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THE MONTGOMERY MODEL REVISITED

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Abstract. We discuss the spectral properties of the operator

$$\mathfrak{h}_{\mathcal{M}}(\alpha) := -\frac{d^2}{dt^2} + \left(\frac{1}{2}\,t^2 - \alpha\right)^2$$

on the line. We first briefly describe how this operator appears in various problems in the analysis of operators on nilpotent Lie groups, in the spectral properties of a Schrödinger operator with magnetic field and in superconductivity. We then give a new proof that the minimum over α of the groundstate energy is attained at a unique point and also prove that the minimum is non-degenerate. Our study can also be seen as a refinement for a specific nilpotent group of a general analysis proposed by J. Dziubański, A. Hulanicki and J. Jenkins.

1. Historical context and main result

1.1. Sublaplacians on nilpotent groups. In the seventies the operator

$$\mathfrak{h}_{\mathcal{M}}(\alpha) := -\frac{d^2}{dt^2} + \left(\frac{1}{2}\,t^2 - \alpha\right)^2 \tag{1.1}$$

appears in the analysis of the analytic hypoellipticity of left invariant operators on stratified nilpotent groups. When considering the stratified nilpotent Lie algebra $\mathbb{N}_4$ of rank 3 and dimension 4 (called the *Engel algebra*) with two generators X_1 and X_2, the Hörmander Laplacian $\mathcal{P}$ on the corresponding nilpotent group reads in exponential coordinates

$$\mathcal{P} := -(X_1^2 + X_2^2) = -\partial_t^2 - \left(\partial_x + t\partial_y + \frac{t^2}{2}\,\partial_z\right)^2. \tag{1.2}$$

Although the hypoellipticity of this operator was well known, the question of its hypoanalyticity was open in 1979. The author, inspired by results of G. Métivier on nilpotent groups of rank 2 and by the celebrated

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example of Baouendi–Goulaouic $D_t^2 + t^2 D_x^2 + D_y^2$ on $\mathbb{R}^3$, shows that the non-hypoanalyticity of this operator will result from the property that, for some $\alpha \in \mathbb{C}$, the operator $d\pi_\alpha(\mathcal{P})$, where π_α corresponds to a family of complexified representations of $\mathbb{N}_4$, becomes non-injective on the space $\mathcal{S}_{\pi_\alpha}$ of C^∞-vectors of the representation. If one has in mind that

$$d\pi_\alpha(\mathcal{P}) = \mathfrak{h}_{\mathcal{M}}(\alpha),$$

and that $\mathcal{S}_{\pi_\alpha} = \mathcal{S}(\mathbb{R})$, one recognizes the operator which is of interest for us.

Note that a positive answer to the question of non-injectivity for some complex α was solved by Pham The Lai and D. Robert in [19]. We refer to the survey [8] for a discussion of this result and a short presentation of the results of Rothschild–Stein and Helffer–Nourrigat in connection with Rockland's conjecture (see also [14] written in 1985).

The analysis of non-hypoanalyticity in this spirit has been pursued in the works by Chanillo, Christ, O. and D. Costin, Hanges, Helffer, Himonas, Laptev, Robert, Trèves, X. P. Wang (see [2, 15, 20] and references therein).

Let us recall very briefly how one can recover $\mathfrak{h}_{\mathcal{M}}(\alpha)$ by hand. Starting from (1.2), we can take the partial Fourier transform in the x, y and z variables and get the family of operators

$$d\pi_{\xi,\eta,\zeta}(\mathcal{P}) := -\frac{d^2}{dt^2} + \left(\xi + t\eta + \frac{t^2}{2}\,\zeta\right)^2,$$

depending on the three parameters (ξ, η, ζ) which give a (redundant) description of irreducible representations of $\mathbb{N}_4$ (except when $\eta = \zeta = 0$). So up to a dilation, $\mathfrak{h}_{\mathcal{M}}(\alpha)$ corresponds to $d\pi(\mathcal{P})$ for the non-degenerate representations (in the sense of Rockland). The question we will analyze below can be formulated in this way: What is the best constant C_{subell} such that the inequality

$$\| \, |D_z|^{1/3} u \|^2 \leq C_{\text{subell}} \langle \mathcal{P}u, u \rangle_{L^2(\mathbb{R}^4)}, \qquad \forall u \in C_0^\infty(\mathbb{R}^4), \tag{1.3}$$

holds? This inequality measures the subellipticity of this Hörmander operator. We note that this type of question can be formulated in a quite general context (see for example the contribution by Dziubański–Hulanicki–Jenkins [4]).

