COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 2

# JACOBI MATRICES ON TREES

BY

AGNIESZKA M. KAZUN (Wrocław) and RYSZARD SZWARC (Wrocław and Opole)

*This work is dedicated to Andrzej Hulanicki, our teacher and friend*

**Abstract.** Symmetric Jacobi matrices on one sided homogeneous trees are studied. Essential selfadjointness of these matrices turns out to depend on the structure of the tree. If a tree has one end and infinitely many origin points the matrix is always essentially selfadjoint independently of the growth of its coefficients. In case a tree has one origin and infinitely many ends, the essential selfadjointness is equivalent to that of an ordinary Jacobi matrix obtained by restriction to the so called radial functions. For nonselfadjoint matrices the defect spaces are described in terms of the Poisson kernel associated with the boundary of the tree.

**Introduction.** The classical moment problem consists in the following.

*Given a sequence of real numbers $m_n$, find a positive bounded measure $\mu$ on the half-line $[0,\infty)$ or on the whole real line such that*

$$m_n = \int x^n \, d\mu(x) \quad \textit{for } n = 0, 1, 2, \ldots .$$

Two main issues are the existence and uniqueness of the measure $\mu$. It is known that such a measure $\mu$ on the real line exists if and only if the numbers $m_n$ form a positive definite sequence. The uniqueness of the measure $\mu$ is closely related to the selfadjointness of some operators. The problem was intensively investigated starting with the work of Thomas Jan Stieltjes (1894, [12], the case of the half-line) and Hans Hamburger (1920, 1921, [4], the case of the real line), through that of Marcel Riesz (1921–23, [8]–[10], a functional analysis approach), Rolf Nevanlinna (1922, [5], a complex function approach) and Marshall H. Stone (1932, [13], Hilbert space methods), until recent results of Barry Simon (e.g. [11], 1998).

One of the key concepts that have arisen in the modern investigations is that of the Jacobi matrix. An infinite matrix $J$ is called a Jacobi matrix if it has a tridiagonal form

2010 *Mathematics Subject Classification*: Primary 47B36; Secondary 42C05.
*Key words and phrases*: Jacobi matrices, orthogonal polynomials, trees, essential selfadjointness.

DOI: 10.4064/cm118-2-7

$$J=\begin{pmatrix}\beta_0 & \lambda_0 & 0 & 0 & 0 & \dots\\ \lambda_0 & \beta_1 & \lambda_1 & 0 & 0 & \dots\\ 0 & \lambda_1 & \beta_2 & \lambda_2 & 0 & \dots\\ 0 & 0 & \lambda_2 & \beta_3 & \ddots & \\ 0 & 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & & \end{pmatrix},$$

where the diagonal entries $\beta_n$ are real, while the off-diagonal entries $\lambda_n$ are positive. There exists a one-to-one correspondence between positive definite sequences $m_n$ and Jacobi matrices $J$ given by

$$m_n=(J^n\delta_0,\delta_0),$$

where $J$ is regarded as a symmetric unbounded operator on $\ell^2(\mathbb{N}_0)$.

Uniqueness of the measure $\mu$ on the line turns out to be equivalent to essential selfadjointness of $J$ on the subspace of finitely supported sequences in $\ell^2(\mathbb{N}_0)$. Moreover, in the case when the moment problem $m_n$ is indeterminate, description of all the solutions $\mu$ is related to description of all selfadjoint extensions of $J$.

Selfadjointness of an unbounded operator is an important notion on more general grounds. If a symmetric operator admits a selfadjoint extension, or even better, is essentially selfadjoint, then the whole machinery of spectral theory becomes available.

We take up the problem of essential selfadjointness of a Jacobi matrix on spaces which are natural generalizations of $\ell^2(\mathbb{N}_0)$. The linear infinite tree $\mathbb{N}_0$ of nonnegative integers has two obvious extensions. We may consider a homogeneous tree branching out from each vertex into a fixed number of edges directed either downwards (the case of a tree $\Gamma$ with one origin) or upwards (the case of a tree $\Lambda$ with one end at infinity). We consider a Jacobi matrix $J$ as a symmetric operator acting in the space of all square-summable functions defined on the partially ordered set of vertices of these trees. The domain of $J$ consists of finitely supported functions. The main goal of this work is to investigate essential selfadjointness of $J$. In order to do this we look at its deficiency space. It is described by a recurrence relation, whose solutions yield systems of orthogonal polynomials. It turns out that essential selfadjointness of $J$ depends on the structure of the tree and, surprisingly, the behavior of $J$ is completely different in the two cases.

In fact, the main result of Section 2 states that

*The matrix $J$ in the case of the tree $\Lambda$ is always essentially selfadjoint regardless of its entries. Furthermore, $J$ has a pure point spectrum, i.e. there is an orthonormal basis consisting of eigenvectors for $J$.*

In the case of $\Gamma$, essential selfadjointness of $J$ depends on its projection on the one-dimensional tree $\mathbb{N}_0$. Namely, we associate to the Jacobi operator $J$ acting on the tree $\Gamma$ some classical Jacobi matrix $J^r$ acting in $\ell^2(\mathbb{N}_0)$ which corresponds to the restriction of $J$ to the functions constant on levels of $\Gamma$.

The main result of Section 1 is

*The operator $J$ is essentially selfadjoint if and only if $J^r$ is essentially selfadjoint.*

One should not be misled by the apparent similarity to the classical case. The picture becomes clearer when we consider the case when $J$ is not essentially selfadjoint. Then its deficiency space is much bigger than in the case of $\ell^2(\mathbb{N}_0)$, when it is just one-dimensional. We give a description of the nontrivial deficiency space of $J$ on the tree $\Gamma$. It resembles the theory of harmonic functions since a Poisson-like kernel shows up there. We prove that functions in the deficiency space are determined by their boundary values via the Poisson integral. The spectral decomposition of selfadjoint extensions of $J$ is given explicitly. In particular, we show that any such extension has a pure point spectrum.

## Preliminaries

**Selfadjoint extensions of symmetric operators.** Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$. Let $A$ be a linear operator with domain $D(A) \subset \mathcal{H}$ which is dense in $\mathcal{H}$. For a symmetric operator $A$ and a fixed complex number $z \notin \mathbb{R}$ we define the *deficiency space* of $A$ by

$$N_z = (\operatorname{Im}(A - \bar{z}I))^{\perp},$$

where $^{\perp}$ denotes the orthogonal complement in $\mathcal{H}$. It is known that the dimension of $N_z$ is constant on each of the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$. The two numbers $\dim N_i$ and $\dim N_{-i}$ are called the *deficiency indices* of $A$.

THEOREM 0.1. *The deficiency space $N_z$ is the eigenspace of the operator $A^*$ associated with the eigenvalue $z$.*

THEOREM 0.2. *A symmetric operator admits a selfadjoint extension if and only if its deficiency indices are equal.*

THEOREM 0.3. *Let $A$ be a symmetric operator and $B$ be a bounded selfadjoint operator. Then $A$ and $A + B$ have the same deficiency indices.*

THEOREM 0.4. *A symmetric operator is essentially selfadjoint if and only if its deficiency space is trivial for any $z \notin \mathbb{R}$ (i.e. its deficiency indices are zero).*

The above facts can be found in many books, for instance in [6], [7], [14].

**Classical Jacobi matrices.** A *Jacobi matrix* $J$, i.e. a matrix of the form

$$J=\begin{pmatrix} \beta_0 & \lambda_0 & 0 & 0 & 0 & \dots \\ \lambda_0 & \beta_1 & \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_1 & \beta_2 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_2 & \beta_3 & \ddots & \\ 0 & 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & & \end{pmatrix}, \tag{0.5}$$

where $\beta_n$ are real and $\lambda_n$ are positive, can be regarded as a linear operator in the Hilbert space $\ell^2(\mathbb{N}_0)$ with domain $D(J)=\operatorname{lin}\{\delta_0,\delta_1,\delta_2,\dots\}$. Here $\delta_n$ is the characteristic function of the point $n$, and

$$J\delta_n=\lambda_{n-1}\delta_{n-1}+\beta_n\delta_n+\lambda_n\delta_{n+1},\qquad n\ge 0 \tag{0.6}$$

(we adopt the convention that $\lambda_{-1}=\delta_{-1}=0$).

There are two sequences $p_n(x)$ and $q_n(x)$ of orthogonal polynomials associated with a Jacobi matrix $J$. They are solutions to the recurrence relation

$$x\cdot a_n=\lambda_{n-1}a_{n-1}+\beta_n a_n+\lambda_n a_{n+1},\qquad n\ge 1, \tag{0.7}$$

with given initial conditions $a_0$ and $a_1$. Taking $a_0=1$ and $a_1=(1/\lambda_0)(x-\beta_0)$ gives $a_n=p_n$; while $a_0=0$ and $a_1=1/\lambda_0$ give $a_n=q_n$. It is known that all roots of these polynomials are real (see e.g. [3]).

The following basic properties of Jacobi matrices can be found, for instance, in [1], [2], [3], [11], [14].

From (0.5) we can see that the operator $J$ is symmetric. In view of Theorem 0.2 the following theorem implies that $J$ has a selfadjoint extension.

THEOREM 0.8. *The deficiency indices of the operator $J$ are either $(0,0)$ or $(1,1)$. In the former case $J$ is essentially selfadjoint. In the latter case a selfadjoint extension of $J$ is not unique.*

THEOREM 0.9 (The Hamburger criterion). *A Jacobi matrix $J$ is essentially selfadjoint if and only if at least one of the series $\sum p_n(0)^2$ and $\sum q_n(0)^2$ is divergent.*

THEOREM 0.10. *Let $\tilde J$ be a selfadjoint extension of $J$ in the indeterminate case and $E(x)$ be the resolution of the identity associated with $\tilde J$. Then the support of the measure*

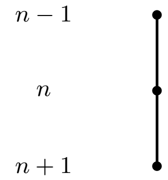
$$d\sigma(x)=d(E(x)\delta_0,\delta_0)$$

*is a discrete set and coincides with the spectrum of the operator $\tilde J$.*

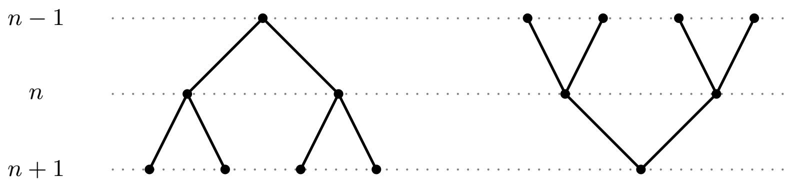
The selfadjointness of $J$ is important in the theory of classical orthogonal polynomials. The measure which is the solution to the moment problem

$m_n = (J^n\delta_0, \delta_0)$ is unique if and only if the Jacobi matrix $J$ is essentially selfadjoint.

**Jacobi matrices on homogeneous trees.** The set $\mathbb{N}_0$ of nonnegative integers can be identified with a linear infinite tree with a *natural* order.

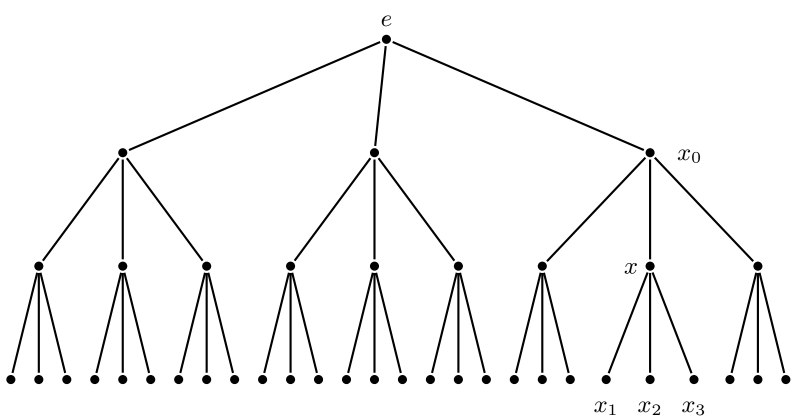



There are two natural generalizations of this configuration: from each vertex there is a fixed number (greater than 1) of edges either pointing downward (a tree with one origin) or upward (a tree with one end).




