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EIGENFUNCTIONS OF THE HARDY–LITTLEWOOD MAXIMAL OPERATOR

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Abstract. We prove that peak shaped eigenfunctions of the one-dimensional uncentered Hardy–Littlewood maximal operator are symmetric and homogeneous. This implies that the norms of the maximal operator on $L(p)$ spaces are not attained.

In [K] it is proved that the centered Hardy–Littlewood maximal operator over balls has nonzero fixed points in $L(p)$ if and only if the dimension of the space is $d \geq 3$ and $d/(d-2) < p \leq +\infty$. Such fixed points are positive superharmonic functions, for example $\inf\{1, |x|^{2-d}\}$. It is also proved that the strong centered maximal operator over parallelograms with sides parallel to the axes has no fixed points in $L(p)$ for every $1 \leq p < +\infty$. These results have been extended to rearrangement invariant spaces in [M-S]. Following this line of research, here we consider the one-dimensional uncentered Hardy–Littlewood maximal operator on locally integrable functions on a finite or infinite interval $a < x < b$,

$$Mf(x) = \sup_{a<w<x<y<b} \frac{1}{y-w} \int_w^y |f(z)|\,dz = \sup_{a<y\neq x<b} \frac{1}{y-x} \int_x^y |f(z)|\,dz.$$

Since the maximal function of a nonconstant function is larger than the function, this maximal operator has no nonconstant fixed points; however, it has eigenfunctions with eigenvalues larger than one, $Mf(x) = \lambda f(x)$. Indeed, since the operator commutes with dilations and reflections, a homogeneity argument shows that the function $|x|^{-\alpha}$ with $0 < \alpha < 1$ is an eigenfunction, with eigenvalue being the value of the maximal function at the point 1. Moreover, since the operator commutes with translations, also translates of homogeneous functions are eigenfunctions. However, there are other eigenfunctions. For example, $\sup_{n\in\mathbb{Z}} |x-n|^{-\alpha}$ is an eigenfunction with the same eigenvalue as $|x|^{-\alpha}$. Our motivation to study these eigenfunctions comes from some extremal problems. The quest for exact norms of operators on function spaces often leads to exploit the symmetries of

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the operator and the supposed extremals. In particular, it has been proved in [G-MS] that the norm of the uncentered one-dimensional maximal operator on $L(p)$, $1 < p < +\infty$, is the positive solution to the equation $(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$, a number between $p/(p-1)$ and $2p/(p-1)$. Moreover, it follows from their proof that supposed extremals are eigenfunctions of the maximal operator. It has also been proved in [B-D] and [C-L-M] that even symmetrization increases the uncentered maximal function. Hence extremals are symmetrically decreasing. As we said, homogeneous functions are eigenfunctions of the maximal operator. More precisely, if $1 < p, \lambda < +\infty$ and $(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$, then

$$M|x|^{-1/p} = \frac{1}{x + \lambda^{-p}x} \int_{-\lambda^{-p}x}^{x} |z|^{-1/p}\, dz = \lambda|x|^{-1/p}.$$

Here we want to prove that, vice versa, peak shaped eigenfunctions are, up to translations, symmetric and homogeneous. In particular, since homogeneous functions are not in $L(p)$, the norms of the maximal operator on these Lebesgue spaces are not attained. On the other hand, it has been proved in [C-L-M] that the norms of the maximal operator on other Lorentz and Marcinkiewicz spaces are attained. In particular, the norm on Weak-$L(p)$ is the same as on $L(p)$ and the homogeneous function $|x|^{-1/p}$ is an extremal. Clearly, nonzero eigenfunctions of the maximal operator with eigenvalues larger than one are positive and cannot be bounded. For this reason, in the following theorem the functions considered are assumed positive and bounded, except for a single peak at a point c, that is, $\sup_{|x-c|<\varepsilon} f(x) = +\infty$ and $\sup_{|x-c|>\varepsilon} f(x) < +\infty$ for every $\varepsilon > 0$. The peak can be inside or at one of the extremes of the interval of definition of the function.

