_t(x) \ge \int_{B_z(t^{1/\alpha}\wedge r_0)} \frac{(t^{1/\alpha} - |y-w|)^{\alpha/2}}{\sqrt{t}}\, p_t(x,y)\,dy.$$

Since

$$p_t(x,y) = t^{-d/\alpha} p_1\left(\frac{x-w}{t^{1/\alpha}}, \frac{y-w}{t^{1/\alpha}}\right),$$

by changing variable $v = (y - w)/t^{1/\alpha}$, we get

$$I_t(x) \geq \int_{B(0,1\wedge r_0)} (1 - |v|)^{\alpha/2} p_1(u, v)\, dv,$$

where $u = t^{-1/\alpha}(x - w) \in B(0,1)$. The latter integral is continuous and strictly positive for $u \in \overline{B(0,1)}$. Thus, $\inf_{x\in D} I_t(x) > 0$. The proof of (10) is complete. ∎

Corollary 2. *If D is a $C^{1,1}$ domain then*

$$p_t^D(x,y) \approx P_x(\tau_D > t)P_y(\tau_D > t)p_t(x,y), \quad 0 < t \leq 1,\ x, y \in \mathbb{R}^d.$$

3. Cones. For $x \in \mathbb{R}^d \setminus \{0\}$ we denote by $\theta(x)$ the angle between x and the point $(0, \ldots, 0, 1) \in \mathbb{R}^d$. We fix $0 < \Theta < \pi$ and consider the right circular cone $\Gamma = \{x \in \mathbb{R}^d \setminus \{0\} : \theta(x) < \Theta\}$. Clearly, $r\Gamma = \Gamma$ for every $r > 0$. By (8),

$$p_t^\Gamma(x,y) = t^{-d/\alpha} p_1^\Gamma(t^{-1/\alpha}x, t^{-1/\alpha}y), \quad x, y \in \mathbb{R}^d,\ t > 0. \tag{11}$$

We fix $x_0 \in \Gamma$ and consider the Martin kernel M for Γ with the pole at infinity, so normalized that $M(x_0) = 1$. It is known that there is $0 \leq \beta < \alpha$ such that

$$M(x) = |x|^\beta M(x/|x|), \quad x \neq 0$$

(see [2, 51, 17]). Since the boundary of Γ is smooth except at the origin, by [51, Lemma 3.3],

$$M(x) \approx \delta_\Gamma^{\alpha/2}(x)|x|^{\beta-\alpha/2}, \quad x \in \mathbb{R}^d. \tag{12}$$

The following result strengthens [2, Lemma 4.2].

Lemma 3. *If Γ is a right circular cone then*

$$P_x(\tau_\Gamma > t) \approx (\delta_\Gamma^{\alpha/2}(t^{-1/\alpha}x) \wedge 1)(|t^{-1/\alpha}x| \wedge 1)^{\beta-\alpha/2}, \quad x \in \mathbb{R}^d,\ t > 0. \tag{13}$$

Proof. Since $P_x(\tau_\Gamma > t) = P_{t^{-1/\alpha}x}(\tau_\Gamma > 1)$, we only need to prove that

$$P_x(\tau_\Gamma > 1) \approx (\delta_\Gamma^{\alpha/2}(x) \wedge 1)(|x| \wedge 1)^{\beta-\alpha/2}, \quad x \in \mathbb{R}^d. \tag{14}$$

If $|x| < 1$ then (14) is a consequence of (12) and [2, Lemma 4.2]. If $|x| \geq 1$ then $P_x(\tau_\Gamma > 1) \approx \delta_\Gamma^{\alpha/2}(x) \wedge 1$. Indeed, considering $C^{1,1}$ domains Γ' and Γ'' such that $\Gamma' \subset \Gamma \subset \Gamma''$ and $\Gamma'' \setminus \Gamma' \subset B(0,1/2)$, we see that $\delta_{\Gamma'}(x) \leq \delta_\Gamma(x) \leq \delta_{\Gamma''}(x) \leq 2\delta_{\Gamma'}(x)$ for such x. Since $P_x(\tau_{\Gamma'} > 1) \leq P_x(\tau_\Gamma > 1) \leq P_x(\tau_{\Gamma''} > 1)$, by using (10) we obtain (14). ∎