**1. A Jacobi operator on a tree with one origin.** For a fixed $d \in \{2, 3, 4, \ldots\}$ we consider an infinite homogeneous tree of degree $d$, i.e. an infinite connected graph with a distinguished vertex (root) $e$ and a partial order such that each vertex $x$ has $d$ successors $x_i$ $(i = 1, \ldots, d)$ and one predecessor $x_0$ (unless $x = e$).

For instance, if $d = 3$, the top levels of the tree look as follows:



The set of all vertices of the tree will be denoted by $\Gamma_d$. There is a natural distance $\operatorname{dist}(\cdot,\cdot)$ in $\Gamma_d$ counting the number of edges in the unique path connecting two fixed vertices. The *length* of a vertex is, by definition, its distance from the root $e$, i.e. $|x| = \operatorname{dist}(x, e)$.

The space $\ell^2(\Gamma_d)$ of all square-summable functions on $\Gamma_d$, i.e.

$$\ell^2(\Gamma_d) = \Big\{ f \in \mathbb{C}^{\Gamma_d} : \sum_{x\in\Gamma_d} |f(x)|^2 < \infty \Big\},$$

is a Hilbert space with the standard inner product

$$(f, g) = \sum_{x\in\Gamma_d} f(x)\,\overline{g(x)}.$$

We write $\delta_x$ for the characteristic function of the one-point set $\{x\}$. Let $\mathcal{F}$ denote the space of all functions with finite support:

$$\mathcal{F} = \operatorname{lin}\{\delta_x : x \in \Gamma_d\}.$$

Let $\lambda_0, \lambda_1, \lambda_2, \ldots$ be fixed positive numbers and $\beta_0, \beta_1, \beta_2, \ldots$ be fixed real numbers. We consider the Jacobi operator $J$ with domain

$$D(J) = \mathcal{F} \subset \ell^2(\Gamma_d),$$

which acts as follows:

$$\begin{aligned} J\delta_e &= \beta_0 \cdot \delta_e + \lambda_0 \cdot (\delta_{e_1} + \cdots + \delta_{e_d}), \\ J\delta_x &= \lambda_{n-1} \cdot \delta_{x_0} + \beta_n \cdot \delta_x + \lambda_n \cdot (\delta_{x_1} + \cdots + \delta_{x_d}), \qquad n \ge 1, \end{aligned} \tag{1.1}$$

where $n = |x|$. We adopt the convention that $\lambda_{-1} = \delta_{e_0} = 0$. Then the action of $J$ can be expressed by the latter formula for all $n \ge 0$.

It is elementary that $J$ thus defined is a symmetric operator.

FACT 1.2. *The deficiency space $N_z(J)$ of the operator $J$ on $\ell^2(\Gamma_d)$ consists of all square-summable functions $v$ on $\Gamma_d$ satisfying*

$$zv(x) = \lambda_{n-1}v(x_0) + \beta_n v(x) + \lambda_n(v(x_1) + \cdots + v(x_d)) \tag{1.3}$$

*for all $x$ with $|x| = n$ and all $n \ge 0$.*

*Proof.* A function $v \in \ell^2(\Gamma_d)$ is orthogonal to $\operatorname{Im}(J - \bar{z}I)$ if and only if for each vertex $x$ with $|x| = n$,

$$\begin{aligned} 0 = (v, (J-\bar{z})\delta_x) &= (v, \lambda_{n-1}\delta_{x_0} + \beta_n\delta_x + \lambda_n(\delta_{x_1} + \cdots + \delta_{x_d}) - \bar{z}\delta_x) \\ &= \lambda_{n-1}v(x_0) + \beta_n v(x) + \lambda_n(v(x_1) + \cdots + v(x_d)) - zv(x). \ \blacksquare \end{aligned}$$

REMARK. Although the domain of $J$ consists of functions with finite support, note that the formula for $J$ can actually be applied to any function on $\Gamma_d$. Therefore we can write

$$N_z(J) = \{v \in \ell^2(\Gamma_d) : Jv(x) = z \cdot v(x),\ x \in \Gamma_d\}.$$

**1.1. The one-dimensional operator.** We call a function on $\Gamma_d$ *radial* if it is constant on each *level* of $\Gamma_d$, that is, on each set of vertices of fixed length. We will denote by $\ell^2_r(\Gamma_d)$ the space of all square-summable radial functions on $\Gamma_d$. Let $\chi_n$ denote the characteristic function of the $n$th level. Note that the normalized functions

$$\mu_n(x) = (\sqrt{d}\,)^{-n} \cdot \chi_n(x) = \begin{cases} d^{-n/2} & \text{for } |x| = n, \\ 0 & \text{for } |x| \neq n, \end{cases}$$

form an orthonormal basis of $\ell^2_r(\Gamma_d)$. Obviously,

$$\chi_n = \sum_{|x|=n} \delta_x.$$

Each vertex of length $n-1$ is a predecessor of exactly $d$ vertices of length $n$. Therefore

$$J\chi_n = d \cdot \lambda_{n-1} \cdot \chi_{n-1} + \beta_n \cdot \chi_n + \lambda_n \cdot \chi_{n+1}.$$

Since $\chi_n = (\sqrt{d}\,)^n \mu_n$, we have

$$(\sqrt{d}\,)^n J\mu_n = d(\sqrt{d}\,)^{n-1}\lambda_{n-1}\mu_{n-1} + (\sqrt{d}\,)^n \beta_n \mu_n + (\sqrt{d}\,)^{n+1}\lambda_n \mu_{n+1}.$$

The restriction of $J$ to $\ell^2_r(\Gamma_d)$ will be denoted by $J^r$. Thus

$$D(J^r) = \operatorname{lin}\{\mu_0, \mu_1, \mu_2, \ldots\} \subset \ell^2_r(\Gamma_d)$$

and

$$J^r \mu_n = \sqrt{d}\,\lambda_{n-1} \cdot \mu_{n-1} + \beta_n \cdot \mu_n + \sqrt{d}\,\lambda_n \cdot \mu_{n+1}, \quad n \geq 0. \tag{1.4}$$

In other words, we can identify $J^r$ with the matrix

$$J^r = \begin{pmatrix} \beta_0 & \sqrt{d}\,\lambda_0 & 0 & 0 & 0 & \ldots \\ \sqrt{d}\,\lambda_0 & \beta_1 & \sqrt{d}\,\lambda_1 & 0 & 0 & \ldots \\ 0 & \sqrt{d}\,\lambda_1 & \beta_2 & \sqrt{d}\,\lambda_2 & 0 & \ldots \\ 0 & 0 & \sqrt{d}\,\lambda_2 & \beta_3 & \sqrt{d}\,\lambda_3 & \ldots \\ 0 & 0 & 0 & \sqrt{d}\,\lambda_3 & \beta_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{1.5}$$

This means that $J^r$ on $\ell^2_r(\Gamma_d)$ can be regarded as a classical one-dimensional Jacobi operator on $\ell^2(\mathbb{N}_0)$. In particular, by Theorem 0.8, its deficiency space $N_z(J^r)$ is either one-dimensional or trivial.

FACT 1.6. *A function $v \in \ell^2_r(\Gamma_d)$ belongs to $N_z(J^r)$ if and only if*

$$zv(x) = \lambda_{n-1}v(x_0) + \beta_n v(x) + \lambda_n(v(x_1) + \cdots + v(x_d)) \tag{1.7}$$

*for each $n \geq 0$ and each $x$ with $|x| = n$. Moreover,*

$$N_z(J^r) \subseteq N_z(J).$$

*Proof.* Let $v \in \ell^2_r(\Gamma_d)$ be orthogonal to $\mathrm{Im}(J^r - \bar{z}I)$, i.e.

$$0 = (v, (J^r - \bar{z})\chi_n), \qquad n \geq 0.$$

We calculate

$$\begin{aligned}
&(v, (J^r - \bar{z})\chi_n) \\
&\quad = (v, d \cdot \lambda_{n-1} \cdot \chi_{n-1} + \beta_n \cdot \chi_n + \lambda_n \cdot \chi_{n+1} - \bar{z}\chi_n) \\
&\quad = d\lambda_{n-1} \sum_{|x|=n-1} v(x) + \beta_n \sum_{|x|=n} v(x) + \lambda_n \sum_{|x|=n+1} v(x) - z \sum_{|x|=n} v(x).
\end{aligned}$$

Since $v$ is radial, we obtain

$$0 = d\lambda_{n-1} \cdot d^{n-1} v(x_0) + \beta_n \cdot d^n v(x) + \lambda_n \cdot d^{n+1} v(x_1) - z \cdot d^n v(x).$$

It follows that $0 = \lambda_{n-1} v(x_0) + \beta_n v(x) + \lambda_n d \cdot v(x_1) - z v(x)$ for each vertex $x$ with $|x| = n$. ∎

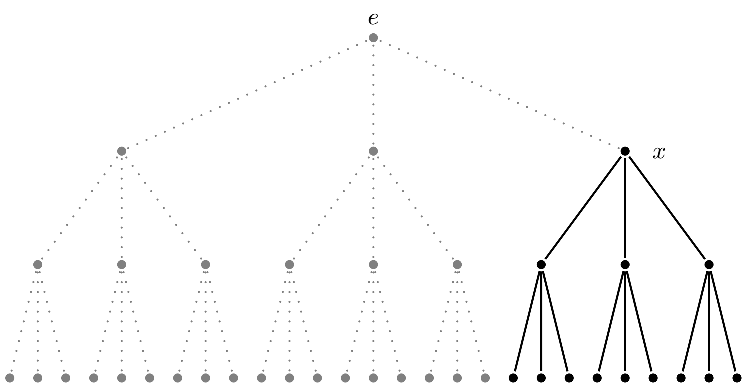
THEOREM 1.8. *The operator $J$ on $\ell^2(\Gamma_d)$ is essentially selfadjoint if and only if the one-dimensional operator $J^r$ on $\ell^2(\mathbb{N}_0)$ is essentially selfadjoint.*

*Proof.* By Theorem 0.4 and Fact 1.6, it suffices to show that if $J$ is not essentially selfadjoint, neither is the matrix $J^r$. To this end, assume that there exists $0 \neq f \in N_z(J)$. We will construct a special function in a deficiency space. This will allow us to show that $J^r$ is not essentially selfadjoint.

Let $x$ be a vertex in the support of $f$ of minimal length, i.e.

$$f(x) \neq 0 \quad \text{and} \quad f(y) = 0 \quad \text{for } |y| < |x|.$$

Let $\Gamma_x$ denote the subtree of $\Gamma_d$ with root at $x$ (see the figure below).




In the proof we are going to apply an averaging operator $E$.

LEMMA 1.9. *The averaging operator*

$$Ef(w) = \frac{1}{d^{|w|}} \sum_{|y|=|w|} f(y) \tag{1.10}$$

*is a selfadjoint projection in* $\ell^2(\Gamma_d)$.