1.2. Schrödinger operators: can one hear the zero-locus of a magnetic field. For the analysis of Schrödinger operators with magnetic fields on compact manifolds, this model was introduced for the first time by Montgomery [16] and was further studied in [12, 18, 6, 9]. Here the toy model (1) proposed by Montgomery is defined on $\mathbb{R}_t \times \mathbb{S}^1_\theta$,

$$h^2 D_t^2 + \left(hD_\theta - \frac{1}{2}\,t^2\right)^2, \tag{1.4}$$

where $h > 0$ is a semi-classical parameter and one is interested in the analysis of the bottom of the spectrum, $E_1(h)$. It is not difficult to see that

$$E_1(h) = h^{4/3} \inf_{n\in\mathbb{Z}} \lambda_1(h^{1/3}n), \tag{1.5}$$

where, for any $\alpha \in \mathbb{R}$, $\lambda_1(\alpha)$ denotes the lowest eigenvalue of the self-adjoint realization $\mathfrak{h}_{\mathcal{M}}(\alpha)$. Let us define

$$\lambda_* := \inf_{\alpha\in\mathbb{R}} \lambda_1(\alpha), \tag{1.6}$$

where $\lambda_j(\alpha)$ denotes the jth eigenvalue of $\mathfrak{h}_{\mathcal{M}}(\alpha)$. To make the link with the problem in (1.3), note that

$$\lambda_* = 1/C_{\mathrm{subell}}. \tag{1.7}$$

1.3. The Montgomery model in superconductivity. In [17, 18] X. B. Pan first observes the role of this operator in the analysis of the onset of superconductivity. This time one looks at the Neumann realization of a Schrödinger operator with constant magnetic field in a bounded regular domain and the analysis of the two-term semi-classical expansion involves λ^*. The result of X. B. Pan was completed in Helffer–Morame [13] and one can find in [5] a presentation of the subject.

1.4. Main result. The main result of this note is the following:

THEOREM 1. *There exists a unique $\alpha_{\min}$ such that*

$$\lambda_* = \lambda_1(\alpha_{\min}). \tag{1.8}$$

Moreover,

$$\alpha_{\min} > 0, \tag{1.9}$$

$$\lambda_1''(\alpha_{\min}) > 0. \tag{1.10}$$

Except the last statement which was conjectured in [10], the theorem was conjectured by Montgomery [16], partially analyzed in [12, 13], stated in Pan–Kwek [18], and verified with a computer assisted (2) but not completely mathematically rigorous proof in [10, 11]. One can also find in [6] and [5] a presentation of some of the results.

(1) Note that this operator is not defined on a compact manifold but has compact resolvent.

(2) By numerical computations of V. Bonnaillie-Noël.

The proof given here will use one essential part obtained in [18] but will avoid the part devoted to the analysis of the solutions of the Riccati equation.

2. Previous results on $\mathfrak{h}_{\mathcal{M}}(\alpha)$. We collect various results obtained in various references [16, 13, 18, 10]. We first note that $\lambda_1(\alpha)$ is continuous and satisfies

$$\lim_{|\alpha|\to+\infty} \lambda_1(\alpha) = +\infty\,. \tag{2.1}$$

This is actually trivial when $\alpha \to -\infty$ (we have indeed $\lambda_1(\alpha) \geq \alpha^2$ if $\alpha < 0$) and results from a semi-classical analysis as $\alpha \to +\infty$. Hence λ_1 admits at least one minimum. Moreover, the map $\alpha \mapsto \lambda_1(\alpha)$ is C^∞ (actually analytic by Kato's theory) and, if we denote by $u_1(t,\alpha)$ the corresponding eigenfunction in $L^2(\mathbb{R})$ such that $\|u_1(\cdot,\alpha)\|^2 = 1$ and $u_1(t,\alpha) > 0$ for all $t \in \mathbb{R}$, then the Feynman–Hellmann formula gives that

$$\lambda_1'(\alpha) = -2 \int_0^{+\infty} \left(\frac{1}{2}\,t^2 - \alpha\right) u_1(t,\alpha)^2\,dt. \tag{2.2}$$

In particular, this implies:

LEMMA 2. *$\alpha \mapsto \lambda_1(\alpha)$ is decreasing for $\alpha \leq 0$ and all the critical points of λ_1 are in $]0,+\infty[$. In addition, λ_1 admits a strictly positive minimum which can only be attained for positive α's.*

Observing that the first eigenfunction is even, we get the following easy lemma (see for example [18]).