An interesting, if trivial, consequence of (13) is that

$$P_x(\tau_\Gamma > t) \approx P_x(\tau_\Gamma > t/2), \quad t > 0,\ x \in \mathbb{R}^d. \tag{15}$$

Theorem 1.

$$p_t^\Gamma(x,y) \approx P_x(\tau_\Gamma > t)P_y(\tau_\Gamma > t)p_t(x,y), \quad x, y \in \mathbb{R}^d,\ t > 0. \tag{16}$$

Proof. We note that the right hand side, say $R_t(x,y)$, of (16) satisfies

$$R_t(x,y) = t^{-d/\alpha} R_1(t^{-1/\alpha}x, t^{-1/\alpha}y).$$

Thus, in view of (11), we only need to prove (16) for $t=1$.

Let Γ' and Γ'' be as in the proof of Lemma 3. Then

$$p_1^{\Gamma'}(x,y) \le p_1^{\Gamma}(x,y) \le p_1^{\Gamma''}(x,y).$$

By (9) we have, for $|x|,|y| \ge 1$,

$$p_1^{\Gamma'}(x,y) \approx (1 \wedge \delta_\Gamma^{\alpha/2}(x))(1 \wedge \delta_\Gamma^{\alpha/2}(y)) p_1(x,y) \approx p_1^{\Gamma''}(x,y).$$

Hence by Lemma 3 we obtain

$$p_1^{\Gamma}(x,y) \approx P_x(\tau_\Gamma > 1) P_y(\tau_\Gamma > 1) p_1(x,y), \quad |x|,|y| \ge 1. \tag{17}$$

In particular, there is a constant c such that

$$p_1^{\Gamma}(x,y) \le c P_x(\tau_\Gamma > 1) p_1(x,y), \quad |x|,|y| \ge 1. \tag{18}$$

If $|x| < 1$ and $|y| \le 4$, then $|x-y| < 5$ and $p_1(x,y) \ge c(1 \wedge |x-y|^{-d-\alpha}) \ge c$. By the semigroup property (7) and (15),

$$\begin{aligned} p_1^{\Gamma}(x,y) &= \int_\Gamma p_{1/2}^{\Gamma}(x,w) p_{1/2}^{\Gamma}(w,y)\, dw \le c \int_\Gamma p_{1/2}^{\Gamma}(x,w)\, dw \\ &= c P_x(\tau_\Gamma > 1/2) \le c P_x(\tau_\Gamma > 1) \\ &\le c P_x(\tau_\Gamma > 1) p_1(x,y), \quad |x| < 1,\ |y| \le 4. \end{aligned} \tag{19}$$

We next assume that $|x| < 1$ and $|y| > 4$. Then $|x-y| > 3$ and $p_1(x,y) \approx |x-y|^{-d-\alpha}$. Define $\Gamma_1 = \Gamma \cap B(0,2)$, $\Gamma_2 = (\Gamma \setminus \Gamma_1) \cap B(0, (|y|+1)/2)$ and $\Gamma_3 = \Gamma \setminus B(0,(|y|+1)/2)$. Using the strong Markov property and the Lévy system (6) with $f(s,z,w) = \mathbf{1}_{\Gamma_1}(z)\mathbf{1}_{\Gamma_1^c}(w) p_{1-s}^{\Gamma}(w,y)$ and $T = 1 \wedge \tau_{\Gamma_1}$, we obtain

$$\begin{aligned} p_1^{\Gamma}(x,y) &= E_x[\tau_{\Gamma_1} < 1; p_{1-\tau_{\Gamma_1}}^{\Gamma}(X_{\tau_{\Gamma_1}}, y)] \\ &= \int_0^1 \int_{\Gamma_1} p_s^{\Gamma_1}(x,z) \int_{\Gamma \setminus \Gamma_1} \nu(w-z) p_{1-s}^{\Gamma}(w,y)\, dw\, dz\, ds \\ &= \int_0^1 \int_{\Gamma_1} p_s^{\Gamma_1}(x,z) \int_{\Gamma_2} \nu(w-z) p_{1-s}^{\Gamma}(w,y)\, dw\, dz\, ds \\ &\quad + \int_0^1 \int_{\Gamma_1} p_s^{\Gamma_1}(x,z) \int_{\Gamma_3} \nu(w-z) p_{1-s}^{\Gamma}(w,y)\, dw\, dz\, ds \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