Theorem 1. *Let $f(x)$ be a locally integrable function on a finite or infinite interval $a < x < b$, with a single peak at a point c. Assume that $f(x)$ is an eigenfunction of the uncentered maximal operator with eigenvalue λ, and let $1 < \lambda, p < +\infty$ be related by the equation $(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$.*

(1) *If $c = a, b$, then $f(x) = d|x-c|^{(1-\lambda)/\lambda}$ for some $d > 0$.*
(2) *If $a < c < b$ and $|x-c| < \min\{c-a, b-c\}$, then $f(x) = d|x-c|^{-1/p}$.*

We split the proof of the theorem into a series of lemmas.

Lemma 1.

(1) *Under the above assumptions, if $c = a, b$ then $f(x)$ is continuous and strictly monotone in $a < x < b$ and*

$$Mf(x) = \frac{1}{x-c} \int_{c}^{x} f(z)\, dz.$$

(2) *If $a < c < b$ then $f(x)$ is continuous and strictly increasing in $a < x < c$ and continuous and strictly decreasing in $c < x < b$. Moreover, for each $x \neq c$ there exists a unique y such that*

$$Mf(x) = \frac{1}{y-x}\int_x^y f(z)\,dz.$$

More precisely, if $a < x < c$ and $Mf(x) \le \lim_{z\to b-} f(z)$ then $y = b$, while if $Mf(x) > \lim_{z\to b-} f(z)$ then $c < y < b$ and $Mf(x) = f(y)$. Similarly, if $c < x < b$ and $Mf(x) \le \lim_{z\to a+} f(z)$, then $y = a$, while if $Mf(x) > \lim_{z\to a+} f(z)$, then $a < y < c$ and $Mf(x) = f(y)$.

Proof. Since (1) and (2) are similar, we only prove (2). The lemma has an easy and intuitive pseudo-proof. A nonzero eigenfunction of the maximal operator cannot have local maxima. Hence, if there is only one peak with $f(c) = +\infty$, the function has to be increasing to the left and decreasing to the right of the peak. Moreover,

$$\frac{d}{dy}\left\{\frac{1}{y-x}\int_x^y f(z)\,dz\right\} = \frac{1}{y-x}\left\{f(y) - \frac{1}{y-x}\int_x^y f(z)\,dz\right\}.$$

Hence the maximum of the averages is attained at the extremes a or b, or at a point y with $f(y) = Mf(x)$. Finally, if $y = y(x)$ is the point which realizes the maximal function at x, then

$$\frac{d}{dx}Mf(x) = \frac{d}{dx}\left\{\frac{1}{y-x}\int_x^y f(z)\,dz\right\} = \frac{f(y)-f(x)}{y-x}.$$

Since this derivative in nonzero, $Mf(x)$ is strictly monotone. The details of this proof can be fixed as follows. The maximal function is lower semicontinuous, that is, for every $t > 0$ the level sets $\{Mf(x) > t\}$ are open. Moreover, in order to evaluate the maximal function in a connected component of one of these level sets, it suffices to consider averages of the function on intervals contained in this connected component. It then follows that in every connected component of $\{f(x) > t\} = \{Mf(x) > \lambda t\}$ there are points with $f(x) > \lambda t$. In particular, every connected component of $\{f(x) > t\}$ contains a connected component of $\{f(x) > \lambda t\}$ and, iterating, one obtains a nested sequence of intervals in $\{f(x) > \lambda^n t\}$, which converges to a peak of the function. Hence, if the function has a single peak, for every $t > 0$ the sets $\{f(x) > t\}$ are nested open intervals, and this implies that the function is unimodal, increasing in $a < x < c$ and decreasing in $c < x < b$. Let $F(x,y)$ be the average of $f(z)$ over the interval with extremes x and y,