LEMMA 3. *$\lambda_1(\alpha) = \lambda_1^N(\alpha)$ where $\lambda_1^N(\alpha)$ is the groundstate energy of the Neumann realization of $-d^2/dt^2 + \left(\frac{1}{2}t^2 - \alpha\right)^2$ in $\mathbb{R}^+$.*

Observing that the second eigenvalue of $\mathfrak{h}_{\mathcal{M}}(\alpha)$ corresponds to the odd spectrum, we also get

$$\lambda_3(\alpha) = \lambda_2^N(\alpha). \tag{2.3}$$

In [10], the following has been proved:

LEMMA 4. *If α_c is a critical point of λ_1 and if*

$$3\lambda_3(\alpha_c) > 7\lambda_1(\alpha_c), \tag{2.4}$$

then

$$\lambda_1''(\alpha_c) > 0. \tag{2.5}$$

Let us recall the proof for completeness. When proving the Feynman–Hellmann formula, we start from

$$(\mathfrak{h}_{\mathcal{M}}(\alpha) - \lambda_1(\alpha))\,\frac{\partial u_1(\cdot,\alpha)}{\partial\alpha} = \left[2\left(\frac{t^2}{2} - \alpha\right) + \frac{\partial\lambda_1}{\partial\alpha}(\alpha)\right] u_1(\cdot,\alpha). \tag{2.6}$$

Multiplying by u_1 and integrating on the line gives (2.2), and differentiating once more we obtain

$$\lambda_1''(\alpha) = 2 - 4\int_{\mathbb{R}} \left(\frac{1}{2}\,t^2 - \alpha\right) u_1(t,\alpha)\partial_\alpha u_1(t,\alpha)\,dt. \tag{2.7}$$

This leads at a critical point α_c of λ_1 to

$$\lambda_1''(\alpha_c) \geq 2 - \frac{4}{\lambda_3(\alpha_c) - \lambda_1(\alpha_c)} \left\| \left(\frac{1}{2}\,t^2 - \alpha_c\right) u_1(\cdot,\alpha_c) \right\|^2. \tag{2.8}$$

Here we have used (2.6) and the fact that $\left(\frac{1}{2}t^2 - \alpha_c\right)u_1(\cdot,\alpha_c)$ and $\partial_\alpha u_1$ are even and orthogonal to u_1.

Of course, we have

$$\|(t^2 - \alpha_c)u_1(\cdot,\alpha_c)\|^2 \leq \lambda_1(\alpha_c)\,,$$

but one can do a little better (according to [18]) using some invariance under dilation. We can observe that the groundstate energy of

$$\rho^{-2}\,\frac{d^2}{dt^2} + \left(\frac{1}{2}\,\rho^2 t^2 - \alpha\right)^2$$

is independent of ρ. Hence we have, for the normalized groundstate $u_1(\cdot,\rho,\alpha)$,

$$\rho^{-2}u_1''(t,\rho,\alpha) + \left(\frac{1}{2}\,\rho^2 t^2 - \alpha\right)^2 u_1(t,\rho,\alpha) = \lambda_1(\alpha)u_1(t,\rho,\alpha).$$

We differentiate this identity with respect to ρ, take the scalar product with $u_1(\cdot,\rho,\alpha)$ and finally set $\rho = 1$ and $\alpha = \alpha_c$. We get

$$2\left\| \left(\frac{1}{2}\,t^2 - \alpha_c\right) u_1(\cdot,\alpha_c) \right\|^2 = \|u_1'(\cdot,\alpha_c\|^2,$$

and consequently

$$3\left\| \left(\frac{1}{2}\,t^2 - \alpha_c\right) u_1(\cdot,\alpha_c) \right\|^2 = \lambda_1(\alpha_c).$$

So finally, we obtain

$$\partial^2_{\alpha\alpha}\lambda_1(\alpha_c) \geq \frac{2}{3}\,\frac{3\lambda_3(\alpha_c) - 7\lambda_1(\alpha_c)}{\lambda_3(\alpha_c) - \lambda_1(\alpha_c)}.$$

This proves the lemma.