We note that for $w \in \Gamma_2$,

$$|w-y| \ge |y| - |w| \ge |y|/4 \ge |x-y|/8.$$

Since $p^{\Gamma_1}_{1-s}(w-y) \le p_{1-s}(w-y) \le c(1-s)|w-y|^{-d-\alpha}$, we obtain

$$\begin{aligned}
\mathrm{I} &\le \int_0^1 \int_{\Gamma_1} p^{\Gamma_1}_s(x,z) \int_{\Gamma_2} \nu(w-z) c \, \frac{1-s}{|w-y|^{d+\alpha}} \, dw \, dz \, ds \\
&\le c|x-y|^{-d-\alpha} \int_0^1 \int_{\Gamma_1} p^{\Gamma_1}_s(x,z) \int_{\Gamma \setminus \Gamma_1} \nu(w-z) \, dw \, dz \, ds \\
&= c|x-y|^{-d-\alpha} P_x(X_{\tau_{\Gamma_1}} \in \Gamma \setminus \Gamma_1, \tau_{\Gamma_1} \le 1) \\
&\le c|x-y|^{-d-\alpha} P_x(X_{\tau_{\Gamma_1}} \in \Gamma \setminus \Gamma_1) \approx M(x) p_1(x,y).
\end{aligned}$$

In the last line we used BHP ([2]). For $z \in \Gamma_1$ and $w \in \Gamma_3$, we have

$$|w-z| \ge |w| - |z| \ge |y|/2 - 3/2 \ge |y|/8 \ge |x-y|/16,$$

hence $\nu(w-z) \le c|x-y|^{-d-\alpha}$, and so

$$\begin{aligned}
\mathrm{II} &\le c|x-y|^{-d-\alpha} \int_0^1 \int_{\Gamma_1} p^{\Gamma_1}_s(x,z) \int_{\Gamma_3} p^{\Gamma}_{1-s}(w,y) \, dw \, dz \, ds \\
&\le c|x-y|^{-d-\alpha} \int_0^1 \int_{\Gamma_1} p^{\Gamma_1}_s(x,z) \, dz \, ds \le c|x-y|^{-d-\alpha} E_x \tau_{\Gamma_1} \le c p_1(x,y) M(x),
\end{aligned}$$

where the last inequality follows from (5) and [2, Lemma 4.6]. By Lemma 3,

$$p^{\Gamma}_1(x,y) \le c P_x(\tau_\Gamma > 1) p_1(x,y), \qquad |x| < 1, \ |y| > 4. \tag{20}$$

Combining (18), (19) and (20), we get

$$p^{\Gamma}_1(x,y) \le c P_x(\tau_\Gamma > 1) p_1(x,y), \qquad x, y \in \mathbb{R}^d.$$

By the symmetry, semigroup property and (15) we obtain

$$\begin{aligned}
p^{\Gamma}_1(x,y) &= \int_{\Gamma} p^{\Gamma}_{1/2}(x,w) p^{\Gamma}_{1/2}(w,y) \, dw \\
&= \int_{\Gamma} 2^{2d/\alpha} p^{\Gamma}_1(x2^{1/\alpha}, w2^{1/\alpha}) p^{\Gamma}_1(w2^{1/\alpha}, y2^{1/\alpha}) \, dw \\
&\le c P_{x2^{1/\alpha}}(\tau_\Gamma > 1) P_{y2^{1/\alpha}}(\tau_\Gamma > 1) \int_{\mathbb{R}^d} p_{1/2}(x,w) p_{1/2}(w,y) \, dw \\
&\le c P_x(\tau_\Gamma > 1) P_y(\tau_\Gamma > 1) p_1(x,y).
\end{aligned}$$