$$F(x,y) = \frac{1}{y-x}\int_x^y f(z)\,dz.$$

In particular, $\sup_{a<y<b} F(x,y) = Mf(x)$. For a fixed $a < x < c$, $F(x,z)$ increases with $z > x$ if $F(x,z) < f(z)$. Indeed, since $F(x,z)$ is continuous and $f(z)$ is lower semicontinuous, the set of z with $F(x,z) < f(z)$ is open and there $F(x,z)$ is strictly increasing. Then $F(x,z)$ stops increasing at the first point $c \le y \le b$ with $f(y) \le F(x,y)$. Indeed, under the assumption $Mf(x) = \lambda f(x)$, this point exists and is finite even in the case $b = +\infty$. Otherwise, $f(z) > F(x,z)$ for all $c < z$ and

$$\frac{z-x}{z-c}\, F(x,z) = \frac{1}{z-c}\int_x^z f(w)\,dw > Mf(z) = \lambda f(z) > \lambda F(x,z).$$

If $(z-x)/(z-c) \le \lambda$ one obtains a contradiction. Hence, for every $a < x < c$ there exists $c \le y \le b$ which defines the maximal function, $F(x,y) = Mf(x)$, with $F(x,z) < Mf(x)$ if $x < z < y$. In particular, if $x < z < y$, then

$$F(x,y) = \frac{z-x}{y-x}\, F(x,z) + \frac{y-z}{y-x}\, F(z,y).$$

If $F(x,z) < Mf(x)$, then $F(z,y) > Mf(x)$ and, a fortiori, $Mf(x) < Mf(z)$. This implies that $Mf(x)$ is strictly increasing in $a < x < c$. Similarly, one can prove that $Mf(x)$ is strictly decreasing in $c < x < b$ and this also implies that the point y with $Mf(x) = F(x,y)$ is uniquely determined. Finally, since a maximal function is lower semicontinuous, $\liminf_{w\to x} Mf(w) \ge Mf(x)$. On the other hand, for unimodal functions also the reverse inequality holds. Indeed, if $x_n \to x < c$ and $Mf(x_n) = F(x_n, y_n)$, then for a subsequence $y_{n_j} \to z \ge c$ and $\limsup_{n_j\to+\infty} Mf(x_{n_j}) = F(x,z) \le Mf(x)$. Hence, $Mf(x)$ is continuous. ∎

Lemma 2. *Under the above assumptions and if $c = a, b$, then $f(x)$ satisfies the equations*

$$\begin{cases} \dfrac{1}{x-c}\displaystyle\int_c^x f(z)\,dz = \lambda f(x), \\ (x-c)\,\dfrac{d}{dx}\, f(x) = \dfrac{1-\lambda}{\lambda}\, f(x). \end{cases}$$

The solutions to these equations are, for some constants d,

$$f(x) = d|x-c|^{(1-\lambda)/\lambda}.$$

Proof. By the previous lemma, if $c = b$ the function increases from x to c and

$$Mf(x) = \frac{1}{x-c}\int_c^x f(z)\,dz.$$

From the equality $Mf(x) = \lambda f(x)$, the integral equation follows and, by differentiation, one obtains the differential equation. ∎

The above lemmas prove part (1) of the theorem. Part (2) is slightly more complicated.

LEMMA 3. *Under the above assumptions, with* $a < c < b$, *let* $f_-(x) = f(c-x)$ *if* $0 < x < c-a$ *and* $f_+(x) = f(c+x)$ *if* $0 < x < b-c$, *also let* $\mu_\pm(t)$ *be the inverse functions of* $f_\pm(x)$. *In* $h < t < +\infty$ *with* $h = \max\{\lim_{x\to a+} f(x), \lim_{x\to b-} f(x)\}$ *these functions satisfy the integral equations*

$$\begin{cases} (\lambda-1)t\mu_-(t) = \int_t^{+\infty} \mu_-(s)\,ds + \int_{\lambda t}^{+\infty} \mu_+(s)\,ds, \\ (\lambda-1)t\mu_+(t) = \int_t^{+\infty} \mu_+(s)\,ds + \int_{\lambda t}^{+\infty} \mu_-(s)\,ds. \end{cases}$$

