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LOCAL ADMISSIBLE CONVERGENCE OF HARMONIC FUNCTIONS ON NON-HOMOGENEOUS TREES

BY

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Abstract. We prove admissible convergence to the boundary of functions that are harmonic on a subset of a non-homogeneous tree equipped with a transition operator that satisfies uniform bounds suitable for transience. The approach is based on a discrete Green formula, suitable estimates for the Green and Poisson kernel and an analogue of the Lusin area function.

1. Introduction. A well known property of the boundary behaviour of harmonic functions in the upper half-plane $\mathbb{R}^2_+$ is the following. For any z in the boundary $\mathbb{R}$ of $\mathbb{R}^2_+$ and any $\alpha > 0$, denote by $\Gamma_\alpha(z)$ the cone with vertex z, i.e.,

$$\Gamma_\alpha(z) = \{(x,y) \in \mathbb{R}^2_+ : |x-z| < \alpha y\}.$$

If $f(x,y)$ is defined in an $\mathbb{R}^2_+$-neighbourhood of a boundary point z, then f has *non-tangential* (or *admissible*) *limit* at z, say equal to l, if, for every $\alpha > 0$, $f(x,y) \to l$ whenever $(x,y) \to z$ and $(x,y) \in \Gamma_\alpha(z)$. Moreover, f is called *non-tangentially* (or *admissibly*) *bounded* at z if, for some α, f is bounded in $\Gamma_\alpha(z)$ (by a constant which depends on α and z). Then, if f is harmonic in $\mathbb{R}^2_+$, the non-tangential boundedness of f and the existence of its non-tangential limits are almost everywhere equivalent: this result is due to Privalov [18]; a weaker form, that holds globally for positive harmonic functions, was proved in [8] and is often called the Fatou non-tangential convergence theorem.

Here is the local version of this statement. For any measurable subset E of $\mathbb{R}$, let $\tilde{E}$ be the union of cones in $\mathbb{R}^2_+$ (of the same arbitrary width) with vertex in E. Then the local admissible convergence theorem asserts that, if f is defined in $\tilde{E}$ and harmonic, then, at almost every point of E, f is non-tangentially bounded if and only if its non-tangential limit exists.

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The area integral introduced by Lusin gives another characterization of non-tangential boundedness of a harmonic function, referred to as the Lusin area theorem in [14]. For every $\alpha > 0$ the area integral of f is a function defined on $\mathbb{R}$ as

$$\int_{\Gamma_\alpha(z)} \|\nabla f\|^2.$$

The area theorem says that, almost everywhere, this integral is finite if and only if f has admissible limit: it was proved by Marcinkiewicz and Zygmund [14] and Spencer [20].

The general version of the local admissible convergence theorem and the area theorem for Euclidean half-spaces is due to A. P. Calderón [4] and E. M. Stein [21] (see also [22, Chapter VII]). Stein's approach uses the Green formula to transform the area integral of a harmonic function over smooth compact domains into an integral over their boundary curves.

The local admissible convergence theorem and the area theorem have a natural extension to the Poincaré half-plane or hyperbolic disc [6]. Observe that in the hyperbolic metric the cone $\Gamma_\alpha(z)$ becomes a tube around a geodesic whose end point is z. Stein's approach has been adapted to general symmetric spaces of rank one in [13]. Further extensions to negatively curved Riemannian manifolds are in [15].

A natural discrete counterpart of semisimple symmetric spaces of non-compact type of rank one is given by homogeneous and semihomogeneous trees (see [19]). A natural Laplace operator on a homogeneous tree is the nearest neighbour isotropic transition operator. Its 1-eigenspace consists of harmonic functions defined on the vertices of homogeneous trees (used in [9] to study representation theory of groups of automorphisms of trees). Non-tangential convergence of harmonic functions on homogeneous trees has been studied in [11]. The Lusin area theorem has recently been extended to homogeneous trees [1], again by making use of the Green formula, proved in [7] in the context of more general (not necessarily homogeneous) trees.

Homogeneous trees are homogeneous spaces of suitable automorphism groups related to semisimple p-adic groups. But in order to state Lusin's theorem on a tree, we do not need any group action. Indeed, the area function has been introduced in [12] on general locally finite trees, not necessarily homogeneous. Its use to obtain results on the boundary behaviour of harmonic functions on a large class of trees was briefly mentioned at the end of [12]. The aim of that reference is to study H^p spaces of harmonic functions on trees. Harmonicity is defined with respect to a nearest neighbour transition operator P whose transition probabilities are bounded away from 0 and $\frac{1}{2}$ (these uniform bounds insure transience). The points at infinity are the geodesic rays in the tree originating at a fixed reference vertex. The area function is given by the series of squares of weighted differences (with weights given by P) of values of f in tubes around a geodesic ray (that is, in a tubular neighbourhood determined by a point at infinity). See more details in Subsection 2.4 below. The methods of [12] are probabilistic and potential-theoretic, based upon hitting probabilities, martingales and their Poisson transform. An independent, more geometric approach, based upon good lambda inequalities, is followed in [7]; it makes use of a natural adaptation to the discrete setting of trees of the Green formula (a tool typical of differential analysis).

Taking advantage of tools that do not require any homogeneity, in this paper we study the local boundary behaviour of harmonic functions on a tree T that is non-homogeneous but regular (in the sense of Subsection 2.2 below) and extend the Lusin area theorem to this environment, by using, as in [1], the Green formula of [7]. Our proof follows the arguments of [1]; we refer the reader to this reference for its links to the proof in the classical, continuous setting. We need estimates for the area function in the tubular subset $W(E)$ of T generated by a measurable subset E of the boundary. These estimates are obtained on appropriate slabs (finite sets that exhaust $W(E)$), by transporting the inequalities to the *boundary* of the slabs via the Green formula. We emphasize the following two facts: the harmonic function is not assumed positive, and we look for conditions equivalent to the existence of its boundary limits locally, on a measurable subset of the boundary Ω of T. Indeed, the function does not really need to be harmonic everywhere, but only in a compact neighbourhood of E in $T \cup \Omega$; for the sake of simplicity, however, all our statements are given for functions harmonic on all of T (indeed, the local admissible convergence theorem is not much more difficult than the global result, a fact also noted in [17] in the context of negatively curved manifolds). Instead, for *positive* harmonic functions on trees, global admissible convergence is not hard to prove: see [11], and particularly its Theorem 2 for the non-homogeneous environment.

Compared to the approach for the half-space or for homogeneous trees, some peculiar differences arise in the present context. For instance, now the Green kernel is not harmonic in the second variable, but only conjugate harmonic. This fact makes it more difficult to extend to the interior of the slabs estimates valid for their boundaries, because we cannot use the maximum principle for harmonic functions.

A part of the proof can be derived from estimates for the area functions given in [7], but the arguments of [7] are more difficult than needed here (they are based on good lambda inequalities aimed at L^p estimates, not just pointwise estimates almost everywhere). Unfortunately, the result follows by the argument of this reference, but not directly from its statement, which includes an additional condition aimed at a deeper result. We include the proof here, with a different exposition to make reading easier.

Simpler pointwise estimates of the same type, based on the analogy with the results of [2, 3, 15] in continuous settings, were also obtained in [16]: its arguments are phrased in classical probabilistic terms instead of the combinatorial probability and potential theory of the parallel approach of [12]. The results in [16] deal with radial convergence and are global rather than local, but, as seen later in the case of manifolds in [17], both these restrictions could be easily bypassed. On the other hand, in contrast to the spirit of the present paper, the analogy between trees and symmetric spaces (that is, the Green formula and differential forms) is not considered in [16].

2. Notation and main result

2.1. Trees. We follow most of the terminology established in [1]. Here is a review. A *tree* T is a connected, simply connected, locally finite graph. With abuse of notation we shall also write T for the set of vertices of the tree. In contrast with [1], here we do not assume that T is homogeneous: the number of edges joining at every vertex of T may vary, but stays finite (because of some other forthcoming assumptions, it will turn out to be bounded). For $x, y \in T$ we write $x \sim y$ if x, y are neighbours. For any $x, y \in T$ there exist a unique $n \in \mathbb{N}$ and a unique minimal finite sequence $(z_0, \ldots, z_n)$ of distinct vertices such that $z_0 = x$, $z_n = y$ and $z_k \sim z_{k+1}$ for all $k < n$; this sequence is called the *geodesic* path from x to y and is denoted by $[x, y]$. The integer n is called the *length* of $[x, y]$ and is denoted by $d(x, y)$; d is a metric on T. We fix a reference vertex $o \in T$ and call it the *origin*. The choice of o induces a partial ordering in T: $x \leq y$ if x belongs to the geodesic from o to y. For $x \in T$, the length $|x|$ of x is defined as $|x| = d(o, x)$. For any vertex x and any integer $k \leq |x|$, x_k is the vertex of length k in the geodesic $[o, x]$. The *sector* $S(x)$ generated by a vertex $x \neq o$ is the set of vertices v such that $x \in [o, v]$.

For $k \in \mathbb{N}$ let S_k be the *circle* $\{x \in T : |x| = k\}$, and B_k the *ball* $\{x \in T : |x| \leq k\}$.