Finally, Pan–Kwek [18] proved the following formula ([3]) (playing the role of Hadamard's formula for the Dirichlet problem in the case of the Neumann

([3]) À la Bolley–Dauge–Helffer [3].

problem): for any critical point α_c of λ_1,

$$2\int_0^{+\infty} t\left(\frac{1}{2}t^2-\alpha_c\right)u(t,\alpha_c)^2\,dt=(\lambda_1(\alpha_c)-\alpha_c^2)u(0,\alpha_c)^2. \tag{2.9}$$

We recall that in this case

$$\int_0^{+\infty}\left(\frac{1}{2}t^2-\alpha_c\right)u_1(t,\alpha_c)^2\,dt=0. \tag{2.10}$$

(2.9) is deduced from the computation (by integration by parts) of

$$\int_0^{+\infty}(\partial_t q)(t,\alpha)u_1(t,\alpha)^2\,dt$$

with $q(t,\alpha)=\left(\frac{1}{2}t^2-\alpha\right)^2$.

3. End of the proof

3.1. A new inequality. A simple new observation is that a tricky combination of (2.9) and (2.10) gives that, for any critical point α_c, we have

$$0\le 2\int_0^{+\infty}(t-\sqrt{2\alpha_c})\left(\frac{1}{2}t^2-\alpha_c\right)u_1(t,\alpha_c)^2\,dt=(\lambda_1(\alpha_c)-\alpha_c^2)u_1(0,\alpha_c)^2. \tag{3.1}$$

As a corollary, we see that, if α_c is a critical point of λ_1, then

$$\alpha_c^2<\lambda_1(\alpha_c). \tag{3.2}$$

Of course this inequality is less efficient that in the analysis of the De Gennes model ([4]) where we have equality, but this will be enough! We will use the information we have obtained in the following way.

LEMMA 5. *If $\alpha_{\min}$ realizes the infimum of λ_1, we have*

$$\alpha_{\min}^2<\lambda_*. \tag{3.3}$$

Then we will complete the proof of the main theorem by proving the following lemma:

LEMMA 6. *There exists (an explicit) λ^* such that*

$$\lambda_*<\lambda^*, \tag{3.4}$$

$$3\lambda_3(\alpha)>7\lambda_1(\alpha),\qquad \forall\alpha\in[0,\sqrt{\lambda^*}]. \tag{3.5}$$

The proof will be a consequence of a good upper bound for $\lambda_1(\alpha)$ and a good lower bound for $\lambda_3(\alpha)$.

([4]) The De Gennes model corresponds to the family of harmonic oscillators $D_t^2+(t+\alpha)^2$ on the half-line with the Neumann condition. It has been shown in [3] that the groundstate energy $\mu(\alpha)$ has a unique minimum and that this minimum is non-degenerate. The proof was based on the identity $\mu'(\alpha)=u_1(0,\alpha)^2(\alpha^2-\mu(\alpha))$.

3.2. Upper bounds for $\lambda_1(\alpha)$. This upper bound will be used two times: to determine an explicit λ^* in (3.4) and then to verify (3.5).

Computing ([5]) the energy $\langle \mathfrak{h}_{\mathcal{M}}(\alpha) u_\rho, u_\rho\rangle_{L^2(\mathbb{R})}$ of the L^2-normalized Gaussian

$$u_\rho = c_\rho \exp -\frac{\rho}{2}\, t^2,$$

we get

$$E(u_\rho, \alpha) = \frac{\rho}{2} + \frac{3}{16}\,\rho^{-2} - \frac{\alpha}{2\rho} + \alpha^2.$$

Hence, for any $\rho > 0$,

$$\lambda_1(\alpha) \leq \frac{\rho}{2} + \frac{3}{16}\,\rho^{-2} - \frac{\alpha}{2\rho} + \alpha^2. \tag{3.6}$$

If we take the minimizing ρ for $\alpha = 0$, we get $\rho_0 = (3/4)^{1/3}$ and the corresponding energy is

$$E(u_{\rho_0}, \alpha) = \frac{1}{2}\left(\frac{3}{4}\right)^{1/3} + \frac{3}{16}\left(\frac{3}{4}\right)^{-2/3} + \alpha^2 - \frac{1}{2}\,\alpha\left(\frac{3}{4}\right)^{-1/3}.$$

Hence we obtain, for this specific ρ_0,

$$\lambda_1(\alpha) \leq \left(\frac{3}{4}\right)^{4/3} - 2^{-1/3}3^{-1/3}\alpha + \alpha^2. \tag{3.7}$$