Proof. To every $0 < x < \mu_-(h) \le c-a$ there is associated a $0 < y < b-c$ such that

$$\begin{aligned} Mf(c-x) = f_+(y) &= \frac{1}{x+y}\left(\int_0^x f_-(z)\,dz + \int_0^y f_+(z)\,dz\right) \\ &= \frac{1}{x+y}\left(xf_-(x) + \int_{f_-(x)}^{+\infty} \mu_-(s)\,ds + yf_+(y) + \int_{f_+(y)}^{+\infty} \mu_+(s)\,ds\right). \end{aligned}$$

If $Mf(c-x) = \lambda f(c-x)$, then $f_+(y) = \lambda f_-(x)$ and this gives

$$(\lambda-1)xf_-(x) = \int_{f_-(x)}^{+\infty} \mu_-(s)\,ds + \int_{\lambda f_-(x)}^{+\infty} \mu_+(s)\,ds.$$

Similarly, to every $0 < z < \mu_+(h) \le b-c$ there is associated a $0 < w < c-a$ with $Mf(c+z) = f_-(w)$ and this gives

$$(\lambda-1)zf_+(z) = \int_{f_+(z)}^{+\infty} \mu_+(s)\,ds + \int_{\lambda f_+(z)}^{+\infty} \mu_-(s)\,ds.$$

Finally, if $f_-(x) = f_+(z) = t$, then $x = \mu_-(t)$ and $z = \mu_+(t)$ and this gives the lemma. Indeed, since each step of this proof can be reversed, these integral equations completely characterize the unimodal eigenfunctions. It also follows that these eigenfunctions are smooth away from their peaks. ∎

LEMMA 4. *Under the above assumptions, the functions* $\mu_\pm(t)$, *defined in* $h < t < +\infty$, *can be extended to the positive real axis* $0 < t < +\infty$ *in such a way that the extensions are positive, decreasing, and satisfy the differential* *equations*

$$\begin{cases} (1-1/\lambda)t \dfrac{d}{dt}\,\mu_-(t) + \mu_-(t) + \mu_+(\lambda t) = 0, \\ (1-1/\lambda)t \dfrac{d}{dt}\,\mu_+(t) + \mu_+(t) + \mu_-(\lambda t) = 0. \end{cases}$$

Proof. The integral equations in the previous lemma, when differentiated, give the differential equations in $h < t < +\infty$. These equations are linear and their solutions are defined for all $0 < t < +\infty$. Indeed, if $\mu_\pm(\lambda t)$ are defined when $\lambda t > k$, solving the equations one obtains an extension of $\mu_\mp(t)$ to $t > k$ and, iterating, one can go backward to zero. By construction, these functions are positive and decreasing at least in $h < t < +\infty$ and, by the equations, if $\mu_\pm(s) > 0$ for $s > t$, then $d\mu_\pm(t)/dt < 0$. Hence, $\mu_\pm(t)$ are positive and decreasing everywhere. ∎

LEMMA 5. *Let* $1 < p, \lambda < +\infty$, $(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$, *and*

$$\begin{cases} z(s) = \lambda^{ps}(\mu_+(\lambda^s) + \mu_-(\lambda^s)), \\ w(s) = \lambda^{ps}(\mu_+(\lambda^s) - \mu_-(\lambda^s)). \end{cases}$$

Then, under the above assumptions, these functions are bounded in $-\infty < s < +\infty$ *and satisfy the constant coefficients differential difference equations*

$$\begin{cases} \dfrac{d}{ds}\,z(s) = \lambda^{1-p}(\lambda-1)^{-1}\log(\lambda)(z(s) - z(s+1)), \\ \dfrac{d}{ds}\,w(s) = \lambda^{1-p}(\lambda-1)^{-1}\log(\lambda)(w(s) + w(s+1)). \end{cases}$$

Proof. Let

$$\begin{cases} (1-1/\lambda)t \dfrac{d}{dt}\,\mu_-(t) + \mu_-(t) + \mu_+(\lambda t) = 0, \\ (1-1/\lambda)t \dfrac{d}{dt}\,\mu_+(t) + \mu_+(t) + \mu_-(\lambda t) = 0. \end{cases}$$