Let Ω be the set of infinite geodesics starting at o. In analogy with the previous notation, for $\omega \in \Omega$ and $n \in \mathbb{N}$, ω_n is the vertex of length n in the geodesic ω. For $x \in T$ the *interval* $U(x) \subset \Omega$, generated by x, is the set $U(x) = \{\omega \in \Omega : x = \omega_{|x|}\}$. The sets $U(\omega_n)$, $n \in \mathbb{N}$, form an open basis at $\omega \in \Omega$. Equipped with this topology, Ω is compact and totally disconnected.

2.2. Very regular transition operators. On the vertices of T we consider a *very regular* transition operator, that is, an operator P with transition probabilities $p(u, v)$ that do not vary too much, in the following sense:

(H1) P is a nearest neighbour operator, that is, $p(u, v) = 0$ unless the vertices u and v are neighbours;

(H2) for some constants δ_-, δ_+, with $0 < \delta_\pm < 1/2$, and for all neighbours u and v, the following inequality holds:

$$\delta_- \le p(u, v) \le 1/2 - \delta_+.$$

Observe that the lower bound $p(u, v) \ge \delta_-$ yields a bound on the homogeneity degree (that is, the number of neighbours) at each vertex: this number is bounded by $1/\delta_-$. Instead, the upper bound $p(u, v) \le 1/2 - \delta_+$ implies that, once a reference vertex o is fixed, the probability of moving forward is larger than the probability of moving backwards, and so it is clear that the random walk induced by P is *transient* (the random vertex X_n at time n of the random walk induced by P moves definitively out of every finite set almost surely). A rigorous proof of transience was given by W. Woess and the author in [12, Appendix]. Therefore this upper bound provides a uniform estimate for the backward and forward flows, that is, for transience; we remark that this uniform condition is sufficient but not necessary for transience, but it is very natural in this context.

REMARK 1. For the sake of simplicity, from now on we shall take $\delta_+ = \delta_-$ and shall call this constant δ, even if this leads to the unnecessary restriction $\delta \le 1/4$; deriving more precise formulas involving two different constants is a simple matter that does not add anything important to our results and is left to the interested reader.

2.3. Harmonic measure. For every positive integer n, the family $\{U(x) : |x| = n\}$ is a partition of Ω into finitely many open and closed sets. These sets generate a σ-algebra on the boundary Ω of T. Taking advantage of transience, we define a measure ν on every set E in this σ-algebra by

$$\nu(E) = \Pr[X_\infty \in E], \tag{2.1}$$

where the limit random boundary point $X_\infty = \lim_{n\to\infty} X_n$ exists almost surely because of transience (since there are no absorbing states). The measure ν extends to a regular Borel probability measure on Ω, called the *boundary hitting distribution*, or *harmonic measure*.

2.4. Harmonic functions, the area function and local admissible convergence. First of all, let us define the *Laplace operator* $\Delta = P - \mathbb{I}$, where $\mathbb{I}$ is the identity operator. The transposed operator of the transition operator P is denoted by P^*; that is, for all vertices x, y,

$$p^*(x, y) = p(y, x).$$

The *conjugate Laplacian* is the transpose operator $\Delta^* = P^* - \mathbb{I}$.

DEFINITION 1. A function $f : T \to \mathbb{R}$ is *harmonic* at $x \in T$ if $Pf(x) \equiv \sum_{y\sim x} p(x,y)\, f(y) = f(x)$. A function is harmonic on T if it is harmonic at every $x \in T$; equivalently, $\Delta f = 0$. A function is *conjugate harmonic* if $\Delta^* f = 0$.

Denote by Λ the set of all the oriented edges (i.e., ordered pairs of neighbours). For $\sigma \in \Lambda$ denote by $b(\sigma)$ the beginning vertex of σ and by $e(\sigma)$ the ending vertex: $\sigma = [b(\sigma), e(\sigma)]$. The choice of a reference vertex $o \in T$ (see 2.1) gives rise to a positive orientation on edges: an edge σ is *positively oriented* if $b(\sigma)$ is the predecessor of $e(\sigma)$.

The beginning and ending vertices induce two maps $b : \Lambda \to T$ and $e : \Lambda \to T$ defined as above. These maps induce two different liftings, $f \circ b$ and $f \circ e$, for any $f : T \to \mathbb{R}$.

DEFINITION 2. For any function $f : T \to \mathbb{R}$, the *gradient* $\nabla f : \Lambda \to \mathbb{R}$ is

$$\nabla f(\sigma) = f(e(\sigma)) - f(b(\sigma)).$$

For $x \in T$, let $\Lambda(x) = \{\sigma \in \Lambda : b(\sigma) = x\}$ be the *star* of x.

DEFINITION 3. For $x \in T$,

$$\|\nabla f(x)\|^2 \equiv \sum_{\sigma\in\Lambda(x)} p(\sigma)|\nabla f(\sigma)|^2.$$

REMARK 2. For future reference, we observe that, for every edge σ, one has $p(\sigma) > \delta$ by the regularity assumption (H2), hence

$$|\nabla f(\sigma)| \le \frac{1}{\sqrt{p(\sigma)}}\, \|\nabla f(x)\| < \frac{1}{\sqrt{\delta}}\, \|\nabla f(x)\|. \tag{2.2}$$

For $x \in T$ and $\omega \in \Omega$ we consider the distance $d(x,\omega) = \min_{j\in\mathbb{N}} d(x,\omega_j)$.

DEFINITION 4. Let $\alpha \ge 0$ be an integer. The *tube* $\Gamma_\alpha(\omega)$ around the geodesic $\omega \in \Omega$ is

$$\Gamma_\alpha(\omega) = \{x \in T : d(x,\omega) \le \alpha\}.$$

DEFINITION 5. The *area function* of f on T is the function on Ω defined by

$$A_\alpha f(\omega) = \Big(\sum_{x\in\Gamma_\alpha(\omega)} \|\nabla f(x)\|^2\Big)^{1/2}.$$

Observe that if $f \in L^1(T)$, then $A_\alpha f(\omega) < \infty$ for every α and ω, because T has bounded homogeneity degree.

DEFINITION 6. A function f on T has *non-tangential limit* at $\omega \in \Omega$ if, for every integer $\alpha \ge 0$, $\lim f(x)$ exists as $|x| \to \infty$ and $x \in \Gamma_\alpha(\omega)$.

We say that f has *non-tangential limit up to width* β if the above limit exists for all $0 \le \alpha \le \beta$.

A function f on T is *non-tangentially bounded* at $\omega \in \Omega$ if, for some $M > 0$, one has $|f(x)| \leq M$ for $x \in \Gamma_0(\omega)$.

Observe that this definition of non-tangential boundedness is equivalent to saying that, for some non-negative integer $\alpha = \alpha(\omega)$, f is bounded on a tube of width α around the geodesic $\Gamma_0(\omega)$ (by a different constant depending on α). This of course does not mean that, for a given ω, boundedness of f on the geodesic ray $\Gamma_0(\omega)$ is equivalent to boundedness on a tube of positive radius around ω; nevertheless, as a consequence of our main theorem below, we shall see that this is true for almost all ω.

Definition 7. The *cone* $W_\alpha(E)$ over a measurable subset E of Ω is

$$W_\alpha(E) = \bigcup_{\omega \in E} \Gamma_\alpha(\omega).$$

For any integer $s > 0$ let $W_\alpha^s(E) = \bigcup_{\omega \in E} \Gamma_\alpha(\omega) \cap [s;\infty)$ (here, with abuse of notation, we denote by $[m,k]$ the corona of vertices of T whose length is between m and k).

The main goal of this paper is the following extension of the Lusin area theorem [14, 20] to non-homogeneous trees. This theorem has been proved for homogeneous trees in [1].

Main Theorem. *Let E be a measurable subset of Ω and f a harmonic function on T. Then the following are equivalent:*

(i) *f is non-tangentially bounded at almost every $\omega \in E$;*
(ii) *f has non-tangential limit at almost every $\omega \in E$;*
(iii) *$A_0 f(\omega) < \infty$ for almost every $\omega \in E$;*
(iv) *for every fixed $\alpha \geq 0$, $A_\alpha f(\omega) < \infty$ for almost every $\omega \in E$.*

The same statement holds if f is harmonic on a connected subset of T whose boundary contains E, or more precisely on some tube $W_\beta(E)$, provided that $\alpha \leq \beta$ in (iv) *and, in* (ii), *f is assumed to be non-tangentially bounded up to width β.*

By the remark at the end of Definition 6, in this discrete setting condition (iii) in the theorem is equivalent to the more familiar statement that for almost every ω there is some integer α such that $A_\alpha f$ is finite at ω.

3. Estimates for the Green and Poisson kernels. The proof of the Main Theorem needs some probabilistic estimates that we develop in this section. Some results are taken without proof from [12].

Let $F(u,v)$ be the probability that the random walk starting at u with law P hits v: that is, if X_n is the random vertex visited at time n, then

$$F(u,v) = \Pr[\exists n > 0 : X_n = v,\ X_j \neq v\ \forall j < n \mid X_0 = u].$$

Since a nearest neighbour random walk from u to v must visit every intermediate vertex, a standard stopping time argument yields the following *multiplicativity rule*: if $u_0, u_1, \dots, u_n$ are consecutive vertices in a geodesic ray, then

$$F(u_0, u_n) = \prod_{i=1}^{n} F(u_{i-1}, u_i). \tag{3.1}$$

Of course, the nth operator power P^n has entries $P^n(u,v)$ given by the probability $p^{(n)}(u,v)$ of moving from u to v in n steps.