Minimizing this upper bound over α we obtain

$$\lambda_* \leq 2^{1/3}3^{-2/3}. \tag{3.8}$$

Hence we will prove the lemma with

$$\lambda^* = 2^{1/3}3^{-2/3} \approx 0.6057. \tag{3.9}$$

The numerically computed infimum (already computed in [16], also later by C. Bolley [12], and quite recently with more accuracy by V. Bonnaillie-Noël) was

$$\lambda_*^{\mathrm{num}} \approx 0.57. \tag{3.10}$$

Approximately (3.7) reads

$$\lambda_1(\alpha) \leq 0.68 + \alpha^2 - 0.55\alpha \tag{3.11}$$

and

$$\alpha^* := \sqrt{\lambda^*} \approx 0.778. \tag{3.12}$$

Numerical computations of Virginie Bonnaillie-Noël (mentioned in [10]) give that the unique minimum is attained for

$$\alpha^{\mathrm{num}} \approx 0.35;$$

([5]) This idea was rather efficient for giving an upper bound in the analogous problem for the De Gennes model.

observe for later reference that

$$\alpha^{\mathrm{num}} < \alpha^* .$$

REMARK 7. One can marginally improve the choice of λ^* by taking

$$\lambda^{**} := \inf_{\rho,\alpha} \left(\frac{\rho}{2} + \frac{3}{16}\,\rho^{-2} - \frac{\alpha}{2\rho} + \alpha^2 \right).$$

This leads to

$$\lambda^{**} = \inf_{\rho} \left(\frac{\rho}{2} + \frac{1}{8}\,\rho^{-2} \right) = 3 \cdot 2^{-7/3}.$$

We observe that $\lambda^{**} < \lambda^*$ and that

$$\lambda^{**} \approx 0.595.$$

3.3. Lower bounds for $\lambda_3(\alpha)$. We develop the idea of comparing our operator with a harmonic oscillator. We first observe that, for any $\gamma \geq 0$, we have

$$\gamma t^2 + \alpha^2 - (\alpha+\gamma)^2 \leq \left(\frac{1}{2}\,t^2 - \alpha \right)^2 .$$

From this we can compare our initial operator to the harmonic oscillator with potential $v_\gamma(t) = \gamma t^2 + \alpha^2 - (\alpha+\gamma)^2$.

We then get, for any γ,

$$\lambda_3(\alpha) \geq 5\gamma^{1/2} - 2\gamma\alpha - \gamma^2 \,. \tag{3.13}$$

Then the question is how to choose γ. Without trying to be optimal, we will see that $\gamma = 1$ will be enough for our purpose and this choice leads to the lower bound

$$\lambda_3(\alpha) \geq 4 - 2\alpha. \tag{3.14}$$

3.4. Verification of (3.5). We will prove that, for any $\alpha \in [0, \alpha^*]$,

$$3(4-2\alpha) > 7(3^{4/3}2^{-8/3} - 2^{-1/3}3^{-1/3}\alpha + \alpha^2). \tag{3.15}$$

Using the calculator, this will be a consequence of

$$7.23 > 2.16\alpha + 7\alpha^2 .$$

The left hand side is increasing with α and a simple calculation shows that

$$2.16\alpha_* + 7\alpha_*^2 \approx 4.27 + 1.68 \approx 5.95.$$

Hence the proof of the lemma is finished.

The proof of the main theorem results because we have shown that there are no α realizing the minimum outside of $[0, \alpha_*]$ and that the only critical points in $[0, \alpha_*]$ are non-degenerate local minima. This shows the uniqueness.

4. Final remarks. This approach gives a stronger result than the claim in [18]. We do not know if Theorem 1 holds for the groundstate energy of the family (indexed by $\alpha \in \mathbb{R}$) of operators $\mathfrak{h}_{\mathcal{M}}^{(k)}(\alpha)$ defined on $\mathbb{R}$ by

$$\mathfrak{h}_{\mathcal{M}}^{(k)}(\alpha) := D_t^2 + \left(\frac{1}{k+1}\, t^{k+1} - \alpha\right)^2 \qquad (k > 1)$$

and considered in [1, 10, 11] but it is not excluded that it could work for some k (odd). Numerically computed graphs of λ_1 realized by V. Bonnaillie-Noël (see [10]) show that it is at least "true" for $k \leq 7$.

Note that a different (but connected) approach (as announced in [11]) permits one to prove that the conclusion of Theorem 1 holds for k large enough and that the minimum, if in addition k is even, is attained uniquely for $\alpha = 0$.

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