If $\mu(t) = \mu_+(t) + \mu_-(t)$, the sum of the two equations gives

$$(1-1/\lambda)t \frac{d}{dt}\,\mu(t) + \mu(t) + \mu(\lambda t) = 0.$$

Moreover, if $z(s) = \lambda^{ps}\mu(\lambda^s)$ and $(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$, then

$$\begin{aligned} \frac{d}{ds}\,z(s) &= \frac{d}{ds}\,(\lambda^{ps}\mu(\lambda^s)) \\ &= p\log(\lambda)\lambda^{ps}\mu(\lambda^s) - \log(\lambda)(1-1/\lambda)^{-1}\lambda^{ps}(\mu(\lambda^s) + \mu(\lambda^{s+1})) \\ &= \lambda^{1-p}(\lambda-1)^{-1}\log(\lambda)(z(s) - z(s+1)). \end{aligned}$$

Similarly, if $\nu(t) = \mu_+(t) - \mu_-(t)$, the difference of the two equations gives

$$(1-1/\lambda)t \frac{d}{dt}\,\nu(t) + \nu(t) - \nu(\lambda t) = 0.$$

Moreover, if $w(s) = \lambda^{ps}\nu(\lambda^s)$, then

$$\frac{d}{ds}\, w(s) = \lambda^{1-p}(\lambda-1)^{-1}\log(\lambda)(w(s)+w(s+1)).$$

Since $\mu(t)$ is positive, $z(\log_\lambda(t)) = t^p\mu(t)$ is bounded by the function

$$t^p\mu(t) + \lambda^{1-p}(\lambda-1)^{-1}\int_t^{\lambda t} s^{p-1}\mu(s)\,ds.$$

On the other hand, this function is constant, as its derivative vanishes:

$$\begin{aligned}\frac{d}{dt}\Big(t^p\mu(t) &+ \lambda^{1-p}(\lambda-1)^{-1}\int_t^{\lambda t} s^{p-1}\mu(s)\,ds\Big)\\ &= t^p\,\frac{d}{dt}\,\mu(t) + (p-\lambda^{1-p}(\lambda-1)^{-1})t^{p-1}\mu(t) + \lambda(\lambda-1)^{-1}t^{p-1}\mu(\lambda t)\\ &= (1-1/\lambda)^{-1}t^{p-1}\Big((1-1/\lambda)t\,\frac{d}{dt}\,\mu(t)+\mu(t)+\mu(\lambda t)\Big) = 0.\end{aligned}$$

Finally, since $|w(s)| \le z(s)$, also $w(s)$ is bounded. ∎

Lemma 6. *Let α be a real number and let*

$$\begin{cases} \alpha\,\dfrac{d}{ds}\,z(s) = z(s) - z(s+1),\\ \alpha\,\dfrac{d}{ds}\,w(s) = w(s) + w(s+1).\end{cases}$$

Then the solutions to the first equation which are uniformly bounded on the real line are constant, and the only bounded solution to the second equation is identically zero.

Proof. It is well known that these equations have lots of solutions, depending on arbitrary functions in intervals of length one. However, it is easy to check that bounded exponential solutions are constant, and this implies that bounded solutions are constant. Indeed, if $z(s)$ is a tempered distribution, then the Fourier transform of the first equation gives

$$(2\pi i\alpha\xi - 1 + \exp(2\pi i\xi))\widehat{z}(\xi) = 0.$$

The only real zero of $2\pi i\alpha\xi - 1 + \exp(2\pi i\xi)$ is at the origin; it is simple if $\alpha \ne -1$ and double if $\alpha = -1$, hence the distribution $\widehat{z}(\xi)$ has support in $\xi = 0$ and $z(s)$ is a polynomial. More precisely, if $\alpha \ne -1$ then $\widehat{z}(\xi)$ is a point mass and $z(s)$ is a constant, while if $\alpha = -1$ then $\widehat{z}(\xi)$ is a linear combination of a point mass and a derivative of a point mass and $z(s)$ is an affine function. In both cases, if $z(s)$ is bounded then it is constant. Similarly, if $w(s)$ is a tempered distribution, then