The *Green kernel* $G(u,v)$ of the transition operator is the expected number of visits to v of the random walk starting at u:

$$G(u,v) = \sum_{n=0}^{\infty} P^n(u,v) = \sum_{n=0}^{\infty} p^{(n)}(u,v). \tag{3.2}$$

REMARK 3. For each $v \in T$, the Green kernel $G(x,v)$ is *harmonic* with respect to x at every vertex $x \neq v$, and *conjugate harmonic* with respect to v at every $v \neq x$.

Indeed, fix v and write $\gamma(x) = G(x,v) \equiv \sum_{n=0}^{\infty} P^n(x,v)$ and $\delta_v(x) = 1$ if $x = v$ and 0 otherwise. Then

$$\begin{aligned} P\gamma(x) &= \sum_{y\sim x} p(x,y)\gamma(y) = \sum_{y\sim x} p(x,y) \sum_{n=0}^{\infty} p^{(n)}(y,v) \\ &= \sum_{n=0}^{\infty} \sum_{y\sim x} P(x,y)P^n(y,v) = \sum_{n=0}^{\infty} P^{n+1}(x,v) = \gamma(x) - \delta_v(x) \end{aligned}$$

(indeed, P^0 is the identity operator that has entries 1 on the diagonal and zero otherwise).

The same argument shows that G is conjugate harmonic in the second variable v except at $v = x$.

$G(u,v)$ has a natural meaning in combinatorics: it is the weighted number of finite paths from u to v. The stopping time argument mentioned above now yields (as in [5, Proposition 2.5])

$$G(u,v) = F(u,v)\, G(v,v) \quad \text{for all } u \neq v. \tag{3.3}$$

Given two positive functions f and g, we write $f \approx g$ if $f < Cg$ and $g < Cf$ for some constant C.

PROPOSITION 3.1 ([12, Corollary 1, Proposition 2]). *If P is very regular, then we have the following equivalence of functions of $x \in T$:*

$$\nu(U(x)) \approx F(o,x) \approx G(o,x).$$

We now obtain upper and lower bounds on F. The lower bound is obvious: given any two neighbours u and v, the regularity assumption (H2) yields

$$F(u,v) > p(u,v) \geq \delta.$$

For the upper bound, for all $u \sim v$ the multiplicativity rule (3.1) gives, for all $u \sim v$,

$$F(v,u) = \frac{p(v,u)}{1 - \sum_{w \sim v, w \neq u} p(v,w) F(w,v)}$$

[12, identity (4)]. Since $p(v,u) < \frac{1}{2} - \delta$ by (H2), $\sum_{w \sim v,\, w \neq u} p(v,w) = 1 - p(v,u) < 1 - \delta$ by (H1) and $F(w,v) \leq 1$, this means that, for $u \sim v$,

$$F(u,v) < \frac{1/2 - \delta}{\delta}.$$

Unfortunately, as already observed, $\delta < 1/4$ and so the right hand side of the last inequality is larger than 1, not useful for the goal of establishing exponential decay of the hitting probabilities $F(u,v)$ when the distance between u and v grows.

Here we have used the fact that the sum $\sum_{w \sim v, w \neq u} p(v,w)$ is the complement to 1 of the remaining transition probability $p(u,v)$, which is larger than δ by (H1). A more accurate analysis can be based on the comparison of the transition operator on vertices v with a transition operator on the integers $|v|$. For the latter operator, the previous sum reduces to one term only, bounded above by $1/2 - \delta$ by (H2); so its complement to 1 is bounded below by $1/2 + \delta$. This yields the stronger estimate

$$F(u,v) < \frac{1/2 - \delta}{1/2 + \delta}.$$

proved rigorously by W. Woess and the author in the Appendix of [12].

Again by the same stopping time argument we see that the function $F(u,v)$ (regarded as a function over geodesic arcs) is multiplicative (it splits as a product over the constituent edges, as noted in [5] and [9, identity (1)], among many other references). Therefore the previous estimate now yields the following result.

Corollary 3.2 ([12, Proposition 2]). *If the transition operator P is very regular, with δ as in* (H2)*, then for all vertices u, v at distance n,*

$$\delta^n < F(u,v) < \left(\frac{1/2 - \delta}{1/2 + \delta}\right)^n \quad \text{and} \quad G(u,v) < \left(\frac{1/2 - \delta}{1/2 + \delta}\right)^n G(v,v).$$

Moreover, $\nu_u(U(v)) := \Pr[X_\infty \in U(v) \mid x_0 = u]$ satisfies the rule

$$\nu_u(U(v)) = F(u,v)\, \frac{1 - F(v,u)}{1 - F(u,v) F(v,u)}.$$

Therefore the harmonic measure ν satisfies the following inequalities: for some $0 < \varepsilon < 1$ (namely, $\varepsilon = (1/2 - \delta)/(1/2 + \delta)$) and for all vertices x in T

and v in the sector $S(x)$ at distance $d(v,x)$ from x,

$$\nu(U(v)) < (1-\varepsilon)^{d(v,x)}\nu(U(x)).$$

By Proposition 3.1, the Green function G on T is equivalent to the function $g(x) = \nu(U(x))$ introduced in [1]. Therefore the statements that yield bounds involving g translate directly into the corresponding results for G. We now list these statements, taken from [1, Section 4]; however, we first clarify our notation as follows:

Definition 8 (Variants of the Green kernel). For every $x \in T$,

$$G(x) = G(o,x), \tag{3.4a}$$

$$g(x) = \nu(U(x)), \tag{3.4b}$$

and for every measurable subset $E \subset \Omega$,

$$g_E(x) = \nu(U(x) \cap E). \tag{3.4c}$$

Definition 9. For every x, $v \in T$ the *Martin kernel* $K(x,v)$ is

$$K(x,v) \equiv \frac{G(x,v)}{G(o,v)} = \frac{F(x,v)}{F(o,v)}.$$

For every $x \in T$ and $\omega \in \Omega$ the *Poisson kernel* $K(x,\omega)$ is defined as

$$K(x,\omega) \equiv \lim_{v\to\infty} \frac{G(x,v)}{G(o,v)} = \lim_{v\to\infty} \frac{F(x,v)}{F(o,v)}$$

(the second identity follows from (3.3)). As a consequence of Remark 3, for every $\omega \in \Omega$, $K(\cdot,\omega)$ is harmonic on T.

Corollary 3.3. *Denote by $\{x_j : j = 0, \ldots, |x|\}$ the vertices in the geodesic arc $[o,x]$.*

(i) *If $v \in S(x)$ then*

$$\left(\frac{1/2+\delta}{1/2-\delta}\right)^{|x|} < K(x,v) < \delta^{-|x|}.$$

The same inequalities are satisfied by $K(x,\omega)$ if $\omega \in U(x)$.

(ii) *If $v \in S(x_j) \setminus S(x_{j+1})$, with $j = 0, \ldots, |x|-1$, then*

$$\delta^{|x|-j}\left(\frac{1/2+\delta}{1/2-\delta}\right)^{j} < K(x,v) < \delta^{-j}\left(\frac{1/2-\delta}{1/2+\delta}\right)^{|x|-j}.$$

The same inequalities are satisfied by $K(u,\omega)$ if $\omega \in U(x_j)\setminus U(x_{j+1})$.

(iii) *If $x < y$ and $v \notin S(x)$ then*

$$\frac{K(y,v)}{K(x,v)} < \left(\frac{1/2-\delta}{1/2+\delta}\right)^{d(x,y)}.$$

The same inequalities hold for $K(y,\omega)/K(x,\omega)$ if $\omega \notin U(x)$.

Proof. By the multiplicativity rule (3.1) and (3.3), for every $x \in T$ and for every j as above, the fraction $G(x,v)/G(o,v) = F(x,v)/F(o,v)$ is constant with respect to v for all v in $S(x_j) \setminus S(x_{j+1})$, that is, it depends only on the bifurcation index $j = N(x,v)$, which is the number of edges in common of the finite geodesic arcs $[o,x]$ and $[o,v]$. Therefore the same estimates that hold for $K(x,v)$ when $v \in S(x)$ or $v \in S(x_j) \setminus S(x_{j+1})$ also hold for $K(x,\omega)$ when $\omega \in U(x)$ or $\omega \in U(x_j) \setminus U(x_{j+1})$. For the same reason, if $v \in S(x)$, then $F(x,v)/F(o,v) = 1/F(o,x)$, and part (i) follows from Corollary 3.2. If, more generally, $v \in S(x_j) \setminus S(x_{j+1})$, with $j = 0, \ldots, |x|-1$, then $K(x,v) = F(x,x_j)/F(o,x_j)$, and the same argument yields part (ii).

We prove part (iii) for $\omega \notin S(x)$ (the argument for $v \notin U(x)$ is the same). Let m be the bifurcation index $m = N(x,\omega)$. Then, if $x < y$, one has

$$K(y,\omega)/K(x,\omega) = \frac{F(y,x_m)}{F(0,x_m)} \Big/ \frac{F(x,x_m)}{F(0,x_m)} = F(y,x) < \left(\frac{1}{2} + \delta\right)^{d(x,y)},$$

by the multiplicativity rule (3.1) and Corollary 3.2. ∎

Therefore the Poisson kernel, being a locally constant function on Ω, belongs to $L^p(\Omega)$ for every $x \in T$, for $1 \le p \le \infty$.