*Proof.* For any $f, g \in \mathcal{F}$ we have

$$\begin{aligned}(Ef, g) &= \sum_{w\in\Gamma_d} Ef(w)\cdot\overline{g(w)} = \sum_{k=0}^{\infty}\sum_{|w|=k}\Big(\frac{1}{d^k}\sum_{|y|=k} f(y)\Big)\cdot\overline{g(w)}\\ &= \sum_{k=0}^{\infty}\frac{1}{d^k}\sum_{|w|=k}\sum_{|y|=k} f(y)\overline{g(w)}.\end{aligned}$$

Reversing the order of summation yields

$$(Ef, g) = \sum_{k=0}^{\infty}\sum_{|y|=k}\Big(\frac{1}{d^k}\sum_{|w|=k}\overline{g(w)}\Big)\cdot f(y) = \sum_{w\in\Gamma_d} f(w)\cdot\overline{Eg(w)} = (f, Eg),$$

which proves the symmetry.

Now, by the Schwarz inequality,

$$\begin{aligned}\|Ef\|^2 &= \sum_{w\in\Gamma_d}|Ef(w)|^2 = \sum_{k=0}^{\infty}\sum_{|w|=k}\Big|d^{-k}\sum_{|y|=k} f(y)\Big|^2\\ &\le \sum_{k=0}^{\infty} d^k d^{-2k}\Big(\sum_{|y|=k}|f(y)|\Big)^2\\ &\le \sum_{k=0}^{\infty} d^{-k}\Big(\sum_{|y|=k}1^2\Big)\Big(\sum_{|y|=k}|f(y)|^2\Big)\\ &= \sum_{k=0}^{\infty}\sum_{|y|=k}|f(y)|^2 = \sum_{y\in\Gamma_d}|f(y)|^2 = \|f\|^2,\end{aligned}$$

whence $\|E\| \le 1$. Moreover, for $f \in \ell^2_r(\Gamma_d)$ we obtain the equality $\|Ef\| = \|f\|$. ■

We denote by $f_x$ the restriction of $f$ to the subtree $\Gamma_x$. Let $k = |x|$. The symbol $E_x$ will denote the averaging operator on $\Gamma_x$. More precisely, $E_x(g)$ is the mean value of a function $g$ on each level of the subtree $\Gamma_x$:

$$E_x : \ell^2(\Gamma_x) \to \ell^2_r(\Gamma_x)$$

and

$$E_x g(y) = d^{-(|y|-k)}\cdot\sum_{\substack{t\in\Gamma_x\\ |t|=|y|}} g(t). \tag{1.11}$$

By Lemma 1.9, it is obvious that $E_x$ is a contraction on $\ell^2(\Gamma_x)$. Thus the function $E_x(f_x)$ is square-summable and radial on $\Gamma_x$. Restricting to $\Gamma_x$ and averaging in $\Gamma_x$ does not change the value at $x$. Therefore $E_x(f_x)$ takes a nonzero value at $x$. In order to belong to a deficiency space it needs to satisfy appropriate equations. Since $f$, as an element of $N_z(J)$, satisfies all the recurrence equations (1.3), its restriction $f_x$ satisfies those of them which are related to the restriction of $J$ to $\Gamma_x$. Indeed, at each vertex of $\Gamma_x$ different from $x$ the equations and values remain unchanged. Therefore, only the equation at $x$ can raise doubts. However, at $x$ we have

$$zf_x(x) = 0 + \beta_k f_x(x) + \lambda_k(f_x(x_1) + \cdots + f_x(x_d)),$$

which is consistent with the convention in (1.1) applied to the operator $J$ with coefficients shifted by $k$. The corresponding radial operator is expressed by the matrix

$$J_k^r = \begin{pmatrix} \beta_k & \sqrt{d}\,\lambda_k & 0 & 0 & 0 & \dots \\ \sqrt{d}\,\lambda_k & \beta_{k+1} & \sqrt{d}\,\lambda_{k+1} & 0 & 0 & \dots \\ 0 & \sqrt{d}\,\lambda_{k+1} & \beta_{k+2} & \sqrt{d}\,\lambda_{k+2} & 0 & \dots \\ 0 & 0 & \sqrt{d}\,\lambda_{k+2} & \beta_{k+3} & \sqrt{d}\,\lambda_{k+3} & \dots \\ 0 & 0 & 0 & \sqrt{d}\,\lambda_{k+3} & \beta_{k+4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{1.12}$$

It is immediate that taking the mean value on levels does not affect the recurrence relation described above. Hence

$$0 \neq E_x(f_x) \in N_z(J_k^r),$$

i.e. the matrix $J_k^r$ is not essentially selfadjoint. We add to $J_k^r$ an extra first column and first row consisting of zeros. We also add an extra first coordinate with value zero to the vector $E_x(f_x)$. We thus get one additional equation in the description of the deficiency space of the new operator (cf. (1.7)) which is trivially satisfied. Hence, the extended matrix is not essentially selfadjoint either. Therefore, the matrix with exactly $k$ extra zero columns and rows

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & & & \\ 0 & 0 & & J_k^r & \\ \vdots & \vdots & & & \end{pmatrix}$$

is not essentially selfadjoint. Next we add to it a symmetric finite-dimensional operator of the form

$$\begin{pmatrix} \beta_0 & \sqrt{d}\,\lambda_0 & 0 & 0 & 0 & \dots \\ \sqrt{d}\,\lambda_0 & \ddots & \ddots & & \vdots & \\ 0 & \ddots & \beta_{k-1} & \sqrt{d}\,\lambda_{k-1} & 0 & \\ 0 & & \sqrt{d}\,\lambda_{k-1} & 0 & 0 & \\ 0 & \dots & 0 & 0 & 0 & \ddots \\ \vdots & & & & \ddots & \ddots \end{pmatrix}.$$

Since it is selfadjoint and bounded, the operator $J^r$, by Theorem 0.3, is not essentially selfadjoint. ■

REMARK. We have associated with $J$ in $\ell^2(\Gamma_d)$ the radial operator $J^r$ acting in $\ell^2_r(\Gamma_d)$, which can be identified with $\ell^2(\mathbb{N}_0)$. The two matrices

$$J = \begin{pmatrix} \beta_0 & \lambda_0 & 0 & 0 \\ \lambda_0 & \beta_1 & \lambda_1 & 0 \\ 0 & \lambda_1 & \beta_2 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad J^r = \begin{pmatrix} \beta_0 & \sqrt{d}\,\lambda_0 & 0 & 0 \\ \sqrt{d}\,\lambda_0 & \beta_1 & \sqrt{d}\,\lambda_1 & 0 \\ 0 & \sqrt{d}\,\lambda_1 & \beta_2 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

do not have to be essentially selfadjoint at the same time. Let us consider an example. For $d = 2$ let $\beta_n = \lambda_n + \lambda_{n-1}$ and $\beta_0 = \lambda_0$. Then

$$J = \begin{pmatrix} \lambda_0 & \lambda_0 & & & \\ \lambda_0 & \lambda_0 + \lambda_1 & \lambda_1 & & \\ & \lambda_1 & \lambda_1 + \lambda_2 & \lambda_2 & \\ & & \lambda_2 & \lambda_2 + \lambda_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

The recurrence relation associated with $J$ (cf. (0.7)) is

$$\begin{aligned} xa_n &= \lambda_{n-1}a_{n-1} + (\lambda_n + \lambda_{n-1})a_n + \lambda_n a_{n+1} \\ &= (a_{n-1} + a_n)\lambda_{n-1} + (a_n + a_{n+1})\lambda_n, \qquad n \geq 1. \end{aligned}$$

In particular, for $x = 0$ we get

$$a_{n+1}(0) = -\frac{\lambda_{n-1}}{\lambda_n}\,(a_{n-1}(0) + a_n(0)) - a_n(0).$$

For the sequence $p_n(0)$ (cf. (0.7)) we get $a_0(0) = p_0(0) = 1$ and $a_1(0) = p_1(0) = -1$. Consequently, by induction, $p_n(0) = (-1)^n$. Hence the series $\sum p_n(0)^2$ is divergent. By the Hamburger criterion (Theorem 0.9), the matrix $J$ is essentially selfadjoint.

The corresponding matrix on the tree $\Gamma_2$ is of the form (cf. (1.5))

$$J^r = \begin{pmatrix} \lambda_0 & \sqrt{2}\,\lambda_0 & & & \\ \sqrt{2}\,\lambda_0 & \lambda_0+\lambda_1 & \sqrt{2}\,\lambda_1 & & \\ & \sqrt{2}\,\lambda_1 & \lambda_1+\lambda_2 & \sqrt{2}\,\lambda_2 & \\ & & \sqrt{2}\,\lambda_2 & \lambda_2+\lambda_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Let $\lambda_n = 2^n$. Then $\lambda_{n-1}+\lambda_n = 3\cdot 2^{n-1}$. Hence for $x=0$ the general solution to the recurrence relation

$$\sqrt{2}\cdot a_{n-1} + 3\cdot a_n + 2\sqrt{2}\cdot a_{n+1} = 0, \qquad n \geq 1,$$

is

$$a_n = \left(\frac{1}{\sqrt{2}}\right)^n (c_1\cdot\cos n\theta + c_2\cdot\sin n\theta).$$

Thus the series

$$\sum |a_n|^2 \leq (|c_1|^2+|c_2|^2)\sum \frac{1}{2^n}$$

is always convergent. Hence both series $\sum p_n(0)^2$ and $\sum q_n(0)^2$ (cf. (0.7)) are convergent. By the Hamburger criterion, $J^r$ is not essentially selfadjoint.

**1.2. Description of the deficiency space.** We are going to write down the nontrivial deficiency space $N_z(J)$ as a sum of spaces associated with vertices of $\Gamma_d$.

Fix a vertex $x$ of length $k$. Let $J_k$ denote the truncated matrix

$$J_k = \begin{pmatrix} \beta_k & \lambda_k & 0 & 0 & 0 & \cdots \\ \lambda_k & \beta_{k+1} & \lambda_{k+1} & 0 & 0 & \cdots \\ 0 & \lambda_{k+1} & \beta_{k+2} & \lambda_{k+2} & 0 & \cdots \\ 0 & 0 & \lambda_{k+2} & \beta_{k+3} & \lambda_{k+3} & \cdots \\ 0 & 0 & 0 & \lambda_{k+3} & \beta_{k+4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Observe that the subtree $\Gamma_x$ of $\Gamma_d$ can be identified in a natural way with the whole tree $\Gamma_d$. Hence $\ell^2_r(\Gamma_x)$ can be identified with $\ell^2_r(\Gamma_d)$.

In this way the matrix $J$ restricted to $\ell^2(\Gamma_x)$ coincides with the operator $J_k$ on $\ell^2(\Gamma_d)$. Moreover, $J$ restricted to $\ell^2_r(\Gamma_x)$ coincides with the operator $J_k$ on $\ell^2_r(\Gamma_d)$. Similarly to (1.4) and (1.5), it can be further identified with $J_k$ on $\ell^2(\mathbb{N}_0)$.

From now on we assume that the operator $J$ in $\ell^2(\Gamma_d)$ is not essentially selfadjoint. Hence

$$N_z(J) = (\mathrm{Im}(J-\bar{z}I))^{\perp} \neq \{0\}.$$

By Theorem 1.8, the operator $J^r$ on $\ell^2(\mathbb{N}_0)$ is not essentially selfadjoint. Furthermore, from the proof of this theorem, the truncated matrix $J^r_k$ on $\ell(\mathbb{N}_0)$ is not essentially selfadjoint either. By the above arguments, $J$ on $\ell^2_r(\Gamma_x)$ is not essentially selfadjoint. Moreover, its deficiency space is one-dimensional (cf. Theorem 0.8).

Let $\tilde{f}_x$ denote a nonzero function in this deficiency space. Observe that $\tilde{f}_x(x) \neq 0$. Indeed, if $v_n$ is the value of $\tilde{f}_x$ on the $n$th level of $\Gamma_d \supset \Gamma_x$, then the condition describing the deficiency space

$$J\tilde{f}_x = z\tilde{f}_x$$

(cf. Fact 1.6) is equivalent to the system of equations

$$\begin{aligned} zv_k &= \beta_k v_k + d\cdot\lambda_k v_{k+1},\\ zv_n &= \lambda_{n-1}v_{n-1} + \beta_n v_n + d\cdot\lambda_n v_{n+1}, \qquad n > k. \end{aligned}$$

Hence, if $v_k = 0$, then $v_{k+1} = 0$, and so $\tilde{f}_x \equiv 0$, which yields a contradiction.