We will now prove the lower bound in (16). We first assume that $|x| < 1$ and $|y| \le 2$, and we let $\Gamma_4 = \Gamma \cap B(0,4)$. Since Γ_4 is bounded, the semigroup $p^{\Gamma_4}_t$ is intrinsically ultracontractive ([47]). In particular,

$$p^{\Gamma}_{1/2}(x,y) \ge p^{\Gamma_4}_{1/2}(x,y) \ge c E_x \tau_{\Gamma_4} E^y \tau_{\Gamma_4}.$$

Furthermore, by [2, Lemma 4.6] and Lemma 3 we obtain

$$E^y \tau_{\Gamma_4} \ge c M(y) \ge c P_y(\tau_\Gamma > 1).$$

We see that

$$p^{\Gamma}_{1/2}(x,y) \geq cP_x(\tau_{\Gamma} > 1)P_y(\tau_{\Gamma} > 1)p_1(x,y), \qquad |x| < 1,\ |y| \leq 2. \tag{21}$$

If $|x| < 1$ and $|y| > 2$, then by the semigroup property, (17) and (21),

$$\begin{aligned} p^{\Gamma}_1(x,y) &= \int_{\Gamma} p^{\Gamma}_{1/2}(x,z)p^{\Gamma}_{1/2}(z,y)\,dz \\ &\geq c\,P_x(\tau_{\Gamma} > 1)P_y(\tau_{\Gamma} > 1) \int_{\Gamma_1\setminus B(0,1)} P_z(\tau_{\Gamma} > 1)^2 p_1(x,z)p_1(z,y)\,dz \\ &\geq c\,P_x(\tau_{\Gamma} > 1)P_y(\tau_{\Gamma} > 1)p_1(x,y) \int_{\Gamma_1\setminus B(0,1)} P_z(\tau_{\Gamma} > 1)^2\,dz. \end{aligned}$$

Hence

$$p^{\Gamma}_1(x,y) \geq cP_x(\tau_{\Gamma} > 1)P_y(\tau_{\Gamma} > 1)p_1(x,y), \qquad |x| < 1,\ |y| > 2. \tag{22}$$

By (17), (21), (22), symmetry (and scaling), we get the lower bound in (16). ∎

We note that Theorem 1 strengthens [2, Corollary 4.8]. Also,

$$p^{\Gamma}_t(x,y) \approx p^{\Gamma}_{t/2}(x,y), \qquad x,y \in \mathbb{R}^d,\ t > 0.$$

In view of Lemma 3, for the right circular cone Γ, (16) is equivalent to

$$\begin{aligned} &p^{\Gamma}_t(x,y) \\ &\quad\approx (\delta^{\alpha/2}_{\Gamma}(t^{-1/\alpha}x) \wedge 1)(|t^{-1/\alpha}x| \wedge 1)^{\beta-\alpha/2}\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\ &\qquad\times (\delta^{\alpha/2}_{\Gamma}(t^{-1/\alpha}y) \wedge 1)(|t^{-1/\alpha}y| \wedge 1)^{\beta-\alpha/2}, \qquad t > 0,\ x,y \in \mathbb{R}^d. \end{aligned} \tag{23}$$

This is explicit except for the exponent β (see [2] in this connection). Recall that $\int_0^\infty p^{\Gamma}_t(x,y)\,dt = G_{\Gamma}(x,y)$, the Green function of Γ. By integrating (23) one can obtain sharp estimates for the Green function of the right circular cone. For $d \geq 2$ the estimates—first given in [51, Theorem 3.10]—are the following:

$$\frac{G_{\Gamma}(x,y)}{|x-y|^{\alpha-d}} \approx 1 \wedge \left\{\frac{\delta^{\alpha/2}_{\Gamma}(x)\delta^{\alpha/2}_{\Gamma}(y)}{|y-x|^{\alpha}}\left(\frac{|x| \wedge |y|}{|x| \vee |y|}\right)^{\beta-\alpha/2}\right\}, \qquad x,y \in \mathbb{R}^d. \tag{24}$$

We skip the details of the integration (similar calculations are given in [57] and [26]). It is noteworthy that $\beta = \alpha/2$ if Γ is a half-space [2]. For the case of dimension $d = 1$, and $\Gamma = (0,\infty)$, we refer the reader to [24], see also [26, Corollary 1.2].