The *Poisson integral* of a function h in $L^1(\Omega)$ is defined by

$$\mathbf{K}h(x) = \int_{\Omega} h(\omega) K(x,\omega)\, d\nu$$

where ν is the harmonic measure introduced in (2.1). More generally, by integrating measures on Ω against the Poisson kernel one obtains harmonic functions on T; this integral representation is called the *Poisson representation*. The measure ν on Ω represents the harmonic function with constant value 1: the Poisson integral of the constant function 1 on Ω is the function 1 on T. The measure ν is also called the *Poisson measure* and its support Ω is the *Poisson boundary* of T.

The following terminology is convenient:

DEFINITION 10. For any vertex x and any integer $k \le |x|$, x_k denotes the vertex of length k in the geodesic $[o,x]$.

Then the Poisson kernel, by its Definition 9, Corollary 3.3 and [10, Corollary 10-22], has the following properties:

(a) $K(x_{k+k_1},\omega) < \left(\frac{1/2-\delta}{1/2+\delta}\right)^{k_1} K(x_k,\omega)$ for each $\omega \notin U(x_k)$;
(b) $\int_\Omega K(x,\omega)\, d\nu = 1$ for every $x \in T$.

LEMMA 3.4. *Let $\varepsilon > 0$ and $h \in L^\infty(\Omega)$. Fix $y \in T$ with $|y| = k_0$ and let $U = U(y)$. Assume that $h(\omega) > r + \varepsilon$ for all $\omega \in U$ and some $r \in \mathbb{R}$. Then there exists $R > 0$ such that $\mathbf{K}h(x) > r$ when $x \ge y$ and $|x| \ge k_0 + R$. The constant R depends only on ε and $\|h\|_\infty$.*

Proof. Write $\eta = (1/2-\delta)/(1/2+\delta)$ and

$$\mathbf{K}h(x) \equiv \int\limits_{\Omega} h(\omega)K(x,\omega)\,d\nu = \int\limits_{U} h(\omega)K(x,\omega)\,d\nu + \int\limits_{\Omega\setminus U} h(\omega)K(x,\omega)\,d\nu.$$

As $x \geq y$ and $|x| \geq k_0 + R$, then, by property (a) above, for each $\omega \notin U$, $K(x,\omega) \leq \eta^R K(y,\omega)$. Therefore, if R is sufficiently large and we set $M = \|h\|_\infty$,

$$\Big|\int\limits_{\Omega\setminus U} h(\omega)K(x,\omega)\,d\nu\Big| \leq M\eta^R \int\limits_{\Omega\setminus U} K(y,\omega)\,d\nu \leq M\eta^R < \frac{\varepsilon}{2}.$$

Moreover, by (b) and the hypothesis $h(\omega) > r+\varepsilon$ in U,

$$\int\limits_{U} h(\omega)K(x,\omega)\,d\nu(\omega) > (r+\varepsilon)\Big(1 - \int\limits_{\Omega\setminus U} K(x,\omega)\,d\nu(\omega)\Big).$$

By (a), the integral on the right hand side is bounded by η^R whenever $y \leq x$ and $|x| \geq k_0 + R$. Hence

$$\int\limits_{U} h(\omega)K(x,\omega)\,d\nu(\omega) > (r+\varepsilon)(1-\eta^R).$$

By assumption, $r+\varepsilon < M$. Therefore $\mathbf{K}h(x) > (r+\varepsilon)(1-\eta^R) - M\eta^R > r+\varepsilon-2M\eta^R = r$, if R has been chosen large enough that $M\eta^R < \varepsilon/2$. ∎

4. Proof of the Main Theorem. The proof of the Main Theorem is rather long; we prove the chain of its implications as separate theorems. Note that (ii)⇒(i) and (iv)⇒(iii) are trivial. Then it is enough to show that (i)⇒(iv), (i)&(iii)⇒(ii) and (iii)⇒(i).

We prove first that (i) implies (iv).

THEOREM 4.1. *For every measurable set $E \subset \Omega$, and every harmonic function f non-tangentially bounded almost everywhere on E, the area function of f of every width $\alpha \geq 0$ is finite almost everywhere on E.*

The proof requires some definitions and results on difference operators on trees, developed in [7], which we outline here.

DEFINITION 11 (Variants of the gradient). For every edge $\sigma = [u,v]$ and all functions f on the vertices of T, we set $p(\sigma) = p(u,v)$. We have already introduced the definition of gradient as

$$\nabla f(\sigma) = (f\circ e - f\circ b)(\sigma).$$

We extend the gradient to functions on edges:

$$\nabla f(\sigma) = f(\sigma^*) - f(\sigma),$$

where $\sigma^* = [e(\sigma), b(\sigma)]$ is the edge with the opposite orientation of σ. Moreover, we set

$$\begin{aligned} \partial f(\sigma) &= p(\sigma)\, \nabla f(\sigma), \\ \partial^* f(\sigma) &= p^*(\sigma)\, \nabla f(\sigma), \\ Df(\sigma) &= p(\sigma)\; f \circ e(\sigma) - f \circ b(\sigma), \\ D^\dagger f &= p^*(\sigma)\; f \circ e(\sigma) - f \circ b(\sigma). \end{aligned}$$

In other words,

$$Df([u,v]) = p(u,v)f(v) - f(u), \tag{4.1}$$

$$D^\dagger f([u,v]) = p(v,u)f(v) - f(u). \tag{4.2}$$

A straightforward computation leads to the following result, a discrete analogue of a well known differential identity in $\mathbb{R}^n$.

LEMMA 4.2.

$$\nabla f^2 = 2 f \circ b\, \nabla f + (\nabla f)^2.$$

REMARK 4. Identity (4.2) is slightly different from the corresponding identity for D because, by definition of transition operator, $\sum_{t\sim v} p(v,t) \equiv 1$ but in general $\sum_{t\sim v} p(t,v) \neq 1$: the latter sum is constantly 1 for an isotropic transition operator on a homogeneous tree, because in this case $P^* = P$, but it is different from 1 already in the simplest non-trivial case, when P is isotropic and the tree is semi-homogeneous (i.e., with two alternating degrees of homogeneity). However, we have already observed that, since P is uniformly bounded below by $\delta > 0$, the homogeneity degree (i.e., the number of neighbours) at every vertex is bounded by $1/\delta$; therefore we have the bound

$$\sum_{t\sim v} p(t,v) \leq \frac{1}{\delta}. \tag{4.3}$$

DEFINITION 12. The *boundary* ∂Q of a subset $Q \subset T$ is $\partial Q = \{\sigma \in \Lambda : b(\sigma) \in Q, e(\sigma) \notin Q\}$. The *trace* $b(A)$ of a subset $A \subset \Lambda$ is $b(A) = \{b(\sigma) : \sigma \in A\}$. For $Q \subset T$, the set $b(\partial Q) = \{x \in Q : y \notin Q$ for some $y \sim x\}$ is also called the *frontier* of Q.

The Green formula, well known in the continuous setup, has been extended to the discrete context of trees in [7].

PROPOSITION 4.3 (The Green formula). *If f and h are functions on T and Q is a finite subset of T, then*

$$\begin{aligned} \sum_Q (h\Delta f - f\Delta^* h) &= \sum_{\partial Q} (h \circ b\, Df - f \circ b\, D^\dagger h) \\ &= \sum_{\partial Q} (h \circ b\, \partial f - f \circ b\, \partial^* h - h \circ b\, f \circ b\, \nabla P). \end{aligned}$$

Proposition 4.4 (The Green identity). *For all real-valued functions f on T and for every $x \in T$,*

$$\Delta(f^2)(x) = \|\nabla f(x)\|^2 + 2f(x)\Delta f(x).$$

Proof of Theorem 4.1. We first prove the following:

Claim. *If f is harmonic on T and bounded, then $\|A_\alpha f\|_{L^2(E)} < C\|f\|_{L^\infty(W_0(E))}$ for some constant $C > 0$ independent of f.*

It was proved in [12, Theorem 5] that $A_\alpha f$ and $A_\beta f$ are equivalent in the L^2 norm for every α, β. Therefore it is enough to restrict attention to $A_0 f$. Then

$$\begin{aligned}
\|A_0 f\|^2_{L^2(E)} &= \int_E \sum_{y\in\Gamma_0(\omega)} \sum_{v\sim y} p(v,y)\,|f(v)-f(y)|^2\,d\nu(\omega) \\
&= \int_E \sum_{y\in W_0(E)} \|\nabla f(y)\|^2 \chi_{\Gamma_0(\omega)}(y)\,d\nu(\omega) \\
&= \sum_{y\in W_0(E)} \|\nabla f(y)\|^2 \int_E \chi_{\Gamma_0(\omega)}(y)\,d\nu(\omega) \\
&= \sum_{y\in W_0(E)} \nu(U(y)\cap E)\|\nabla f(y)\|^2 \\
&= \sum_{y\in W_0(E)} g_E(y)\|\nabla f(y)\|^2
\end{aligned} \tag{4.4}$$

(notation as in (3.4c)).