Choose a function $\tilde{f}_x$ such that $\tilde{f}_x(x) = 1$. For each vertex $x \in \Gamma_d$ we define $f_x \in \ell^2(\Gamma_d)$ by saying that $\operatorname{supp} f_x \subseteq \Gamma_x$ and $f_x$ coincides with $\tilde{f}_x$ on $\Gamma_x$.

For each vertex $x \in \Gamma_d$ we also define the linear subspace

$$A_x = \Big\{\sum_{i=1}^d a_i\cdot f_{x_i} : a_i \in \mathbb{C},\ \sum_{i=1}^d a_i = 0\Big\}.$$

For $i \neq j$ the functions $f_{x_i}$ and $f_{x_j}$ are orthogonal as their supports are disjoint. Note that the condition $\sum_{i=1}^d a_i = 0$ guarantees that each element $g \in A_x$ ($x \in \Gamma_d$ and $|x| = n$) satisfies, in addition, the recurrence relation (1.3) at $x$, namely

$$\begin{aligned} 0 &= z\cdot g(x)\\ &= \lambda_{n-1}g(x_0) + \beta_n g(x) + \lambda_n(g(x_1) + \cdots + g(x_d))\\ &= 0 + 0 + \lambda_n\sum_{i=1}^d a_i = 0. \end{aligned}$$

This means that all $A_x$ are $(d-1)$-dimensional subspaces of $N_z(J)$.

Set

$$A_0 = \{a\cdot f_e : a \in \mathbb{C}\}.$$

Obviously, it is a one-dimensional subspace of $N_z(J)$.

We are going to exhibit some properties of the spaces $A_x$. First, we establish the following technical lemma.

LEMMA 1.13. *Let $x \in \Gamma_d$ and $|x| = n$. If $g \in A_x$, then*

$$\sum_{\substack{y \in \Gamma_x \\ |y|=k}} g(y) = 0$$

*for all $k \geq n+1$.*

*Proof.* Observe that for two different vertices $x_i$ and $x_j$ with the same predecessor $x$ the values of $f_{x_i}$ and $f_{x_j}$ on the corresponding levels of $\Gamma_{x_i}$ and $\Gamma_{x_j}$ are equal. This is because, by definition, the value $f_y(x)$ depends only on the lengths of $y$ and $x$. It follows that the sum of the values of $g = f_{x_i} - f_{x_j}$ on each level of $\Gamma_x$ vanishes. It is easily seen that any function $g \in A_x$ is a linear combination of the functions $f_{x_i} - f_{x_j}$. Therefore the values of $g \in A_x$ also vanish on all levels of $\Gamma_x$. ∎

FACT 1.14. *Let $x, y \in \Gamma_d \cup \{0\}$ and $x \neq y$. Then $A_x \perp A_y$.*

*Proof.* Let $g_x \in A_x$ for some vertex $x \in \Gamma_d$, where $|x| = n$. Since $f_e$ is radial, we write $f_e(|t|) = f_e(t)$ for $t \in \Gamma_d$. Then

$$(g_x, f_e) = \sum_{k=n+1}^{\infty} \sum_{\substack{t \in \Gamma_x \\ |t|=k}} g_x(t)\, \overline{f_e(t)} = \sum_{k=n+1}^{\infty} \overline{f_e(k)} \sum_{\substack{t \in \Gamma_x \\ |t|=k}} g_x(t).$$

By Lemma 1.13, all the sums $\sum_{t \in \Gamma_x,\, |t|=k} g_x(t)$ vanish, whence

$$(g_x, f_e) = 0.$$

Consider a function $g_y \in A_y$ for some vertex $y$ different from $x$. If $x \notin \Gamma_y$ and $y \notin \Gamma_x$, then the functions $g_x$ and $g_y$ have disjoint supports and thus they are orthogonal. On the other hand, if $x \in \Gamma_y$, then

$$|x| > |y| \quad \text{and} \quad \operatorname{supp}(g_x) \subset \Gamma_y.$$

Hence

$$(g_x, g_y) = \sum_{t \in \Gamma_x} g_x(t)\, \overline{g_y(t)}$$

and on levels of $\Gamma_x$ the function $g_y$ has constant values $g_y(k)$. Therefore, applying Lemma 1.13 once more, we obtain

$$(g_x, g_y) = \sum_{k=n+1}^{\infty} \sum_{\substack{t \in \Gamma_x \\ |t|=k}} g_x(t)\, \overline{g_y(t)} = \sum_{k=n+1}^{\infty} \overline{g_y(k)} \sum_{\substack{t \in \Gamma_x \\ |t|=k}} g_x(t) = 0.$$

Clearly, the case when $y \in \Gamma_x$ is similar. ∎

FACT 1.15. *Assume that $f \in N_z(J)$ and $f \perp A_x$ for all $x \in \Gamma_d \cup \{0\}$. Then $f \equiv 0$.*

*Proof.* We are going to show that $f$ vanishes on the successive levels of $\Gamma_d$ starting from the root $e$. The function $f_e$ is radial on $\Gamma_d$, whence $E(f_e) = f_e$ (cf. (1.10)). By Lemma 1.9, we thus get

$$0 = (f, f_e) = (f, E(f_e)) = (E(f), f_e).$$

By the same lemma, $E(f)$ is square-summable. Moreover, both $f_e$ and $E(f)$ are in $N_z(J^r)$ because taking the mean value on levels does not affect the recurrence relation (1.7). As $N_z(J^r)$ is one-dimensional, $E(f)$ is a constant multiple of $f_e$. Let $E(f) = \alpha f_e$. Then

$$0 = (E(f), f_e) = (\alpha f_e, f_e) = \alpha \|f_e\|^2,$$

whence $\alpha = 0$. Thus $E(f) = 0$ and in particular

$$f(e) = (Ef)(e) = 0.$$

Summarizing, the orthogonality of $f$ to $f_e$ implies that $f$ vanishes at the root $e$, i.e. on the zero level of the tree $\Gamma_d$. Similarly, the orthogonality of $f$ to the successive spaces $A_x$ enables us to show that $f$ is equal to zero at the corresponding vertices. Indeed, assume that $f(x) = 0$ for each $|x| \le n$. Fix a vertex $x$ of length $n$. Since $f \in N_z(J)$ and $f(x) = f(x_0) = 0$, the recurrence equation (1.3) at $x$,

$$zf(x) = \lambda_{n-1} f(x_0) + \beta_n f(x) + \lambda_n (f(x_1) + \cdots + f(x_d)),$$

gives

$$f(x_1) + \cdots + f(x_d) = 0. \tag{1.16}$$

Fix $g \in A_x$. Since $g$ is radial on each subtree $\Gamma_{x_i}$,

$$E_{x_d} E_{x_{d-1}} \ldots E_{x_1}(g) = g.$$

By the symmetry of these averaging operators (cf. Lemma 1.9),

$$0 = (f, g) = (f, E_{x_d} E_{x_{d-1}} \ldots E_{x_1} g) = (E_{x_1} \ldots E_{x_d} f, g).$$

By (1.16), the function

$$\mathbb{1}_{\Gamma_x} \cdot E_{x_1} \ldots E_{x_d} f,$$

where $\mathbb{1}_{\Gamma_x}$ denotes the characteristic function of $\Gamma_x \supseteq \operatorname{supp} g$, belongs to and is orthogonal to the space $A_x$ at the same time. Hence it must be zero. Therefore

$$f(x_i) = \mathbb{1}_{\Gamma_x} \cdot E_{x_1} \ldots E_{x_d} f(x_i) = 0$$

for all $i = 1, \ldots, d$. ■

We see that the sets $A_x$, in a sense, fill up the whole deficiency space $N_z(J)$. To be more precise, the above facts can be summarized as follows.

THEOREM 1.17. *The algebraic direct sum*

$$\bigoplus_{x\in \Gamma_d\cup\{0\}} A_x = \operatorname{lin}\{g_x : g_x \in A_x,\, x \in \Gamma_d \cup \{0\}\}$$

*of the pairwise orthogonal spaces* $A_x$ *is dense in the nontrivial deficiency space* $N_z(J)$.

REMARK. In the case when $d = 2$, not only $A_0$ but also all the remaining spaces $A_x$ for $x \in \Gamma_2$ are one-dimensional. Moreover, the functions

$$g_x = \frac{f_{x_1} - f_{x_2}}{\|f_{x_1} - f_{x_2}\|}, \qquad x \in \Gamma_d,$$

along with the function $g_0 = f_0/\|f_0\|$ form an orthonormal basis in $N_z(J)$ on the tree $\Gamma_2$.

Let us now calculate norms of elements of $A_x$ in the case when $d \ge 2$ is arbitrary.

Let $p_n$ be the orthogonal polynomials (cf. (0.7)) associated with the matrix

$$J^r = \begin{pmatrix} \beta_0 & \sqrt{d}\,\lambda_0 & 0 & 0 & 0 & \dots \\ \sqrt{d}\,\lambda_0 & \beta_1 & \sqrt{d}\,\lambda_1 & 0 & 0 & \dots \\ 0 & \sqrt{d}\,\lambda_1 & \beta_2 & \sqrt{d}\,\lambda_2 & 0 & \dots \\ 0 & 0 & \sqrt{d}\,\lambda_2 & \beta_3 & \sqrt{d}\,\lambda_3 & \dots \\ 0 & 0 & 0 & \sqrt{d}\,\lambda_3 & \beta_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \tag{1.18}$$

i.e. let the numbers $p_n(z)$ satisfy the equations

$$zp_n(z) = \sqrt{d}\,\lambda_{n-1}p_{n-1}(z) + \beta_n p_n(z) + \sqrt{d}\,\lambda_n p_{n+1}(z), \quad n \ge 0, \tag{1.19}$$
$$p_{-1}(z) = 0, \quad p_0(z) = 1.$$

Dividing by $(\sqrt{d}\,)^n$ gives

$$z \cdot \frac{p_n(z)}{\sqrt{d^n}} = \lambda_{n-1} \cdot \frac{p_{n-1}(z)}{\sqrt{d^{n-1}}} + \beta_n \cdot \frac{p_n(z)}{\sqrt{d^n}} + d \cdot \lambda_n \cdot \frac{p_{n+1}(z)}{\sqrt{d^{n+1}}}, \qquad n \ge 0.$$

These are exactly the equations describing the unique radial function in $N_z(J)$ (cf. (1.7)), hence

$$f_0(x) = f_0(|x|) = \frac{p_{|x|}(z)}{\sqrt{d^{|x|}}}. \tag{1.20}$$

Then

$$\|f_0\|^2 = \sum_{n=0}^{\infty} d^n |f_0(n)|^2 = \sum_{n=0}^{\infty} d^n \left|\frac{p_n(z)}{\sqrt{d^n}}\right|^2 = \sum_{n=0}^{\infty} |p_n(z)|^2.$$

The norm of an arbitrary $g \in A_0$ can be expressed as

$$\|g\| = \alpha_0(z) \cdot |g(e)|,$$

where

$$\alpha_0(z) = \Big(\sum_{n=0}^{\infty} |p_n(z)|^2\Big)^{1/2}.$$

Let $q_n$ be the orthogonal polynomials of the second kind (cf. (0.7)) associated with the matrix $J^r$, i.e.

$$zq_n(z) = \sqrt{d}\,\lambda_{n-1}q_{n-1}(z) + \beta_n q_n(z) + \sqrt{d}\,\lambda_n q_{n+1}(z), \qquad n \geq 1,$$
$$q_0(z) = 0, \qquad q_1(z) = 1/\lambda_0.$$