As stated in the Introduction, we expect (1) to be true quite generally. In particular, the approximation should hold for domains above the graph of a Lipschitz function for all times $t > 0$. Corollary 2 confirms this conjecture for $C^{1,1}$ domains and small times, while Theorem 1 proves it for the right circular cones and all times. By inspecting the relevant proofs in [51], the reader may also verify without difficulty that Theorem 1 and (24) hold as well for all those *generalized* cones ([2]) in $\mathbb{R}^d$, $d \geq 2$, which are $C^{1,1}$ except at the origin.

On the other hand, if D is a bounded $C^{1,1}$ domain, and if we denote by $-\lambda_1$ the first eigenvalue of $\Delta^{\alpha/2}$ on D (i.e. when acting on functions vanishing off D), then by the intrinsic ultracontractivity (see, e.g., [26, Theorem 1.1]),

$$p_t^D(x,y) \approx \delta_D^{\alpha/2}(x)\delta^{\alpha/2}(y)e^{-\lambda_1 t}, \quad t > 1,\ x, y \in \mathbb{R}^d,$$

and so

$$P_x(\tau_D > t) \approx \delta_D^{\alpha/2}(x)e^{-\lambda_1 t}, \quad t > 1,\ x \in \mathbb{R}^d.$$

Therefore (1) fails for large times t if D is bounded (see also [47]).

Acknowledgments. We thank Michał Ryznar and Mateusz Kwaśnicki for discussions and suggestions. The research was partially supported by KBN (MNiI 1 P03A 026 29).

REFERENCES

[1] H. Aikawa and T. Lundh, *The 3G inequality for a uniformly John domain*, Kodai Math. J. 28 (2005), 209–219.

[2] R. Bañuelos and K. Bogdan, *Symmetric stable processes in cones*, Potential Anal. 21 (2004), 263–288.

[3] —, —, *Lévy processes and Fourier multipliers*, J. Funct. Anal. 250 (2007), 197–213.

[4] R. Bañuelos and T. Kulczycki, *Trace estimates for stable processes*, Probab. Theory Related Fields 142 (2008), 313–338.

[5] R. Bañuelos and R. G. Smits, *Brownian motion in cones*, ibid. 108 (1997), 299–319.

[6] M. T. Barlow, A. Grigor'yan, and T. Kumagai, *Heat kernel upper bounds for jump processes and the first exit time*, J. Reine Angew Math. 626 (2009), 135–157.

[7] C. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Ergeb. Math. Grenzgeb. 87, Springer, New York, 1975.

[8] J. Bertoin, *Lévy Processes*, Cambridge Tracts in Math. 121, Cambridge Univ. Press, Cambridge, 1996.

[9] R. M. Blumenthal and R. K. Getoor, *Some theorems on stable processes*, Trans. Amer. Math. Soc. 95 (1960), 263–273.

[10] —, —, *Markov Processes and Potential Theory*, Pure Appl. Math. 29, Academic Press, New York, 1968.

[11] K. Bogdan, *Sharp estimates for the Green function in Lipschitz domains*, J. Math. Anal. Appl. 243 (2000), 326–337.

[12] K. Bogdan, K. Burdzy, and Z.-Q. Chen, *Censored stable processes*, Probab. Theory Related Fields 127 (2003), 89–152.

[13] K. Bogdan and T. Byczkowski, *Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains*, Studia Math. 133 (1999), 53–92.

[14] —, —, *Potential theory of Schrödinger operator based on fractional Laplacian*, Probab. Math. Statist. 20 (2000), 293–335.