We know from Proposition 3.1 that $g_E < CG$ for some positive constant C. Let us set $B_n = \{y : |y| \le n\}$. It follows from the last identity, the Green identity (Proposition 4.4), the Green formula (Proposition 4.3) and the fact that the Green function g is conjugate harmonic except at o that

$$\begin{aligned}
\|A_0 f\|^2_{L^2(E)} &= \lim_{n\to\infty} \sum_{y\in B_n\cap W_0(E)} g_E(y)\|\nabla f(y)\|^2 \\
&< C \lim_{n\to\infty} \sum_{y\in B_n\cap W_0(E)} G(y)\Delta(f^2)(y) \\
&= C \lim_{n\to\infty} \sum_{y\in B_n\cap W_0(E)} G(y)\Delta(f^2)(y) - (f^2)(y)\Delta^* G(y) \\
&= C \lim_{n\to\infty} \sum_{\sigma=[y^-,y]\in\partial B_n\cap W_0(E)} (G(y^-)D(f^2)(\sigma)(f^2)(y^-)\,D^\dagger G(\sigma)).
\end{aligned}$$

Now from (4.1) we see that, for every edge $\sigma = [u,v]$,

$$\begin{aligned}
G\circ b(\sigma)\,Df^2(\sigma) &= G(u)p(u,v)(f^2(v) - f^2(u)) \\
&= G(u)p(u,v)(f(v)-f(u))(f(v)+f(u)).
\end{aligned}$$

Therefore

$$\sum_{\sigma\in\partial B_n\cap W_0(E)} |G\circ b(\sigma)\, Df^2(\sigma)| \le (2\|f\|^2_{L^\infty(W_0(E))}) \sum_{|y|=n} G(y).$$

Moreover, clearly,

$$\sum_{\sigma\in\partial B_n\cap W_0(E)} f^2\circ b(\sigma)\, |D^\dagger g|(\sigma) \le \|f\|^2_{L^\infty(W_0(E))} \sum_{\partial B_n} |D^\dagger g|.$$

It follows from (4.2) and (4.3) that $|D^\dagger g(\sigma)| \le Cg(b(\sigma))$ for some constant $C > 0$. Therefore $\sum_{b(\partial B_n)} |D^\dagger g|$ is bounded by $C\sum_{b(\partial B_n)} g$. Now, by Proposition 3.1, the latter sum is bounded, and the claim is proved.

In particular, *if f is bounded in $W_\alpha(E)$ then $A_\alpha f$ is finite almost everywhere in E.*

Let us now consider the assumption of the Theorem: f is non-tangentially bounded almost everywhere on $W_0(E)$. Let

$$E_k = \{\omega\in E : |f| < k \text{ on } \Gamma_0(\omega)\}.$$

The assumption is equivalent to $E = \bigcup_k E_k$ except for a null set. For $\varepsilon > 0$ choose K such that the set $E(K) = \bigcup_{k\le K} E_k$ satisfies the inequality $\nu(E\setminus E(K)) < \varepsilon$. On $E(K)$ the function f is bounded by K, hence $A_\alpha f$ is finite by the claim and the remarks after it. Since ε is arbitrary, $A_\alpha f$ is finite almost everywhere in E. ■

Next, we prove that (iii) and (i) imply (ii). For this we need some preparation. We start with the solution of the Dirichlet problem on a connected region of a tree, which, surprisingly, does not seem to be in print anywhere in its full form. The boundary $\mathcal{F}Q$ of a connected subset $Q\subset T$ is the set of infinite geodesic rays contained in Q (the *boundary at infinity* $\mathcal{F}_\infty Q$) and of those vertices in Q that belong to an edge whose other vertex is outside Q (the finite part of the boundary). A function on $\mathcal{F}Q$ is measurable if it is measurable on $\mathcal{F}_\infty Q$ with respect to the Borel σ-algebra of Ω (on the finite part of the boundary, which is discrete, all functions are measurable). It is clear that harmonic functions satisfy the maximum principle: if f is harmonic in Q, the maximum of $|f|$ cannot be attained at an interior vertex. As a consequence, we have the first part of the next lemma (see also the final remarks in [23]). The rest of the statement follows from the proof of [11, Theorem 3] (see also [5, Theorem 3.3]).

LEMMA 4.5. *Let Q be a connected region in T, and let $\mathcal{F}Q$ be its boundary. Every function h in $L^\infty(\mathcal{F}Q)$ has a harmonic continuation f to the whole of Q whose maximum is attained on $\mathcal{F}Q$. The function f has non-tangential limit equal to h almost everywhere on the infinite part of the boundary. Of course, if Q is not connected, the same statement holds separately for each connected component.*

The next preliminary result is a uniformization statement, adapted from [1, Proposition 4.2]. This lemma proves that a certain inclusion holds for tubes of width β and cones of width α, for every α and β, in a suitable closed subset D of E (that depends on α, β and ε) such that $\nu(E \setminus D) < \varepsilon$. In [16, Corollary 4], a similar property is proved to hold almost everywhere in E in the case $\alpha = 0$ and $\beta = 1$. Then an adaptation of [16, Corollary 4] should actually prove that the inclusion holds almost everywhere in E. But we do not need this stronger result and do not wish to obtain it by following the arguments of [16], based upon probability. Indeed, we aim to show that a proof of our Main Theorem can be given by following entirely the argument developed in [13] for rank one hyperbolic spaces, based on analysis and differential geometry.

LEMMA 4.6. *Let E be a measurable subset of Ω and $\varepsilon > 0$. There exists a closed set D with $D \subset E$ and $\nu(E \setminus D) < \varepsilon$ such that for all integers α and β, there exists an integer s such that $\Gamma_\beta(\omega) \cap [s;\infty) \subset W_\alpha(E)$ for every $\omega \in D$ (notation as in Definition 7).*

Proof. Since $W_0(E) \subset W_\alpha(E)$ for every $\alpha > 0$, it is sufficient to prove the assertion for $\alpha = 0$. Denote by ω_j the jth vertex of the geodesic ray ω. Let us make use again of notation as in Definition 7: we denote the corona of vertices x with $|x| \geq n$ by $[n, \infty)$. Also, for the sake of simplicity, for $|x| > \beta$ we denote by $x(-\beta)$ the vertex of length $|x| - \beta$ in the geodesic arc from o to x. Suppose that for some $x \in \Gamma_\beta(\omega) \cap [s;\infty)$, with $\omega \in D$, there is no $\omega' \in E$ such that $x \in \Gamma_0(\omega')$. Then $U(x) \cap E = \emptyset$. Let ω_j be the confluence point of $[o, x]$ and ω; since $x \in \Gamma_\beta(\omega)$, we have $d(x, \omega_j) \leq \beta$. Moreover, $j \geq |x| - \beta \geq s - \beta$ (since $\omega \in U(x(-\beta))$), $|x| \geq s$, $U(\omega_j) \supset U(x)$ and

$$\frac{\nu(U(\omega_j) \cap E)}{\nu(U(\omega_j))} \leq \frac{\nu(U(\omega_j)) - \nu(U(x))}{\nu(U(\omega_j))} \leq 1 - (1-\varepsilon)^\beta$$

where ε is as in Corollary 3.2. The rest of the proof is obtained by contradiction as in [1, Proposition 4.2]: we summarize the details here for the sake of completeness. It is again a consequence of the martingale convergence theorem, as in [12, p. 225] (see also (4.16)), that

$$\lim_n \frac{\nu(E \cap U(\omega_n))}{\nu(U(\omega_n))} = 1$$

for almost every $\omega \in E$. Then, for every $\eta < 1$ and $\varepsilon > 0$, by Egoroff's theorem there exists a set $D \subset E$ such that $\nu(E \setminus D) < \varepsilon$ and an integer m, independent of ω, such that

$$\frac{\nu(U(\omega_j) \cap E)}{\nu(U(\omega_j))} > \eta \tag{4.5}$$

for $j \geq m$ and all $\omega \in D$ (again by the fact that ν is regular we can choose D closed). So we have a contradiction. ∎

DEFINITION 13. Let $k_0 > k \in \mathbb{N}$; in what follows we shall take $k > k_0 + R$, where R is the constant of Lemma 3.4. We introduce the *slabs*

$$Q_k = Q_{k,k_0}(E) = W_0(E) \cap \{k_0 \leq |x| \leq k\}. \tag{4.6}$$

The boundary $I_k = \partial Q_k$ splits as a disjoint union of its inward, lateral and outward terminal parts as follows:

$$\begin{aligned} I_k^- &= I_k \cap \{\sigma : |b(\sigma)| = k_0\}, \\ I_k^{\|} &= I_k \cap \{\sigma : k_0 < |b(\sigma)| < k\}, \\ I_k^+ &= I_k \cap \{\sigma : |b(\sigma)| = k\}. \end{aligned} \tag{4.7}$$

A similar disjoint decomposition holds for the set of boundary vertices: $b(I_k) = b(I_k^-) \cup b(I_k^{\|}) \cup b(I_k^+)$. Observe that the number of vertices in $b(I_k^-)$ (a subset of the circle of radius k_0) is independent of k.

REMARK 5. Observe that, for every $\omega \in E$, the geodesic ray $\Gamma_0(\omega)$ intersects I_k^+ (and does so only once). This is equivalent to the following property, to be used later: the vertices $\omega_j \in \Gamma_0(\omega)$ are such that $\max\{j : \omega_j \in Q_{k,k_0}(E)\} = k$.

The estimate for harmonic measures of the next lemma was used in [7] and [1] but never stated or proved explicitly.

LEMMA 4.7. *Let $E \subset \Omega$ be a measurable subset, $k > 0$, $x \in b(I_k^{\|}) \cup b(I_k^+)$ and*

$$\Omega(x) = \{\omega \in U(x) : \text{for all } j > |x|,\ \omega_j \notin b(I_k^{\|}) \cup b(I_k^+)\}. \tag{4.8}$$

Then $\nu(\Omega(x)) \approx \nu(U(x))$.