As before, dividing by $(\sqrt{d}\,)^{n-1}$ gives

$$z \cdot \frac{\lambda_0 q_n(z)}{\sqrt{d^{n-1}}} = \lambda_{n-1} \cdot \frac{\lambda_0 q_{n-1}(z)}{\sqrt{d^{n-2}}} + \beta_n \cdot \frac{\lambda_0 q_n(z)}{\sqrt{d^{n-1}}} + d \cdot \lambda_n \cdot \frac{\lambda_0 q_{n+1}(z)}{\sqrt{d^n}}, \qquad n \geq 1.$$

Therefore, for a fixed $i$,

$$f_{e_i}(x) = f_{e_i}(|x|) = \lambda_0 \cdot \frac{q_{|x|}(z)}{\sqrt{d^{|x|-1}}}. \tag{1.21}$$

Hence

$$\|f_{e_i}\|^2 = \sum_{x \in \Gamma_{e_i}} |f_{e_i}(x)|^2 = \sum_{n=1}^{\infty} d^{n-1}|f_{e_i}(n)|^2$$
$$= \sum_{n=1}^{\infty} d^{n-1}\lambda_0^2 \bigg|\frac{q_n(z)}{\sqrt{d^{n-1}}}\bigg|^2 = \lambda_0^2 \sum_{n=1}^{\infty} |q_n(z)|^2.$$

Let

$$\alpha_1(z) = \lambda_0 \Big(\sum_{n=1}^{\infty} |q_n(z)|^2\Big)^{1/2}. \tag{1.22}$$

Since the functions $f_{e_i}$ are pairwise orthogonal, the norm of an arbitrary function $g \in A_e$ is equal to

$$\|g\| = \Big\| \sum_{i=1}^{d} g(e_i) f_{e_i} \Big\| = \alpha_1(z) \cdot \sqrt{|g(e_1)|^2 + \cdots + |g(e_d)|^2}.$$

Now we consider a function $g \in A_x$ for a fixed vertex $x \neq e$, i.e. $|x| = k \geq 1$. Since

$$g = \sum_{i=1}^{d} g(x_i) \cdot f_{x_i},$$

where $f_{x_i}$ are pairwise orthogonal, we get

$$\|g\|^2 = \|f_{x_i}\|^2 \cdot \sum_{i=1}^{d} |g(x_i)|^2,$$

because the values of $f_{x_i}$ on the subtree $\Gamma_{x_i}$ depend only on the length of vertices and on $k$ which is the length of the root of this subtree. These values are determined by the equations

$$z f_{x_i}(n) = \lambda_{n-1} f_{x_i}(n-1) + \beta_n f_{x_i}(n) + d\lambda_n f_{x_i}(n+1), \qquad n \ge k+1,$$

and

$$f_{x_i}(k) = 0, \qquad f_{x_i}(k+1) = 1$$

(cf. (1.7) and the definition of $f_x$). Note that the numbers

$$\frac{\lambda_k (p_k(z) q_n(z) - q_k(z) p_n(z))}{\sqrt{d^{n-(k+1)}}}, \qquad n \ge k, \tag{1.23}$$

satisfy these equations. Indeed, $p_n(z)$ and $q_n(z)$ satisfy the recurrence starting with $n = 1$, in particular, for $n \ge k$. Therefore, the same holds for any linear combination of them. Since $|x_i| = k+1$, there are exactly $n-(k+1)$ vertices on the $n$th level of $\Gamma_{x_i}$. This accounts for the exponent in the denominator. Furthermore, for $n = k$ the value of the expression (1.23) is 0. Finally, from the formula

$$p_n(z) q_{n+1}(z) - p_{n+1}(z) q_n(z) = 1/\lambda_n$$

(see e.g. [1] or [14]), we get the value 1 for $n = k+1$. Consequently,

$$f_{x_i}(n) = \frac{\lambda_k (p_k(z) q_n(z) - q_k(z) p_n(z))}{\sqrt{d^{n-(k+1)}}}, \qquad n \ge k+1, \tag{1.24}$$

and thus

$$\|f_{x_i}\|^2 = \sum_{n=k+1}^{\infty} d^{n-(k+1)} |f_{x_i}(n)|^2 = \lambda_k^2 \sum_{n=k+1}^{\infty} |p_k(z) q_n(z) - q_k(z) p_n(z)|^2.$$

Hence

$$\|g\|^2 = (|g(x_1)|^2 + \cdots + |g(x_d)|^2) \cdot \lambda_k^2 \sum_{n=k+1}^{\infty} |p_k(z) q_n(z) - q_k(z) p_n(z)|^2.$$

Let $\alpha_{k+1}(z)$ denote the positive number such that

$$\alpha_{k+1}^2(z) = \lambda_k^2 \sum_{n=k+1}^{\infty} |p_k(z) q_n(z) - q_k(z) p_n(z)|^2, \qquad k \ge 1. \tag{1.25}$$

Then

$$\|g\| = \alpha_{k+1}(z) \cdot \sqrt{|g(x_1)|^2 + \cdots + |g(x_d)|^2}$$

for any $g \in A_x$, where $|x| = k \ge 1$.

Note that for $k = 0$ the right hand side of (1.25) gives exactly the number $\alpha_1(z)$ defined already by (1.22) so the numbers $\alpha_k(z)$ may be defined by the common formula (1.25) for all $k \ge 0$.

The following fact summarizes the previous considerations concerning norms.

FACT 1.26. *We have*

$$\|f_x\| = \alpha_{|x|} \quad \textit{for } x \in \Gamma_d \cup \{0\},$$

*and*

$$\|g\| = \begin{cases} \alpha_0(z) \cdot |g(e)| & \textit{if } g \in A_0, \\ \alpha_{|x|+1}(z) \cdot \Big(\sum_{i=1}^{d} |g(x_i)|^2\Big)^{1/2} & \textit{if } g \in A_x,\ x \in \Gamma_d, \end{cases}$$

*where the coefficients $\alpha_k(z)$ do not depend on functions and are as follows:*

$$\alpha_0(z) = \sum_{n=0}^{\infty} |p_n(z)|^2,$$

$$\alpha_k(z) = \lambda_{k-1}^2 \sum_{n=k}^{\infty} |p_{k-1}(z)q_n(z) - q_{k-1}(z)p_n(z)|^2 \quad \textit{for } k \geq 1.$$

**1.3. The deficiency space and the boundary of the tree.** A *path* in a tree is, by definition, a sequence $\{x_n\}$ of vertices such that for any $n$, the vertices $x_n$ and $x_{n+1}$ are joined by an edge. The *boundary* $\Omega = \partial\Gamma_d$ of the tree $\Gamma_d$ is the set of all infinite paths starting at the root $e$.

Note that at each level on the way downward from the root $e$ we have to choose one of $d$ edges, hence the boundary $\Omega$ can be identified with the Cantor set

$$\Omega \simeq \{0, 1, \ldots, d-1\}^{\mathbb{N}}$$

(which is the classical Cantor set in an interval when $d = 2$). Clearly, to each vertex $x$ and thereby to each subtree $\Gamma_x$, there is associated a *cylinder* $\Omega_x \subseteq \Omega$, i.e. the set of all those paths which contain $x$.

Let $\mu$ be the probability measure on $\{0, 1, \ldots, d-1\}$ such that

$$\mu = \frac{1}{d} \cdot (\delta_0 + \delta_1 + \cdots + \delta_{d-1}).$$

Let $d\omega$ denote the natural probability product measure on the boundary $\Omega$,

$$d\omega = \bigotimes_{i=0}^{\infty} d\mu_i, \qquad \mu_i = \mu,$$

i.e. the values of $d\omega$ on cylinders are given by

$$d\omega(\Omega_x) = d^{-|x|}, \qquad x \in \Gamma_d.$$

For each subspace $A_x \subset N_z(J)$ we define the corresponding subspace $B_x \subseteq L^2(\Omega, d\omega)$. Namely, let $B_0$ denote the one-dimensional linear subspace of constant functions on $\Omega$ and for $x \in \Gamma_d$ we put

$$B_x = \Big\{\sum_{i=1}^{d} b_i \cdot \mathbb{1}_{\Omega_{x_i}} : b_i \in \mathbb{C},\ \sum_{i=1}^{d} b_i = 0\Big\}.$$

Similarly to $A_x$, each $B_x$ is a linear space of dimension $d-1$. Another analogy is given by the following property of any element $F$ of $B_x$:

$$\int_{\Omega} F(\omega)\,d\omega = \sum_{i=1}^{d} \int_{\Omega_{x_i}} F(\omega)\,d\omega = d^{-|x_i|} \cdot \sum_{i=1}^{d} b_i = 0.$$

FACT 1.27. *The subspaces $B_x$ for $x \in \Gamma_d \cup \{0\}$ are pairwise orthogonal and fill up the whole $L^2(\Omega, d\omega)$, i.e. the algebraic direct sum*

$$\bigoplus_{x\in\Gamma_d\cup\{0\}} B_x = \operatorname{lin}\{G_x : G_x \in B_x,\ x \in \Gamma_d \cup \{0\}\}$$

*is dense in $L^2(\Omega, d\omega)$.*

*Proof.* Let $G_x \in B_x$ and $G_y \in B_y$ for $x \neq y$. Two cylinders with different vertices are either disjoint or one is a proper subset of the other. If $\Omega_x \cap \Omega_y = \emptyset$, then $G_x$ and $G_y$ are orthogonal. On the other hand, if $\Omega_y \subsetneq \Omega_x$, then there exists $i$ such that $\Omega_y \subseteq \Omega_{x_i}$. Let $b_i$ denote the value of $G_x$ on $\Omega_{x_i}$. Then

$$(G_x, G_y) = \int_{\Omega_y} G_x(\omega) G_y(\omega)\,d\omega = b_i \cdot \int_{\Omega_y} G_y(\omega)\,d\omega = 0,$$

which completes the proof of orthogonality.

Assume that $F \in L^2(\Omega, d\omega)$ is orthogonal to every $B_x$ for $x \in \Gamma_d \cup \{0\}$. In particular, for the function $G_0 \equiv 1$ belonging to $B_0$ we obtain

$$0 = (F, G_0) = \int_{\Omega} F(\omega)\,d\omega = \sum_{i=1}^{d} \int_{\Omega_{e_i}} F(\omega)\,d\omega. \tag{1.28}$$

The orthogonality of $F$ to $\mathbb{1}_{\Omega_{e_i}} - \mathbb{1}_{\Omega_{e_j}} \in B_e$ for $i \neq j$ gives

$$0 = (F, \mathbb{1}_{\Omega_{e_i}} - \mathbb{1}_{\Omega_{e_j}}) = \int_{\Omega_{e_i}} F(\omega)\,d\omega - \int_{\Omega_{e_j}} F(\omega)\,d\omega,$$

whence

$$\int_{\Omega_{e_i}} F(\omega)\,d\omega = \int_{\Omega_{e_j}} F(\omega)\,d\omega.$$

Since all the numbers $\int_{\Omega_{e_i}} F(\omega)\,d\omega$ are equal and sum up to 0 (see (1.28)), they all vanish. Similar considerations applied to $x = e_j$ and its successors $x_i$ yield $\int \Omega_y F(\omega)\,d\omega = 0$ for $\operatorname{dist}(y, e) = 2$. In this way one can show that integrating $F$ over an arbitrary cylinder gives 0. Hence $F = 0$ $d\omega$-almost everywhere. ■

Let $x \in \Gamma_d$. To the function $f_x$ corresponds the function $F_x \in L^2(\Omega, d\omega)$ defined by

$$F_x = \alpha_{|x|} \cdot \sqrt{d^{|x|}} \cdot \mathbb{1}_{\Omega_x}.$$

Then

$$\|F_x\|^2 = \int_\Omega F_x(\omega)\,\overline{F_x(\omega)}\,d\omega = \alpha_{|x|}^2(z)d^{|x|}\cdot d\omega(\Omega_x) = \alpha_{|x|}^2(z) = \|f_x\|^2.$$

Clearly, if $x_i \neq x_j$ have a common predecessor, then $F_{x_i}$ and $F_{x_j}$ are orthogonal as their supports are disjoint. Thus

$$A_x \ni \sum_{i=1}^d a_i f_{x_i} = g \;\leftrightarrow\; G = \sum_{i=1}^d a_i F_{x_i} \in B_x \tag{1.29}$$

is a one-to-one correspondence. Furthermore,

$$\begin{aligned}\|G\| &= \|F_{x_i}\|\cdot\sqrt{|a_1|^2+\cdots+|a_d|^2}\\ &= \|f_{x_i}\|\cdot\sqrt{|a_1|^2+\cdots+|a_d|^2} = \|g\|.\end{aligned}$$

It follows that the mapping $G \mapsto g$ is a linear bijection between $B_x$ and $A_x$ which, in addition, preserves the norm. We also define a mapping from $A_0$ onto $B_0$ by

$$f_0 \mapsto F_0 = \alpha_0 \cdot \mathbb{1}_\Omega, \tag{1.30}$$

which, clearly, is also a norm preserving linear bijection.