[15] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, and Z. Vondraček, *Potential Analysis of Stable Processes and its Extensions*, Lecture Notes in Math. 1980, Springer, Berlin, 2009.
[16] K. Bogdan, W. Hansen, and T. Jakubowski, *Time-dependent Schrödinger perturbations of transition densities*, Studia Math. 189 (2008), 235–254.
[17] K. Bogdan and T. Jakubowski, *Problème de Dirichlet pour les fonctions α-harmoniques sur les domaines coniques*, Ann. Math. Blaise Pascal 12 (2005), 297–308.
[18] —, —, *Estimates of heat kernel of fractional Laplacian perturbed by gradient operators*, Comm. Math. Phys. 271 (2007), 179–198.
[19] K. Bogdan, T. Kulczycki, and M. Kwaśnicki, *Estimates and structure of α-harmonic functions*, Probab. Theory Related Fields 140 (2008), 345–381.
[20] K. Bogdan, A. Stós, and P. Sztonyk, *Harnack inequality for stable processes on d-sets*, Studia Math. 158 (2003), 163–198.
[21] K. Bogdan and P. Sztonyk, *Estimates of the potential kernel and Harnack's inequality for the anisotropic fractional Laplacian*, ibid. 181 (2007), 101–123.
[22] K. Burdzy, *Brownian paths and cones*, Ann. Probab. 13 (1985), 1006–1010.
[23] D. L. Burkholder, *Exit times of Brownian motion, harmonic majorization, and Hardy spaces*, Adv. Math. 26 (1977), 182–205.
[24] H. Byczkowska and T. Byczkowski, *One-dimensional symmetric stable Feynman–Kac semigroups*, Probab. Math. Statist. 21 (2001), 381–404.
[25] Z.-Q. Chen and P. Kim, *Green function estimate for censored stable processes*, Probab. Theory Related Fields 124 (2002), 595–610.
[26] Z.-Q. Chen, P. Kim, and R. Song, *Heat kernel estimates for Dirichlet fractional Laplacian*, J. Eur. Math. Soc., to appear.
[27] —, —, —, *Two-sided heat kernel estimates for censored stable-like processes*, Probab. Theory Related Fields, to appear.
[28] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Related Fields 140 (2008), 277–317.
[29] Z.-Q. Chen and R. Song, *Estimates on Green functions and Poisson kernels for symmetric stable processes*, Math. Ann. 312 (1998), 465–501.
[30] S. Cho, *Two-sided global estimates of the Green's function of parabolic equations*, Potential Anal. 25 (2006), 387–398.
[31] K. L. Chung, *Green's function for a ball*, in: Seminar on Stochastic Processes (Charlottesville, VA, 1986), Progr. Probab. Statist. 13 Birkhäuser Boston, Boston, MA, 1987, 1–13.
[32] E. B. Davies, *The equivalence of certain heat kernel and Green function bounds*, J. Funct. Anal. 71 (1987), 88–103.
[33] R. D. DeBlassie, *Exit times from cones in $\mathbb{R}^n$ of Brownian motion*, Probab. Theory Related Fields 74 (1987), 1–29.
[34] —, *The first exit time of a two-dimensional symmetric stable process from a wedge*, Ann. Probab. 18 (1990), 1034–1070.
[35] C. Dellacherie and P.-A. Meyer, *Probabilities and Potential. B. Theory of Martingales*, North-Holland Math. Stud. 72, North-Holland, Amsterdam, 1982.
[36] B. Dyda, *A fractional order Hardy inequality*, Illinois J. Math. 48 (2004), 575–588.
[37] A. Grigor'yan and J. Hu, *Off-diagonal upper estimates for the heat kernel of the Dirichlet forms on metric spaces*, Invent. Math. 174 (2008), 81–126.
[38] M. Grüter and K.-O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. 37 (1982), 303–342.
[39] T. Grzywny and M. Ryznar, *Estimates of Green functions for some perturbations of fractional Laplacian*, Illinois J. Math. 51 (2007), 1409–1438.