Proof. First of all, observe that if a geodesic ray ω intersects Q_k, then the intersection is contained in the geodesic arc $\omega \cap [k_0, k]$ (and if $\omega \in E$ then the intersection is $\omega \cap [k_0, k]$—see Remark 5—but we do not need this here). Therefore, if $x \in b(I_k^+)$, then $\Omega(x) = U(x)$. Instead, if $x \in b(I_k^{\|})$, then there are vertices $y \notin Q_k$ such that $x = y^-$ and $\Omega(x)$ is the union of $U(y)$ over all such y. Therefore $\Omega(x) \subset U(x)$ and, for each such y, $U(y) \subset \Omega(x)$. Then $\nu(\Omega(x)) \approx \nu(U(x))$ by Corollary 3.2. ∎

Trivially, $\Omega(x) = \emptyset$ if $x \in I_k \setminus b(I_k^{\|}) \cup b(I_k^+)$, but we shall not need this.

We are now ready to prove the implication (iii)&(i)⇒(ii). This implication follows from [16, Theorem 5], but, in line with the spirit of the present paper, we follow an independent and more geometric approach.

THEOREM 4.8. *Let f be a harmonic function on T and E a measurable subset of Ω. Assume that f is non-tangentially bounded on E and*

$A_0 f(\omega) < \infty$ for almost all $\omega \in E$. Then f has non-tangential limit almost everywhere on E.

Proof. By Lemma 4.6, it is enough to show that $f(x)$ has radial limits almost everywhere in E as $|x| \to \infty$; although not strictly necessary for the proof, this simplifies the argument considerably. The radial convergence is a consequence of [16, Theorem 5], but we proceed with our independent argument inspired by differential geometry and line integrals over boundaries of finite sets.

We know that f is bounded on each $\Gamma_0(\omega)$ with $\omega \in E$. Without loss of generality, we may assume that f is uniformly bounded independently of ω, that is, bounded in $W_0(E)$. Indeed, this follows from the same uniformization procedure already explained at the end of the proof of Theorem 4.1: if f is non-tangentially bounded almost everywhere on E, the sets E_k introduced in the proof of Theorem 4.1 are once again an exhausting family of nested subsets of E. So we can assume, without loss of generality, that f is bounded on $W_0(E)$.

Let $\varepsilon > 0$ be given and let R be the offset given in Lemma 3.4 corresponding to ε and to $\|h\|_\infty = \|f\|_{L^\infty(W_0(E))}$.

With δ as in the regularity assumption (H2), let $\eta = \delta\varepsilon^2/(4R^2)$. For almost every $\omega \in E$ there exists an integer $k = k(\omega)$ such that

$$\sum_{\Gamma_0(\omega)\cap\{|x|\geq k\}} \|\nabla f(x)\|^2 < \eta.$$

We would like to have $k(\omega)$ constant. We can assume that this is true on an arbitrarily large subset of E, by the following uniformization argument.

Let $E_j = \{\omega \in E : \sum_{\Gamma_0(\omega)\cap\{|x|\geq j\}} \|\nabla f(x)\|^2 < \eta\}$. Then the E_j form an increasing nested family of sets, and $\nu(E \setminus \bigcup_j E_j) = 0$. So, for every $\rho > 0$, there is an integer k_0 so that $\nu(E \setminus E_{k_0}) < \rho$ and $\sum_{\Gamma_0(\omega)\cap\{|x|\geq k_0\}} \|\nabla f(x)\|^2 < \eta$ for every $\omega \in E_{k_0}$. For the sake of simplicity, let us write H instead of E_{k_0}.

Observe that $\|\nabla f(x)\| < \sqrt{\eta}$ for every $x \in \Gamma_0(\omega)$ with $\omega \in H$ and $|x| \geq k_0$. This fact leads to a useful control on the oscillation of f in the truncated cone $W^{k_0}(H)$ introduced in Definition 7. Indeed, if $k > k_0 + R$ and $x, y \in W^{k_0}(H)$, then $x \in \Gamma_0(\omega)$ and $y \in \Gamma_0(\tilde\omega)$ for some ω, $\tilde\omega \in H$. Suppose that $|x| = |y| = k$. If $\tilde\omega \in U(\omega_{k-R})$, then the geodesic arc γ that joins x and y lies inside $\Gamma_0(\omega) \cup \Gamma_0(\tilde\omega)$ and its length $l(\gamma)$ is less than or equal to $2R$. By the triangular inequality and (2.2), we obtain

$$|f(y) - f(x)| \leq l(\gamma) \max_{\sigma\subset\gamma} |\nabla f(\sigma)| < 2R \frac{1}{\sqrt{\delta}} \max_{\gamma} \|\nabla f(x)\| < \varepsilon. \tag{4.9}$$

Now we need to approximate the boundary values of f by the locally constant functions on Ω determined by the restriction of f to the forward part $b(I_k^+)$ of the vertex-boundary of the slab Q_k introduced in Definition 13. This motivates the following definition:

DEFINITION 14 (Lifting of a function from a finite set to the boundary of the tree). Let $J \subset V(T)$ be a finite set of vertices. Define f^J on Ω by

$$f^J(\omega) = \begin{cases} 0 & \text{if } \omega_j \notin \partial(J) \text{ for all } j, \\ f(\omega_m) & \text{if } m = \max\{j : \omega_j \in \partial(J)\}. \end{cases} \tag{4.10}$$

Finally, let $J_k = b(I_k^+)$ and $f_k = f^{J_k}$.

We introduce three harmonic functions. The first two are the following: F_k is the Poisson integral of f_k, and Φ the Poisson integral of the characteristic function χ_{H^C} of the complement H^C of H. The third, h, is the solution of the Dirichlet problem (Lemma 4.5) in the unbounded corona $T \setminus B_{k_0-1}$, with boundary data 1 on the circle $\{|x| = k_0\}$ and 0 at infinity.

We *claim* that, for some positive constants A, B independent of k, the following inequalities hold on Q_k:

$$F_k + A\Phi + Bh > f - 2\varepsilon. \tag{4.11}$$

As all functions in this inequality are harmonic, it suffices to prove it on the boundary $b(I_k)$.

Since $h > 0$, to prove (4.11) on I_k^+, it is enough to show that, for every $\omega \in H$ and $\tilde{\omega} \in U(\omega_{k-R})$, the following inequality holds:

$$(f_k + A\chi_{H^C})(\tilde{\omega}) > f_k(\omega) - \varepsilon. \tag{4.12}$$

Indeed, (4.11) follows from (4.12) and Lemma 3.4 by taking the Poisson integral with respect to the variable $\tilde{\omega}$. To prove (4.12), let $x \in Q_k$ with $|x| = k$ and $\omega \in H$ be such that $\omega_k = x$. Let $\tilde{\omega} \in U(\omega_{k-R})$, and observe that the distance of $\tilde{\omega}_k$ and x is at most $2R$. Consider the two cases $\tilde{\omega} \in H$ and $\tilde{\omega} \in H^C$ separately. In the first case, $f_k(\tilde{\omega}) > f_k(\omega) - \varepsilon$ by (4.9), and (4.11) follows from Lemma 3.4. In the second case, $\tilde{\omega} \in H^C$, we still have $f_k(\tilde{\omega}) + A\chi_{H^C}(\tilde{\omega}) > f_k(\omega) - \varepsilon$ provided A is large enough: it is sufficient to take $A = 2\|f\|_{W_0(H)}$ (we recall the assumption that f is bounded on $W_0(E) \supset W_0(H)$).

Let us now consider the lateral part of the boundary. Let $x \in b(I_k^{\|})$. Then $x = z^-$ for some z that does not belong to $\Gamma_0(\omega)$ for any $\omega \in H$. This is equivalent to $U(z) \subset H^C$, and so

$$\Phi(x) = \int_{\Omega} K(x,\omega)\chi_{H^C}(\omega)\, d\nu \geq \int_{U(z)} K(x,\omega)\, d\nu.$$

As $x = z^-$, we know by Corollary 3.3(i) that $K(x,\omega) > \left(\frac{1/2+\delta}{1/2-\delta}\right)^{|x|}$ for all

$\omega \in U(z)$. On the other hand, by Proposition 3.1 and Corollary 3.2,

$$\nu(U(z)) > \left(\frac{1/2+\delta}{1/2-\delta}\right)^{|z|} = \left(\frac{1/2+\delta}{1/2-\delta}\right)^{|x|+1}.$$

Hence $\int_{U(z)} K(x,\omega)\,d\nu \approx 1$. So $\Phi(x)$ is bounded below on $b(I_k^{\|})$ independently of k (the same argument shows that it is also bounded above, but we do not need this).

Note that F_k is bounded in $W_0(H)$ because so is assumed to be f. Since $h > 0$, the estimate on Φ shows that (4.11) holds on $b(I_k^{\|})$ if A is large enough, independently of k.