In view of Theorem 1.17 and Fact 1.27, all these bijections have a unique extension to an injective isometry

$$U\colon L^2(\Omega, d\omega) \xrightarrow{\text{onto}} N_z(J). \tag{1.31}$$

For a fixed vertex $y \in \Gamma_d$ we define a functional on $L^2(\Omega, d\omega)$ by

$$F \mapsto (UF)(y).$$

As it is linear and bounded, it determines, by the Riesz Theorem, a unique function $P_z(y,\omega) \in L^2(\Omega, d\omega)$ such that for all $F \in L^2(\Omega, d\omega)$,

$$(UF)(y) = \int_\Omega P_z(y,\omega)F(\omega)\,d\omega. \tag{1.32}$$

In particular, for a function $g \in A_x$ and the corresponding $G \in B_x$,

$$g(y) = \int_\Omega P_z(y,\omega)G(\omega)\,d\omega. \tag{1.33}$$

In view of this formula, it is natural to call $P_z(y,\omega)$ the *Poisson kernel*. It describes a relationship between functions in the deficiency space and functions on the boundary of the tree.

We now state some of the properties of this kernel.

FACT 1.34. *For a fixed $y \in \Gamma_d$,*

$$(JP_z(\cdot,\omega))(y) = z \cdot P_z(y,\omega)$$

*$d\omega$-almost everywhere.*

*Proof.* Let $y \in \Gamma_d \cup \{0\}$. It is sufficient to show that for all $x \in \Gamma_d \cup \{0\}$ and $G \in B_x$,

$$(JP_z(y,\cdot), G) = (zP_z(y,\cdot), G).$$

Set $g = UG \in A_x$. Since $A_x \subset N_z(J)$, $g$ satisfies, in particular, the recurrence relation (1.3) at $y$, i.e.

$$(Jg)(y) = z \cdot g(y).$$

In view of (1.33), we have

$$\Big(J\Big(\int_{\Omega} P_z(\cdot,\omega)G(\omega)\,d\omega\Big)\Big)(y) \;= z \cdot \int_{\Omega} P_z(y,\omega)G(\omega)\,d\omega.$$

By the linearity of the integral, we get

$$\int_{\Omega} JP_z(y,\omega)G(\omega)\,d\omega = \int_{\Omega} zP_z(y,\omega)G(\omega)\,d\omega,$$

completing the proof. ∎

Now we are going to make use of automorphisms of $\Gamma_d$. By an *automorphism* we mean a mapping $k\colon \Gamma_d \to \Gamma_d$ such that

$$(\forall x, y \in \Gamma_d) \qquad \operatorname{dist}(kx, ky) = \operatorname{dist}(x,y).$$

Observe that the root $e$ is the only vertex in $\Gamma_d$ of degree $d$. Clearly, automorphisms preserve the degree of vertices. Therefore $k(e) = e$. In addition, each automorphism of $\Gamma_d$ extends to the boundary $\Omega$.

Note that $J$ commutes with all isometries of $\Gamma_d$. Each automorphism of $\Gamma_d$ acts on functions in $A_x$ and in $B_x$ in a natural way. Namely, let $k\colon \Gamma_d \to \Gamma_d$ be an automorphism of $\Gamma_d$. For any $g \in A_x$ we put

$$({}_k g)(y) = g(k^{-1}y).$$

Then

$${}_k g \in A_{kx} \quad \text{and} \quad {}_k g(ky) = g(y).$$

Similarly, $k$ acts on functions in $B_x$. Hence for the corresponding function $G \in B_x$ we have

$${}_k G \in B_{kx} \quad \text{and} \quad ({}_k G)(k\omega) = G(\omega).$$

LEMMA 1.35. *For any fixed vertex $y \in \Gamma_d$ and an arbitrary automorphism $k$ of $\Gamma_d$,*

$$P_z(ky, k\omega) = P_z(y,\omega)$$

*$d\omega$-almost everywhere.*

*Proof.* Let $x \in \Gamma_d \cup \{0\}$ and $g \in A_x$. By (1.33) and the property of the automorphism $k$, we have

$$\int_\Omega P_z(y,\omega)G(\omega)\,d\omega = g(y) = {}_kg(ky) = U({}_kG)\,(ky) = \int_\Omega P_z(ky,\omega)\cdot({}_kG)(\omega)\,d\omega.$$

Replacing $\omega$ by $k\omega$ in the last integral yields

$$\begin{aligned}\int_\Omega P_z(y,\omega)\cdot G(\omega)\,d\omega &= \int_\Omega P_z(ky,k\omega)\cdot({}_kG)(k\omega)\,d(k\omega)\\ &= \int_\Omega P_z(ky,k\omega)\cdot G(\omega)\,d(k\omega).\end{aligned}$$

By the invariance of the measure $d\omega$, we obtain

$$\int_\Omega P_z(y,\omega)\cdot G(\omega)\,d\omega = \int_\Omega P_z(ky,k\omega)\cdot G(\omega)\,d\omega.$$

Since $G$ was an arbitrary function, the conclusion follows. ∎

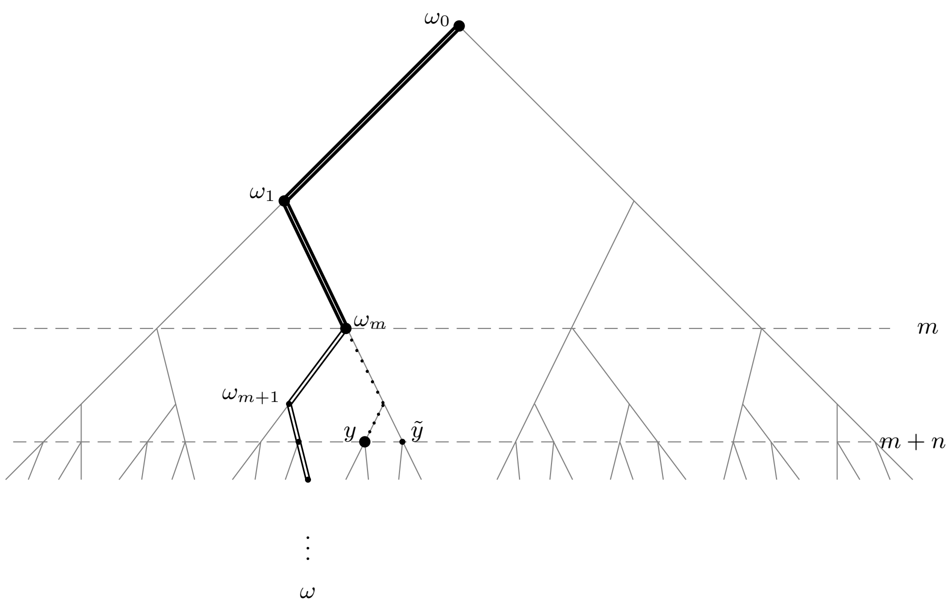
Fix a path $\omega \in \Omega$ and a vertex $y \in \Gamma_d$. Number the consecutive vertices in $\omega$ with the numbers $0, 1, 2, \dots$ starting with the root $e$,

$$\omega = \{\omega_0, \omega_1, \omega_2, \dots\}, \qquad \omega_0 = e.$$

The relative position of the path $\omega$ and the vertex $y$ can be described by two nonnegative integers. Let $n = n(y,\omega)$ denote the distance between $y$ and $\omega$ and let $m = m(y,\omega)$ be such that the vertex $\omega_m \in \omega$ realizes this distance,

$$n = \mathrm{dist}(y,\omega) = \mathrm{dist}(y,\omega_m).$$

Obviously, $|y| = m + n$. One can say that on the way from the root $e$ to the vertex $y$ we do exactly $m$ steps along the path $\omega$ and exactly $n$ steps off $\omega$.



For instance, for $y$ and $\omega$ in the figure above ($d = 2$) we get $n = m = 2$. Note that replacing $y$ by $\tilde{y}$ gives the same numbers $n$ and $m$.

FACT 1.36. *The value of the Poisson kernel $P_z(y,\omega)$ depends only on the numbers $m(y,\omega)$ and $n(y,\omega)$ defined above.*

*Proof.* Let $y, y' \in \Gamma_d$ and $\omega, \omega' \in \Omega$ satisfy

$$m(y,\omega) = m(y',\omega') \quad \text{and} \quad n(y,\omega) = n(y',\omega').$$

Let $k_1$ be any automorphism on $\Gamma_d$ mapping $\omega$ to $\omega'$. By assumption, $\operatorname{dist}(k_1 y, \omega') = \operatorname{dist}(y, \omega')$ and both distances are realized by the same point in the path $\omega'$. Hence, there exists an automorphism $k_2$ which fixes $\omega'$ but maps $k_1 y$ to $y'$. By Lemma 1.35, we thus get

$$P_z(y',\omega') = P_z(k_2(k_1 y), k_2\omega') = P_z(k_1 y, \omega') = P_z(k_1 y, k_1\omega) = P_z(y,\omega). \ \blacksquare$$

To give an explicit formula for $P_z(y,\omega)$, we introduce some projections.

Let $\pi_z$ denote the projection of $\ell^2(\Gamma_d)$ onto the deficiency space

$$N_z(J) = \overline{\bigoplus_{x\in\Gamma_d\cup\{0\}} A_x}$$

(cf. Theorem 1.17). Then for every $f \in \ell^2(\Gamma_d)$ we have

$$\pi_z(f) = \sum_{x\in\Gamma_d\cup\{0\}} \pi_{z,x}(f),$$

where $\pi_{z,x}$ denotes the projection of $\ell^2(\Gamma_d)$ onto $A_x$; in particular, $\pi_{z,0}$ is the projection onto $A_0$.