[40] T. Grzywny and M. Ryznar, *Two-sided optimal bounds for Green functions of half-spaces for relativistic α-stable process*, Potential Anal. 28 (2008), 201–239.
[41] Q.-Y. Guan, *Integration by parts formula for regional fractional Laplacian*, Comm. Math. Phys. 266 (2006), 289–329.
[42] W. Hansen, *Uniform boundary Harnack principle and generalized triangle property*, J. Funct. Anal. 226 (2005), 452–484.
[43] —, *Global comparison of perturbed Green functions*, Math. Ann. 334 (2006), 643–678.
[44] N. Jacob, *Pseudo Differential Operators and Markov Processes. Vol. I*, *Fourier Analysis and Semigroups*, Imperial College Press, London, 2001.
[45] T. Jakubowski, *The estimates for the Green function in Lipschitz domains for the symmetric stable processes*, Probab. Math. Statist. 22 (2002), 419–441.
[46] T. Kulczycki, *Properties of Green function of symmetric stable processes*, ibid. 17 (1997), 339–364.
[47] —, *Intrinsic ultracontractivity for symmetric stable processes*, Bull. Polish Acad. Sci. Math. 46 (1989), 325–334.
[48] T. Kulczycki and B. Siudeja, *Intrinsic ultracontractivity of the Feynman–Kac semigroup for relativistic stable processes*, Trans. Amer. Math. Soc. 358 (2006), 5025–5057.
[49] M. Kwaśnicki, *Intrinsic ultracontractivity for stable semigroups on unbounded open sets*, Potential Anal. 31 (2009), 57–77.
[50] P. J. Méndez-Hernández, *Exit times of symmetric α-stable processes from unbounded convex domains*, Electron. J. Probab. 12 (2007), no. 4, 100–121.
[51] K. Michalik, *Sharp estimates of the Green function, the Poisson kernel and the Martin kernel of cones for symmetric stable processes*, Hiroshima Math. J. 36 (2006), 1–21.
[52] S. C. Port, *Hitting times for transient stable processes*, Pacific J. Math. 21 (1967), 161–165.
[53] M. Rao, R. Song, and Z. Vondraček, *Green function estimates and Harnack inequality for subordinate Brownian motions*, Potential Anal. 25 (2006), 1–27.
[54] L. Riahi, *Estimates of Green functions and their applications for parabolic operators with singular potentials*, Colloq. Math. 95 (2003), 267–283.
[55] K.-i. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Stud. Adv. Math. 68, Cambridge Univ. Press, Cambridge, 1999.
[56] R. L. Schilling and T. Uemura, *On the Feller property of Dirichlet forms generated by pseudo differential operators*, Tohoku Math. J. (2) 59 (2007), 401–422.
[57] B. Siudeja, *Symmetric stable processes on unbounded domains*, Potential Anal. 25 (2006), 371–386.
[58] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton Univ. Press, Princeton, NJ, 1970.
[59] N. T. Varopoulos, *Gaussian estimates in Lipschitz domains*, Canad. J. Math. 55 (2003), 401–431.
[60] T. Watanabe, *Asymptotic estimates of multi-dimensional stable densities and their applications*, Trans. Amer. Math. Soc. 359 (2007), 2851–2879.
[61] K. Yosida, *Functional Analysis*, Classics in Math., Springer, Berlin, 1995 (reprint of the sixth (1980) ed.).
[62] Q. S. Zhang, *The boundary behavior of heat kernels of Dirichlet Laplacians*, J. Differential Equations 182 (2002), 416–430.
[63] —, *The global behavior of heat kernels in exterior domains*, J. Funct. Anal. 200 (2003), 160–176.

[64] Z. X. Zhao, *Green function for Schrödinger operator and conditioned Feynman–Kac gauge*, J. Math. Anal. Appl. 116 (1986), 309–334.

Krzysztof Bogdan, Tomasz Grzywny
Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: Krzysztof.Bogdan@pwr.wroc.pl
Tomasz.Grzywny@pwr.wroc.pl

Received 5 December 2008;
revised 12 March 2009 (5138)