Finally, it is easy to deal with the inward part of the boundary. Indeed, Φ is the Poisson integral of a characteristic function, therefore $0 \le \Phi \le 1$, and, as just observed, F_k is bounded independently of k. Let $x \in b(I_k^-)$; then $|x| = k_0$, and, by the way h is defined, $h(x) = 1$. Therefore (4.11) holds on $b(I_k^-)$ if B is large enough, independently of k. The claim is proved. The rest of the proof follows closely [1, Theorem 5.2]; we outline the argument. By applying the claim also to $-f$, on Q_k we have

$$F_k - A\Phi - Bh - 2\varepsilon < f < F_k + A\Phi + Bh + 2\varepsilon.$$

The sequence $\{f_k\}$ is uniformly bounded by $M = \|f\|_{W_0(E)}$. By passing to a subsequence, the Banach–Alaoglu theorem shows that, as $k \to \infty$, $\int_\Omega p f_k \to \int_\Omega p f_\infty$, for some bounded function f_∞ and every function $p \in L^1(\Omega)$. Denote by F the Poisson integral of f_∞. Since $K(x,\omega) \in L^1(\Omega)$ for every x, it follows that F_k converges pointwise to F. Thus on $W_0^{k_0}(H)$ one has

$$F - A\Phi - Bh - 2\varepsilon < f < F + A\Phi + Bh + 2\varepsilon.$$

The functions F and Φ have non-tangential limits because they are the Poisson integral of bounded functions and the non-tangential limit of Φ is 0 almost everywhere in H [11, Theorem 1]. The same is true for h by Lemma 4.5. It follows that f has radial limit almost everywhere in E. ∎

Finally, we show that that (iii) implies (i).

As observed in Section 1, this part of the Main Theorem is virtually known: the proof is an adaptation of the proof of [7, Proposition 8] (see also [1, Theorem 5.2]). To make the exposition (a bit) friendlier than in that reference, we collect all the geometric arguments, based on the Green formula, in the following lemma.

LEMMA 4.9. *Let f be a harmonic function on T and $D \subset \Omega$ measurable. Let k, k_0 be positive integers, $Q_k = W_0(D) \cap [k_0, k]$ the slab introduced in* (4.6)*, and $I_{\mathrm{out}}(k) = I_k^{\|} \cup I_k^+$ the outer part of its boundary* (*notation as in* (4.7)). *Assume that $\|\nabla f\|$ is bounded in $W_0(D)$ and $\sum_{x\in Q_k} \|\nabla f(x)\|^2\, g(x)$*

is uniformly bounded with respect to k. Then the L^2 norms $\|f_k\|^2_{L^2(\omega)}$ of the boundary liftings $f_k = f^{b(I_{\text{out}}(k))}$ of Definition 14 are uniformly bounded with respect to k.

Proof. By suitably renormalizing f we may assume that $\|\nabla f\| \le 1$ in $W_0(D)$ and $\sum_{x\in Q_k} \|\nabla f(x)\|^2 g(x) \le 1$ for every k. So, since f is harmonic, the Green identity (Proposition 4.4) yields

$$0 \le \sum_{Q_k} \|\nabla f(x)\|^2 g(x) = \sum_{Q_k} g\Delta f^2.$$

Let $C \equiv \sum_{Q_k} g\Delta f^2$. We have just shown that

$$0 \le C \le 1. \tag{4.13}$$

By the Green formula (Proposition 4.3)

$$C = \sum_{I_k} (g\circ b\,\partial f^2 - f^2\circ b\,\partial^* g - g\circ b\, f^2\circ b\,\nabla p),$$

where ∂, ∂^* are as in Definition 11. Now it follows from that definition and Lemma 4.2 that

$$\begin{aligned}
&\sum_{I_k} (g\circ b\,\partial f^2 - f^2\circ b\,\partial^* g - g\circ b\, f^2\circ b\,\nabla P)\\
&\quad= \sum_{\sigma=[x,y]\in I_k} (p(\sigma)g(x)(\nabla f^2)(\sigma) - p^*(\sigma)(g(y)-g(x))f^2(x)\\
&\qquad\qquad\qquad - (p^*(\sigma)-p(\sigma))g(x)f^2(x))\\
&\quad= \sum_{\sigma=[x,y]\in I_k} (p(\sigma)g(x)(\nabla f)^2(\sigma) + f^2(x)(p(\sigma)g(x) - p^*(\sigma)g(y))\\
&\qquad\qquad\qquad + 2p(\sigma)g(x)f(x)\nabla f(\sigma))\\
&\quad= \sum_{I_k} (p\, g\circ b(\nabla f)^2 + f^2\circ b(pg\circ b - p^* g\circ e) + 2pg\circ bf\circ b\nabla f).
\end{aligned}$$

Let $I_{\text{in}}(k) = I_k^-$ be as in (4.7). The boundary I_k of Q_k splits as $I_k = I_{\text{in}}(k)\cup I_{\text{out}}(k)$. Then $C = C_1 + C_2 + C_3$ where $C_1 = \sum_{I_k} -(f^2\circ b)p^*(\sigma)\nabla g$, $C_2 = \sum_{I_k} p(\sigma)(g\circ b)(\nabla f)^2$ and $C_3 = 2\sum_{I_k} p(\sigma)(g\circ b)(f\circ b)\nabla f$. Moreover, we let

$$\begin{aligned}
C_1^- &= \sum_{\sigma=[x,y]\in I_{\text{in}}(k)} f^2(x)(p(\sigma)g(x) - p^*(\sigma)g(y))\\
&= \sum_{I_{\text{in}}(k)} (f^2\circ b)(p(\sigma)g\circ b - p^*(\sigma)g\circ e)\\
C_1^+ &= \sum_{I_{\text{out}}(k)} (f^2\circ b)(p(\sigma)g\circ b - p^*(\sigma)g\circ e),
\end{aligned}$$

$$C_2^- = \sum_{I_{\rm in}(k)} p(\sigma)(g\circ b)(\nabla f)^2, \qquad C_2^+ = \sum_{I_{\rm out}(k)} p(\sigma)(g\circ b)(\nabla f)^2,$$
$$C_3^- = 2\sum_{I_{\rm in}(k)} p(\sigma)(g\circ b)(f\circ b)\nabla f, \qquad C_3^+ = 2\sum_{I_{\rm out}(k)} p(\sigma)(g\circ b)(f\circ b)\nabla f.$$

Then by (4.13),

$$0 \le C = C_1^+ + C_1^- + C_2^+ + C_2^- + C_3^+ + C_3^- \le 1.$$

From now on, M will denote a generic constant, not always the same but always independent of k. We observe that C_1^-, C_2^- and C_3^- are uniformly bounded, since $I_{\rm in}(k)$ is contained in the bounded set $\{\sigma : |b(\sigma)| = k_0\}$. Thus the sum on $I_{\rm in}(k)$ is uniformly bounded: $|C_1^- + C_2^- + C_3^-| \le M$. Since $|C_1^-| - |C_2^-| - |C_3^-| \le |C_1^- + C_2^- + C_3^-|$, it follows that

$$|C_1^-| \le |C_2^-| + |C_3^-| + M. \tag{4.14}$$

Now observe that, if $\sigma \in I_{\rm out}(k)$, then $|e(\sigma)| = |b(\sigma)|+1$, and so $g(e(\sigma)) = \nu(U(e(\sigma)) < \nu(U(b(\sigma)) = g(b(\sigma))$; moreover, by Corollary 3.2, $g(e(\sigma)) \ge (1-\varepsilon)g(b(\sigma)) = \frac{2\delta}{1/2+\delta} g(b(\sigma))$. By the regularity assumption (H2), we have $p^*(\sigma) \le 1/2 - \delta$. Let $\beta = 2(1/2-\delta)/(1/2+\delta)$; note that $0<\beta<1$ since $0<\delta<1/2$. Then, for $\sigma = [x,y] \in I_{\rm out}(k)$,

$$p^*(\sigma) g(y) \le \left(\frac{1}{2} - \delta\right)\frac{2\delta}{1/2+\delta} g(x) = \delta\beta g(x) \le \beta p(\sigma) g(x),$$

hence, with $\eta = \frac{1}{1-\beta}$,

$$p(\sigma) g\circ b \le \eta(p(\sigma) g\circ b - p^*(\sigma) g\circ e).$$

By this inequality and (4.14) one has

$$\begin{aligned} 0 < \sum_{I_{\rm out}(k)} f^2\circ b\, g\circ b &\le \frac{1}{\delta}\sum_{I_{\rm out}(k)} p(\sigma)\, f^2\circ b\, g\circ b \\ &\le \eta C_1^- \le \eta\Big(M + \sum_{I_{\rm out}(k)} g\circ b\,(\nabla f)^2 + 2\sum_{I_{\rm out}(k)} g\circ b\,|f\circ b|\,|\nabla f|\Big). \end{aligned}$$

Since $\|\nabla f(x)\| \le 1$ in $W_0(D)$, by (2.2) we have $|\nabla f(\sigma)| \le 1/\sqrt{\delta}$ for all $x \in W_0(D)$ and all edges σ starting at x, and so on $I_{\rm out}(k)$. Then

$$\sum_{I_{\rm out}(k)} f^2\circ b\, g\circ b \le \eta\bigg(M + \frac{1}{\delta}\sum_{I_{\rm out}(k)} g\circ b + \frac{1}{\sqrt{\delta}}\sum_{I_{\rm out}(k)} g\circ b\,|f\circ b|\bigg). \tag{4.15}$$

By [7, Proposition 3], $\sum_{I_{\rm out}(k)} g\circ b \le \sum_{b(I_{\rm in}(k))} g(x)$, and the latter sum is bounded by $\sum_{|x|=k_0} g(x) = \sum_{|x|=k_0} \nu(E(x)) = 1$. Moreover, for every set J of vertices and every positive function h on J, $\sum_{\partial J} h\circ b \sim \sum_{b(\partial J)} h$, because every edge in ∂J contains a unique beginning vertex in $b(\partial J)$, and conversely every vertex in $b(\partial J)$ belongs to at most $1/\delta$ edges in ∂J (see

Subsection 2.2). Therefore, if we write $M' = \eta M$, (4.15) becomes

$$\sum_{b(I_{\mathrm{out}}(k))} f^2 g \le M' + M' \sum_{b(I_{\mathrm{out}}(k))} g|f|.$$

We can bound the last term on the right hand side by Schwarz's inequality:

$$\sum_{b(I_{\mathrm{out}}(k))} g|f| \le \Big(\sum_{b(I_{\mathrm{out}}(k))} g\Big)^{1/2} \Big(\sum_{b(I_{\mathrm{out}}(k))} g f^2\Big)^{1/2}.$$

Hence

$$\sum_{b(I_{\mathrm{out}}(k))} f^2 g \le M' + M' \Big(\sum_{b(I_{\mathrm{out}}(k))} f^2 g\Big)^{1/2}.$$

This shows that the left hand side is bounded uniformly with respect to k.