FACT 1.37. *Let $x$ be a vertex in $\Gamma_d$ and $k = |x|+1$. If $y \in \Gamma_x$ is different from $x$ and $y \in \Gamma_{x_i}$, then*

$$\pi_{z,x}(\delta_y) = \frac{\overline{f_{x_i}(y)}}{\alpha_k^2(z)} \cdot \left[ f_{x_i} - \frac{1}{d}\sum_{j=1}^{d} f_{x_j} \right].$$

*Moreover, for all $y \in \Gamma_d$ we have*

$$\pi_{z,0}(\delta_y) = \frac{\overline{f_e(y)}}{\alpha_0^2(z)} \cdot f_e.$$

*Proof.* Since $\pi_{z,x}(\delta_y) \in A_x$, there exist constants $a_j$ such that

$$\pi_{z,x}(\delta_y) = \sum_{j=1}^{d} a_j f_{x_j}.$$

Let $g = \sum_{j=1}^{d} b_j f_{x_j}$ be any element of $A_x$. Then

$$(g, \delta_y) = (\pi_{z,x}(g), \delta_y) = (g, \pi_{z,x}(\delta_y)).$$

Since $\operatorname{supp}(f_{x_j}) \subseteq \Gamma_{x_j}$ and $y \in \Gamma_{x_i}$, we obtain

$$(g, \delta_y) = \sum_{j=1}^{d} b_j f_{x_j}(y) = b_i f_{x_i}(y).$$

On the other hand, $(f_{x_i}, f_{x_j}) = 0$ for $i \neq j$. Hence

$$(g, \pi_{z,x}(\delta_y)) = \sum_{j=1}^{d} b_j\, \overline{a_j}\, \|f_{x_j}\|^2 = \alpha_k^2(z) \cdot \sum_{j=1}^{d} b_j\, \overline{a_j}.$$

We thus get the equation

$$b_i f_{x_i}(y) = \alpha_k^2(z) \cdot \sum_{j=1}^{d} b_j\, \overline{a_j}$$

for any coefficients $b_j$ such that $\sum_{j=1}^{d} b_j = 0$. Let $b_i = 1$. Then setting $b_{j_0} = -1$ for an arbitrary $j_0 \neq i$ yields

$$f_{x_i}(y) = \alpha_k^2(z)(\overline{a_i} - \overline{a_{j_0}}).$$

This means that the coefficients $a_j$ for all $j \neq i$ have the same value, as $j_0$ was chosen arbitrarily. Set $a = a_j$ for $j \neq i$. Since the sum of all the coefficients $a_j$ vanishes, we have $a_i = -(d-1)a$. It follows that

$$f_{x_i}(y) = \alpha_k^2(z) \cdot [-(d-1)\overline{a} - \overline{a}] = -d\, \overline{a} \cdot \alpha_k^2(z),$$

whence

$$a = -\frac{\overline{f_{x_i}(y)}}{d\alpha_k^2(z)}.$$

Summarizing,

$$\begin{aligned}
\pi_{z,x}(\delta_y) &= \frac{(d-1)\, \overline{f_{x_i}(y)}}{d\alpha_k^2(z)} \cdot f_{x_i} - \sum_{j \neq i} \frac{\overline{f_{x_i}(y)}}{d\alpha_k^2(z)} \cdot f_{x_j} \\
&= \frac{\overline{f_{x_i}(y)}}{\alpha_k^2(z)} \cdot \left[ \left(1 - \frac{1}{d}\right) \cdot f_{x_i} - \frac{1}{d} \cdot \sum_{j \neq i} f_{x_j} \right] \\
&= \frac{\overline{f_{x_i}(y)}}{\alpha_k^2(z)} \cdot \left[ f_{x_i} - \frac{1}{d} \cdot \sum_{j=1}^{d} f_{x_j} \right].
\end{aligned}$$

The formula for $\pi_{z,0}(\delta_y)$ is clear as the vector generating the one-dimensional subspace $A_0$ is equal to $(\alpha_0(z))^{-1} f_e$. Hence

$$\pi_{z,0}(\delta_y) = \frac{(\delta_y, f_e)}{\|f_e\|^2} \cdot f_e = \frac{\overline{f_e(y)}}{\alpha_0^2(z)} \cdot f_e. \quad \blacksquare$$

In order to describe the action of $\pi_z$ on $\delta_y$ it is necessary to consider all the subspaces $A_x$ such that $y \in \Gamma_x$.

Corollary 1.38. *Let $y \in \Gamma_d$ have length $n \geq 0$. Let $y_0, y_1, \ldots, y_n$ be the path from the root $e$ to $y = y_n$. Then*

$$\pi_z(\delta_y) = \frac{\overline{f_e(y)}}{\alpha_0^2(z)} \cdot f_e + \sum_{i=1}^{n} \frac{\overline{f_{y_i}(y)}}{\alpha_i^2(z)} \cdot \Big[ f_{y_i} - \frac{1}{d} \sum_{j=1}^{d} f_{(y_{i-1})_j} \Big],$$

*where $(y_{i-1})_j$ for $j = 1, 2, \ldots$ are all the successors of the vertex $y_{i-1}$.*

*Proof.* It is sufficient to apply Fact 1.37 to the sum

$$\pi_z(\delta_y) = \sum_{x \in \Gamma_d \cup \{0\}} \pi_{z,x}(\delta_y) = (\pi_{z,y_0} + \pi_{z,y_1} + \cdots + \pi_{z,y_n})(\delta_y). \ \blacksquare$$

Note that in view of (1.29) and (1.30), the isometry $U$ defined by (1.31) can be expressed by the formula

$$U\Big(\alpha_k(z)\sqrt{d^k} \cdot \sum_{i=1}^{d} a_i \cdot \mathbb{1}_{\Omega_{x_i}}\Big) = \sum_{i=1}^{d} a_i f_{x_i},$$

where $k = |x_i| = |x| + 1$, and

$$U(\alpha_0(z) \cdot \mathbb{1}_{\Omega}) = f_e. \tag{1.39}$$

As the supports are disjoint for $i \neq j$, the first formula can be written as

$$U(\alpha_k(z)\sqrt{d^k} \cdot \mathbb{1}_{\Omega_{x_i}}) = f_{x_i}. \tag{1.40}$$

Here comes the promised explicit formula for the Poisson kernel.

Theorem 1.41. *Let $\omega \in \Omega$ and $y \in \Gamma_d$ be of length $n$. Then*

$$P_z(y, \omega) = \frac{f_e(y)}{\alpha_0(z)} \cdot \mathbb{1}_{\Omega} + \sum_{i=1}^{n} \frac{f_{y_i}(y)}{\alpha_i(z)} \cdot \sqrt{d^i} \cdot \Big[ \mathbb{1}_{\Omega_{y_i}} - \frac{1}{d} \mathbb{1}_{\Omega_{y_{i-1}}} \Big],$$

*where $\{y_0, y_1, \ldots, y_n\}$ is the only path from the root $e = y_0$ to $y = y_n$.*

*Proof.* Applying (1.39) and (1.40) in Corollary 1.38 yields

$$\pi_z(\delta_y) = U(S(y)),$$

where

$$S(y) = \frac{\overline{f_e(y)}}{\alpha_0(z)} \cdot \mathbb{1}_{\Omega} + \sum_{i=1}^{n} \frac{\overline{f_{y_i}(y)}\sqrt{d^i}}{\alpha_i(z)} \cdot \Big[ \mathbb{1}_{\Omega_{y_i}} - \frac{1}{d} \mathbb{1}_{\Omega_{y_{i-1}}} \Big].$$

For any $F \in L^2(\Omega, d\omega)$ we have

$$UF(y) = (UF, \delta_y) = (UF, \pi_z(\delta_y)).$$

On the other hand, by (1.32),

$$UF(y) = \int_{\Omega} P_z(y, \omega) F(\omega)\, d\omega = (F, \overline{P_z(y, \cdot)}).$$

Hence

$$(F, \overline{P_z(y,\cdot)}) = (UF, \pi_z(\delta_y)) = (UF, U(S(y))) = (F, S(y)),$$

which completes the proof. ∎

**1.4. The spectrum of a selfadjoint extension.** Our next aim is to describe the spectral properties of $J$.

Recall that we are considering the case when $J$ is not essentially selfadjoint. For a fixed vertex $x \in \Gamma_d$ we define the linear subspace $H_x$ of $\ell^2(\Gamma_d)$ to consist of those functions $f \in \ell^2(\Gamma_d)$ which satisfy

(1) $\operatorname{supp}(f) \subset \Gamma_x \setminus \{x\}$,
(2) $\sum_{i=1}^d f(x_i) = 0$,
(3) $f$ is radial on each subtree $\Gamma_{x_i}$,
(4) the value of $f$ on a level of $\Gamma_{x_i}$ equals the value of $f$ on the corresponding level of $\Gamma_{x_j}$ multiplied by $f(x_i)/f(x_j)$.

Moreover, we set $H_0 = \ell^2_r(\Gamma_d)$.

Fact 1.42. *The spaces $H_x \subset \ell^2(\Gamma_d)$, where $x \in \Gamma_d \cup \{0\}$, satisfy*

(1) *$J[H_x] \subseteq H_x$ for every $x$,*
(2) *$H_x$ is closed for every $x$,*
(3) *$H_x \perp H_y$ for $x \neq y$,*
(4) *$\bigoplus_{x\in\Gamma_d\cup\{0\}} H_x = \operatorname{lin}\{f \in H_x : x \in \Gamma_d \cup \{0\}\}$ is dense in $\ell^2(\Gamma_d)$.*

*Proof.* Properties (1) and (2) are clear. Property (3) may be proved in much the same way as Fact 1.14. To prove (4), assume that $g \in \ell^2(\Gamma_d)$ is orthogonal to every $H_x$ for $x \in \Gamma_d \cup \{0\}$. In particular, as $\delta_e \in H_0$,

$$0 = (g, \delta_e) = g(e).$$

The orthogonality to $\delta_{e_i} - \delta_{e_j} \in H_e$ gives

$$0 = (g, \delta_{e_i} - \delta_{e_j}) = g(e_i) - g(e_j),$$

whence the values on the first level are all equal. Furthermore, since the characteristic function $\chi_1$ of the first level is an element of $H_0$, we get

$$0 = (g, \chi_1) = \sum_{i=1}^d g(e_i).$$

This means that $g$ vanishes also on the first level of $\Gamma_d$. Similar considerations show that $g$ is equal to 0 at each level of $\Gamma_d$. ∎

Let $J_x$ denote the restriction of $J$ to $H_x \cap D(J)$. For $x = 0$ the operator $J_x$ has the matrix $J^r = J^r_0$. For $x \in \Gamma_d$ the action of $J_x$ is associated with the restricted matrix $J^r_n$, where $n = |x| + 1$ (cf. (1.12)).

Since $J$ is not essentially selfadjoint, neither is any of the matrices $J^r_n$ (cf. the beginning of Section 1.2). It is known that there exists a selfadjoint extension $\tilde{J}_n$ of $J_n^r$ and the spectrum of each selfadjoint extension is a discrete set (cf. Theorem 0.10).

Let $\tilde{J}_x$ be the operator with domain $D(\tilde{J}_x) \subseteq H_x$ associated with the selfadjoint extension $\tilde{J}_{|x|+1}$. Hence its spectrum $\sigma(\tilde{J}_{|x|+1})$ is a discrete set, so $\tilde{J}_x$ has a pure point spectrum (i.e. there exists a basis consisting of eigenvectors). Define

$$\tilde{J}(f) = \sum_{x \in \Gamma_d \cup \{0\}} \tilde{J}_x(f)$$

with domain

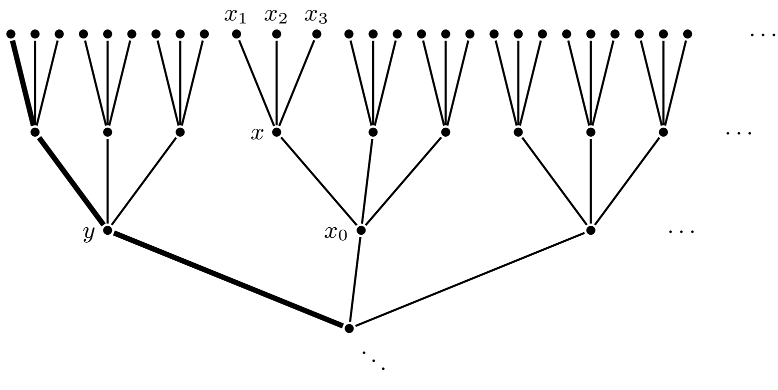
$$D(\tilde{J}) = \bigoplus D(\tilde{J}_x) = \mathrm{lin}\{f \in \ell^2(\Gamma_d) : f \in D(\tilde{J}_x)\}.$$

Since the $H_x$ are invariant under $J$ and their Hilbert orthogonal sum is the whole $\ell^2(\Gamma_d)$, the operator $\tilde{J}$ is a selfadjoint extension of $J$. Moreover, the spectrum of this extension

$$\sigma(\tilde{J}) = \overline{\bigcup \sigma(\tilde{J}_x)}$$

is also a pure point spectrum.