$$(2\pi i\alpha\xi - 1 - \exp(2\pi i\xi))\widehat{w}(\xi) = 0.$$

Since $2\pi i\alpha\xi - 1 - \exp(2\pi i\xi)$ has no real zeroes, $\widehat{w}(\xi)$ has to be zero. In conclusion, if $z(s)$ is constant and $w(s)$ is zero, then $\mu_+(t) = \mu_-(t) = kt^{-p}$, so $f_-(x) = f_+(x) = dx^{-1/p}$ for all x with $f_-(x) = f_+(x) > h$, that is, $0 < x < \min\{c-a, b-c\}$. Hence $f(x) = d|x-c|^{-1/p}$ if $|x-c| < \min\{c-a, b-c\}$. ∎

From these lemmas, part (2) of Theorem 1 follows. ∎

THEOREM 2.

(1) *The norm of the maximal operator on $L(p)$, $1 < p < +\infty$, is the positive solution to the equation $(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$.*
(2) *This norm is not attained, that is, for every nonzero function f,*

$$\Big\{\int_{-\infty}^{+\infty} |Mf(x)|^p\,dx\Big\}^{1/p} < \lambda\Big\{\int_{-\infty}^{+\infty} |f(x)|^p\,dx\Big\}^{1/p}.$$

Proof. Part (1) is due to [G-MS]; however, in order to prove (2), here we present an alternative proof due to J. Duoandikoetxea. Let $M_\pm f(x)$ be the left- and right-sided Hardy–Littlewood maximal operators,

$$M_-f(x) = \sup_{y<x} \frac{1}{x-y}\int_y^x |f(z)|\,dz, \qquad M_+f(x) = \sup_{y>x} \frac{1}{y-x}\int_x^y |f(z)|\,dz.$$

Let also $Mf(x) = \max\{M_\pm f(x)\}$ and $Nf(x) = \min\{M_\pm f(x)\}$. Then, by Riesz's sunrise lemma,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (|Mf(x)|^p + |Nf(x)|^p)\,dx \\
&\qquad = \int_{-\infty}^{+\infty} (|M_-f(x)|^p + |M_+f(x)|^p)\,dx \\
&\qquad = \int_0^{+\infty} pt^{p-1}(|\{M_-f(x) > t\}| + |\{M_+f(x) > t\}|)\,dt \\
&\qquad = \int_0^{+\infty} pt^{p-1}\Big(t^{-1}\int_{\{M_-f(x)>t\}} |f(x)|\,dx + t^{-1}\int_{\{M_+f(x)>t\}} |f(x)|\,dx\Big)\,dt \\
&\qquad = \frac{p}{p-1}\int_{-\infty}^{+\infty} |f(x)|(|M_-f(x)|^{p-1} + |M_+f(x)|^{p-1})\,dx \\
&\qquad = \frac{p}{p-1}\int_{-\infty}^{+\infty} |f(x)|(|Mf(x)|^{p-1} + |Nf(x)|^{p-1})\,dx.
\end{aligned}$$

Hence, by Hölder's inequality,

$$\begin{aligned}(p-1)\int_{-\infty}^{+\infty}|Mf(x)|^p\,dx+(p-1)\int_{-\infty}^{+\infty}|Nf(x)|^p\,dx\\ \le p\Big\{\int_{-\infty}^{+\infty}|f(x)|^p\,dx\Big\}^{1/p}\Big\{\int_{-\infty}^{+\infty}|Mf(x)|^p\,dx\Big\}^{(p-1)/p}\\ +p\Big\{\int_{-\infty}^{+\infty}|f(x)|^p\,dx\Big\}^{1/p}\Big\{\int_{-\infty}^{+\infty}|Nf(x)|^p\,dx\Big\}^{(p-1)/p}.\end{aligned}$$