On the other hand, $f_k = 0$ outside of $b(I_k^+)$, by Definition 14. Let $\Omega(x)$ be as in (4.8). Then $\Omega(x) \subseteq U(x)$ and by Remark 5 we know that E is the disjoint union of the $\Omega(x)$ (here is why we introduced slabs!). Hence, by Lemma 4.7,

$$\|f_k\|^2_{L^2(\omega)} = \sum_{x \in b(I_{\mathrm{out}}(k))} f^2(x)\, \nu(\Omega(x)) \le \sum_{x \in b(I_{\mathrm{out}}(k))} f^2(x)\, g(x) \le M'. \ \blacksquare$$

Let x_k be as in Definition 10; remember also that x^- is the predecessor of the vertex x.

Theorem 4.10. *Let f be a harmonic function on T and E a measurable subset of Ω. Assume that $A_0 f(\omega) < \infty$ for almost every $\omega \in E$. Then f is non-tangentially bounded almost everywhere on E.*

Proof. Let

$$E_k = \{\omega \in E : A_0 f(\omega) \le k\}.$$

Observe that $E_k \subset E_{k+1}$, and all the E_k are closed since $A_0 f$ is lower semicontinuous [7, p. 259]. The assumption is equivalent to $E = \bigcup_k E_k$ except for a null set. As ν is a regular measure, for every $\varepsilon > 0$ there exists a closed (hence compact) subset $D_\varepsilon \subset E$ such that $\nu(E \setminus D_\varepsilon) < \varepsilon$ and a constant $L = L(\varepsilon) > 0$ such that $A_0 f \le L$ on D_ε. Moreover, by the martingale convergence theorem, there exists a closed set $D \subset D_\varepsilon$ with $\nu(D_\varepsilon \setminus D) < \varepsilon$ and an integer $m > 0$ such that, for all $k \ge m$ and $\omega \in D$, one has

$$\nu(U(\omega_k) \cap E) \ge \frac{1}{2}\, \nu(U(\omega_k)) \tag{4.16}$$

[1, Lemma 4.1 and (5.2)].

So it is enough to prove the statement for the subset D instead of E; therefore, without loss of generality, we can assume that $A_0 f$ is bounded almost everywhere on E, and by renormalizing f we can assume $A_0 f \le 1$ almost everywhere on E. Hence $\|\nabla f(x)\| \le 1$ in $W_0(D)$.

From now on, the proof is almost the same as in [7, Proposition 8, p. 266–270].

Let $s \geq m$. With g and g_E as in (3.4b), we make the following *claim*:

$$\sum_{W_0^s(D)} \|\nabla f(x)\|^2 g(x) \leq 2. \tag{4.17}$$

Let us prove the claim. For $\omega \in E$, let χ be the characteristic function of $\Gamma_0(\omega)$. By the same argument of (4.4),

$$\begin{aligned}\sum_{y \in W_0(E)} g_E(y)\|\nabla f(y)\|^2 &= \sum_{y \in W_0(E)} \|\nabla f(y)\|^2 \int_E \chi(y)\, d\nu(\omega)\\ &= \int_E (A_0 f(\omega))^2\, d\nu \leq \nu(E) \leq 1.\end{aligned}$$

Observe that $x \in \Gamma_0(\omega)$ if and only if $U(x)$ intersects E. Therefore $\int_E \chi(x,\omega) = \nu\{E \cap U(x)\}$. If $x \in W_0^s(D)$, then by (4.16),

$$\nu\{E \cap U(x)\} \geq \frac{1}{2}\, \nu(U(x)) = g(x).$$

Thus

$$\frac{1}{2} \sum_{W_0^s(D)} \|\nabla f(x)\|^2 g(x) \leq \int_E (A_0 f(\omega))^2\, d\nu \leq \nu(E) \leq 1.$$

This proves the claim. The last inequalities should be compared with those in [16, Section 7].

We want to prove that f is non-tangentially bounded almost everywhere in D. For this, we first show that $|f|$ is bounded by the Poisson integral of a function in $L^2(\Omega,\nu)$. We now approximate the truncated cone $W_0^{k_0}(D)$ with the slabs $Q_k = Q_{k,k_0}(D)$ introduced in Definition 4.6. We choose and fix $k_0 > 0$, so that $o \notin Q_k$. The claim and the remarks before it show that the conditions of Lemma 4.9 hold. Although not strictly necessary, it is convenient to modify now the terminology introduced after Definition 14, in order to obtain a positive lifting to the boundary, as follows: now we let $f_k = |f|^{b(I_{\mathrm{out}}(k))}$ be the boundary lifting of $|f|$ restricted to $b(I_{\mathrm{out}}(k))$ (compare with Definition 14). So $\{f_k\}$ is a non-negative uniformly bounded sequence in $L^2(\Omega)$. Thus there exists a subsequence of $\{f_k\}$ that converges weakly in $L^2(\Omega)$ to, say, a non-negative $\tilde{f} \in L^2(\Omega)$. Let F_k be the Poisson integral of f_k, and $\tilde{F}$ the Poisson integral of $\tilde{f}$. Since the Poisson kernels belongs to $L^2(\Omega)$, this means that F_k converges pointwise to $\tilde{F}$. As already observed, it is enough to prove that, for an appropriate constant M independent of k,

$$|f| \leq M + M\tilde{F}$$

for some M. Therefore it suffices to prove that, on Q_k,

$$|f| \leq M + MF_k. \tag{4.18}$$

By harmonicity and the maximum principle it is sufficient to prove (4.18) on $b(\partial Q_k) = b(I_{\text{in}}(k)) \cup b(I_{\text{out}}(k))$ (see also the remark in [7, p. 269]).

Since the cardinality of $b(I_{\text{in}}(k))$ is finite, we can choose M so large that $|f| \le M$ on $b(I_{\text{in}}(k))$. On the other hand, if $\Omega(v)$ is the boundary subset introduced in (4.8), then

$$F_k(x) = \int_{\Omega} f_k(\omega) K(x,\omega)\, d\nu = \sum_{v\in b(I_{\text{out}}(k))} |f(v)| \int_{\Omega(v)} K(x,\omega)\, d\nu \ge |f(x)|$$

(remember that now $f_k \ge 0$ is the boundary lift of $|f|$).

Observe that if $\omega \in D$ and $x \in I_{\text{out}}(k)$ is the last vertex of the geodesic ω which belongs to Q_k, then $\omega \in \Omega(x) \subseteq U(x)$, and so $K(x,\omega) = 1/F(o,x)$ by Proposition 3.1. Therefore

$$F_k(x) = |f(x)| \, \frac{\nu(\Omega(x))}{F(o,x)}.$$

On the other hand, for $x \in b(I_{\text{out}}(k))$ and $\omega \in \Omega(x)$, by Lemma 4.7 and Proposition 3.1, $\nu(\Omega(x)) \approx \nu(U(x)) \approx F(o,x) = 1/K(x,\omega)$. Therefore $F_k(x) \approx |f(x)|$. This proves (4.18).

Now, if $\tilde f$ were bounded, then $\tilde F(x) = \int K(x,\omega)\, \tilde f(\omega)\, d\nu(\omega)$ would also be bounded, because $\|K(x,\cdot)\|_1 = 1$ by property (b) before Lemma 3.4. Since $\tilde f \in L^2$, $\tilde f$ is finite almost everywhere. Then we can repeat the exhaustion procedure of the beginning of the proof. Let D_k be the subset of D where $|\tilde f| < k$; the sets D_k are a nested family and $\bigcup_k D_k = D$. This means that, if for every $\varepsilon > 0$ we limit attention to a subset $D_\varepsilon \subset D$ with $\nu(D \setminus D_\varepsilon) < \varepsilon$, then we can assume that $\tilde f$ is bounded, and so f is bounded by (4.18). Therefore $\|f\|_{L^\infty(\Gamma_0(\omega))}$ is finite for almost every ω.

We have finished the proof by showing that the Poisson integral $\tilde F$ of the L^2 function $\tilde f$ is non-tangentially bounded almost everywhere on Ω. This fact follows also, via a different argument, by [12, Proposition 5]. ∎

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