If $\|Mf\|_{L(p)}/\|f\|_{L(p)}=\lambda$ and $\|Nf\|_{L(p)}/\|f\|_{L(p)}=\mu$, then

$$(p-1)\lambda^p-p\lambda^{p-1}\le-(p-1)\mu^p+p\mu^{p-1}.$$

As $Nf(x)\ge|f(x)|$ almost everywhere, it follows that $\mu\ge1$ and $-(p-1)\mu^p+p\mu^{p-1}\le1$. This proves that the norm of the uncentered maximal operator on $L(p)$ is smaller than or equal to the positive solution to the equation $(p-1)\lambda^p-p\lambda^{p-1}-1=0$. On the other hand, by testing the operator on suitable truncations of homogeneous functions, $|x|^{-1/p}\chi_{\{\varepsilon<|x|<\delta\}}(x)$ with $\varepsilon\to0$ and $\delta\to+\infty$, one checks that the norm of the operator is larger than or equal to the solution of $(p-1)\lambda^p-p\lambda^{p-1}-1=0$. This proves (1).

In order to prove (2), one has to analyze when the above inequalities reduce to equalities. Assuming $f(x)$ nonnegative, there is equality in Hölder's inequality only if $f(x)$ is proportional to $Mf(x)$ and to $Nf(x)$, hence $Mf(x)=\lambda f(x)$ and $Nf(x)=\mu f(x)$ almost everywhere. The equality $-(p-1)\mu^p+p\mu^{p-1}=1$ holds only if $\mu=1$. Then, by modifying $f(x)$ if necessary on a set of measure zero, one can assume $Nf(x)=f(x)$ everywhere and this implies that $f(x)$ is unimodal. Indeed, the equality $\min\{M_\pm f(x)\}=f(x)$ and lower semicontinuity of maximal functions imply that in any interval, $\inf_{a\le x\le b}f(x)=\min\{f(a),f(b)\}$. The fact that the supposed extremals are eigenfunctions of the maximal operator also follows from [G-MS], and the unimodality and symmetry also follow from [C-L-M]. If $f(x)$ is unimodal, then $Mf(x)$ is continuous and it follows that $Mf(x)=\lambda f(x)$ everywhere. Finally, by the previous theorem, a unimodal eigenfunction of the maximal operator is homogeneous, hence it is not in $L(p)$. ∎

Remark 1. If $Mf(x)=\lambda f(x)$ on the line, then Theorem 1 applies to every connected component of the level set $\{Mf(x)>t\}$ which contains only one peak. Moreover, the peak is the midpoint of the connected component. Anyhow, as we said, there are eigenfunctions of the uncentered maximal operator which are not unimodal. A simple example is $\sup_{n\in\mathbb{Z}}|x-n|^{-\alpha}$, but it is also possible to construct bimodal eigenfunctions, with level sets of finite measure. Let $1<p,\lambda<+\infty$ with $(p-1)\lambda^p-p\lambda^{p-1}-1=0$ and let $f(x)$ be an even function, with two peaks at ±1, continuous and decreasing in $1<x<+\infty$ and equal to $\sup\{|x+1|^{-1/p},|x-1|^{-1/p}\}$ in $|x|\le1+(\lambda(p-1)/p)^p$. A suitable definition in $|x|>1+(\lambda(p-1)/p)^p$ will give the desired eigenfunction. The average of this function over an interval $\varepsilon - 1 < x < \varepsilon + 1$ with $|\varepsilon| < 1$ is $p/(p-1)$. This implies that, in order to compute the maximal function, if $|x| < 1 + (\lambda(p-1)/p)^p$ one has to average over one peak and there $Mf(x) = \lambda f(x)$, while if $|x| > 1 + (\lambda(p-1)/p)^p$ one has to average over two peaks. If $0 < t < 1$ let $\mu(t) = |\{f(x) > t\}|/2 - 1$, and if $t \geq 1$ let $\mu(t) = |\{f(x) > t\}|/4 = t^{-p}$. As in the proof of Theorem 1, the condition that $Mf(x) = \lambda f(x)$ if $|x| > 1 + (\lambda(p-1)/p)^p$ can be translated into an integral equation in $0 < t < p/(\lambda(p-1))$,

$$\frac{2p/(p-1) + t\mu(t) + \int_t^{+\infty} \mu(s)\, ds + \lambda t\mu(\lambda t) + \int_{\lambda t}^{+\infty} \mu(s)\, ds}{2 + \mu(t) + \mu(\lambda t)} = \lambda t.$$

From this, simplifying and differentiating, one obtains

$$(1 - 1/\lambda)t \frac{d}{dt} \mu(t) + \mu(t) + \mu(\lambda t) + 2 = 0.$$

In order to construct an eigenfunction it suffices to find a function $\mu(t)$ continuous and decreasing in $0 < t < +\infty$, with $\mu(t) = t^{-p}$ if $t \geq p/\lambda(p-1)$ and which satisfies the differential equation in $0 < t < p/(\lambda(p-1))$. Observe that a solution, if positive at infinity, has to be positive and decreasing in $0 < t < +\infty$. Hence, the solution of this equation gives the desired eigenfunction.

Remark 2. An eigenfunction of the maximal operator with a single peak is homogeneous, hence it is not in $L(p)$ for any $1 \leq p \leq +\infty$. There are nonunimodal eigenfunctions with more than one peak, but in any case no nonzero eigenfunction is in $L(p)$. To see this, first check that if $1 < p$, $\lambda < +\infty$ and $(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$, then $d\lambda/dp < 0$. Also observe that if $Mf(x) = \lambda f(x)$, then $\|Mf\|_{L(q)} = \lambda \|f\|_{L(q)}$ for every $1 \leq q \leq +\infty$. But an eigenvalue cannot be larger than the norm of the operator, hence a nonzero eigenfunction cannot be in $L(q)$ if $q > p$. If one can prove that $f(x) \geq C|x|^{-1/p}$ for some $C > 0$ and every $|x| \geq 1$, then it also follows that an eigenfunction is not in $L(q)$ if $q \leq p$. Let C be such that

$$C \int_{-1}^{1} |z|^{-1/p}\, dz < \min\left\{ \int_{-1}^{0} f(z)\, dz, \int_{0}^{1} f(z)\, dz \right\}.$$

Since $f(x)$ is lower semicontinuous, if the inequality $f(x) \geq C|x|^{-1/p}$ fails in $|x| \geq 1$, then there exist $\varepsilon > 0$ and $|x| \geq 1$ with the property that $f(x) = C(1-\varepsilon)|x|^{-1/p}$, while $f(z) > C(1-\varepsilon)|z|^{-1/p}$ for every $1 < |z| < |x|$. On the other hand,

$$\begin{aligned} f(x) = \lambda^{-1} Mf(x) &\geq \frac{\lambda^{-1}}{x + \lambda^{-p}x} \int_{-\lambda^{-p}x}^{x} f(z)\, dz \\ &> C\, \frac{(1-\varepsilon)\lambda^{-1}}{x + \lambda^{-p}x} \int_{-\lambda^{-p}x}^{x} |z|^{-1/p}\, dz = C(1-\varepsilon)|x|^{-1/p}. \end{aligned}$$

REMARK 3. There are analogs of Theorems 1 and 2 for the left and right maximal operators. If $f(x)$ is an eigenfunction of the left-sided maximal operator, $M_{-}f(x) = \lambda f(x)$, with a single peak at a point c, then $f(x) = d(x-c)_{+}^{(1-\lambda)/\lambda}$. A similar statement holds for the right maximal operator, with $\pm$ interchanged. As before, there is a relation between eigenvalues and norms. The norm of these maximal operators on $L(p)$, $1 < p < +\infty$, is $p/(p-1)$, which is the eigenvalue λ associated to the power $-1/p$. The norm is not attained and indeed it can be proved that there are no nonzero eigenfunctions in $L(p)$. Finally, after the one-sided and uncentered maximal operators, one maximal operator is still missing, the centered one.

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