**2. A Jacobi operator on a tree with one end.** For a fixed number $d = 2, 3, 4, \ldots$ we consider an infinite homogeneous tree of degree $d$ which is partially ordered and locally looks like the one in the previous section but upside down. For instance, if $d = 3$, the top levels of the tree look as follows:




In view of the figure above, it is intuitively clear what the partial order in this tree is. All vertices with only one edge are on the zero level. Those at distance 1 from the zero level have length 1, and so on. To be more precise, this time we distinguish not a vertex but an infinite path $\omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$ where $\omega_0$ is any vertex with only one adjacent edge. The natural distance $\mathrm{dist}(\cdot,\cdot)$ enables one to calculate the distance between a given vertex $x$ and the path $\omega$, i.e. $\mathrm{dist}(x,\omega)$. Then, by the *length* of a vertex $x$ we mean

$$|x| = n - \mathrm{dist}(x,\omega),$$

where $n$ is the index of the element of $\omega$ which realizes the distance

$$\mathrm{dist}(x,\omega) = \mathrm{dist}(x,\omega_n).$$

In the figure above the fixed path $\omega$ is indicated by a bold line. We have $|x| = 3 - 2 = 1$ and for $y = \omega_2$ we have $|y| = 2 - 0 = 2$. It is clear that the length $|\cdot|$ defined in this way is independent of the choice of $\omega$.

The set of all vertices with this partial order is denoted by $\Lambda_d$.

In the tree $\Lambda_d$, each vertex of length at least 1 has exactly $d$ predecessors and one successor. Each origin, i.e. a vertex with no predecessor, has length 0 and exactly one adjacent (downward) edge so also one successor. This time there are infinitely many vertices of length 0. At each vertex, however, there is just one downward edge so $\Lambda_d$ can be said to be a homogeneous tree with one end.

Just as for $\Gamma_d$, the predecessors of a vertex $x$ are denoted by $x_1, \ldots, x_d$ and the successor by $x_0$. In analogy with the previous section we also define the action of the Jacobi operator $J$ on the characteristic function $\delta_x$ of a vertex $|x| = n$, namely

$$J\delta_x = \lambda_{n-1}(\delta_{x_1} + \cdots + \delta_{x_d}) + \beta_n\delta_x + \lambda_n\delta_{x_0}.$$

The domain of $J$ consists of functions with finite support, i.e.

$$D(J) = \mathrm{lin}\{\delta_x : x \in \Lambda_d\} \subseteq \ell^2(\Lambda_d).$$

We keep the convention that $\lambda_{-1} = 0$, which makes the formula for $J$ valid also for vertices of length 0.

Fact 2.1. *The deficiency space $N_z(J)$ of the operator $J$ on $\ell^2(\Lambda_d)$ consists of all square-summable functions $v$ on $\Lambda_d$ satisfying*

$$zv(x) = \lambda_{n-1}(v(x_1) + \cdots + v(x_d)) + \beta_n v(x) + \lambda_n v(x_0) \tag{2.2}$$

*for all $x$ with $|x| = n$ and all $n \geq 0$.*

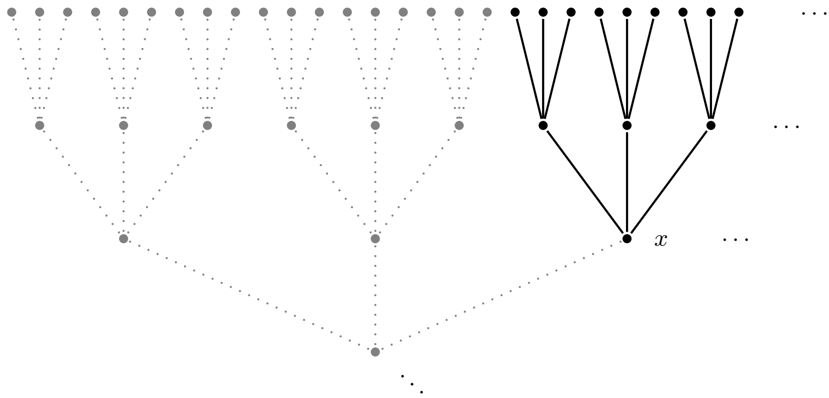
*Proof.* Just as for $\Gamma_d$, this follows by calculation:

$$\begin{aligned} 0 &= (v, (J - \bar{z})\delta_x) \\ &= (v, \lambda_{n-1}(\delta_{x_1} + \cdots + \delta_{x_d}) + \beta_n\delta_x + \lambda_n\delta_{x_o} - \bar{z}\delta_x) \\ &= \lambda_{n-1}(v(x_1) + \cdots + v(x_d)) + \beta_n v(x) + \lambda_n v(x_0) - zv(x). \ \blacksquare \end{aligned}$$

Remark. Clearly, an equivalent formulation of the assertion is:

$$N_z(J) = \{v \in \ell^2(\Lambda_d) : Jv(x) = z \cdot v(x),\ x \in \Lambda_d\}.$$

Let $\Lambda_x$ denote the subtree of $\Lambda_d$ which ends at the vertex $x$. The subtree $\Lambda_x$ is emphasized in the figure below.




The following technical lemma is a direct preparation for the main theorem of this section.

LEMMA 2.3. *Assume that $v \in \ell^2(\Lambda_d)$ satisfies the recurrence relation (2.2) for some $z \in \mathbb{C}$ (possibly real). Let $x \in \Lambda_d$ have length $n$. Assume that the values $p_k(z)$ of the orthogonal polynomials associated with the matrix $J^r$ are nonzero for $k = 1, \ldots, n$. Then the values of $v$ are constant on each level of $\Lambda_x$. Moreover, if $y \in \Lambda_x$ and $|y| = k \geq 0$, then*

$$v(y) = \sqrt{d^k}\, p_k(z) \cdot v_0,$$

*where $v_0$ is the value of $v$ on the zero level of $\Lambda_x$.*

*Proof.* The proof is by induction on $n$.

(1) Fix a vertex $x$ with $|x| = n = 1$. For each $i = 1, \ldots, d$, by (2.2) for $x = x_i$ we have

$$(z - \beta_0)v(x_i) = \lambda_0 v(x).$$

Hence

$$v(x) = \frac{z - \beta_0}{\lambda_0} \cdot v(x_i) = \sqrt{d}\, p_1(z) \cdot v_0$$

as $p_0(z) = 1$ and

$$z\, p_0(z) = \beta_0 p_0(z) + \sqrt{d}\, \lambda_0\, p_1(z).$$

(2) Assume that the assertion holds for some $n \geq 1$. Let $x$ be any vertex in $\Lambda_x$ of length $n + 1$. Each of its predecessors $x_i$ has length $n$ so the values of $v$ on the $k$th level of $\Lambda_{x_i}$ are constant and equal to $\sqrt{d^k} \cdot p_k(z) \cdot v_0^i$, where $v_0^i$ is the value of $v$ on the zero level of $\Lambda_{x_i}$. By assumption, the recurrence equation (2.2) at $x_i$,

$$(z - \beta_n) \cdot v(x_i) = \lambda_{n-1} \cdot \sum_{j=1}^{d} v((x_i)_j) + \lambda_n \cdot v(x),$$

yields

$$(z - \beta_n) \cdot \sqrt{d^n}\, p_n(z) \cdot v_0^i = d\lambda_{n-1} \cdot \sqrt{d^{n-1}}\, p_{n-1}(z) \cdot v_0^i + \lambda_n \cdot v(x).$$

Hence

$$v(x)=\frac{(z-\beta_n)\sqrt{d^n}\,p_n(z)-d\lambda_{n-1}\sqrt{d^{n-1}}\,p_{n-1}(z)}{\lambda_n}\cdot v_0^i.$$

Since $\{p_n(z)\}$ satisfies (1.19), we get

$$v(x)=v_0^i\cdot\sqrt{d^{n+1}}\cdot p_{n+1}(z).\ \blacksquare$$

Here is the main theorem.

THEOREM 2.4. *The operator $J$ on $\Lambda_d$ is always essentially selfadjoint.*

*Proof.* For a complex number $z\notin\mathbb{R}$ all the coefficients $\sqrt{d^k}p_k(z)$ appearing in Lemma 2.3 are nonzero, as all roots of the orthogonal polynomials $p_n$ are real. By Lemma 2.3, if there existed a function satisfying all the equations (2.2), it would have to be nonzero and constant on levels of the whole tree $\Lambda_d$. However, there are infinitely many vertices on each level, so such a function cannot be square-summable. Therefore, $N_z(J)=\{0\}$. $\blacksquare$

THEOREM 2.5. *The Jacobi operator $J$ on $\Lambda_d$ has a pure point spectrum, i.e. there is an orthonormal basis consisting of eigenvectors for $J$. Moreover, $\sigma(J)$ coincides with the closure of the set of all roots of the orthogonal polynomials $p_n$ associated with the matrix $J$.*

*Proof.* Since $J$ is essentially selfadjoint it suffices to find a set of eigenvectors which is linearly dense in $D(J)$.

Fix $x\in\Lambda_d$ of length $n\ge 1$. We consider the subspace $M_x\subset D(J)$ consisting of the functions with support in $\Lambda_x$. Clearly,

$$\dim M_x=1+d+d^2+\cdots+d^n.$$

It is known that $p_n$ has exactly $n$ real simple roots $t_1,\dots,t_n$. For a fixed predecessor $x_i$ of $x$ and for a fixed $t_j$ let $f_{i,j}\in M_x$ be given by

$$f_{i,j}(y)=\begin{cases}\sqrt{d^k}\cdot p_k(t_j) & \text{for } y\in\Lambda_{x_i} \text{ and } |y|=k,\\ 0 & \text{for } y\notin\Lambda_{x_i}.\end{cases}$$

Of course, $f_{i,j}$ satisfies (2.2) with $z=t_j$ and for any $x\in\Lambda_{x_i}$ different from $x_i$. Furthermore, since

$$f_{i,j}(x)=0=\sqrt{d^n}\cdot p_n(t_j),$$

the equation (2.2) also holds at $x_i$. Hence, the linear combinations

$$f_{1,j}-f_{i,j}\quad \text{for } i=2,\dots,d$$

satisfy (2.2) at $x$, i.e.

$$0=(z-\beta_n)(f_{1,j}(x)-f_{i,j}(x))=\lambda_{n-1}(f_{1,j}(x_1)-f_{i,j}(x_i))+\lambda_n\cdot 0$$

because $f_{1,j}(x_1)=f_{i,j}(x_i)$.

By the above, when $j$ is fixed and $i$ varies from 2 to $d$, the functions $f_{1,j} - f_{i,j}$ satisfy (2.2) for $z = t_j$ at every vertex of $\Lambda_d$, i.e.

$$J(f_{1,j} - f_{i,j}) = t_j \cdot (f_{1,j} - f_{i,j}), \qquad i = 2, \dots, d.$$

Hence they are eigenfunctions associated with the eigenvalue $t_j$. Clearly, there are $d-1$ of them and they form a linearly independent system since the functions $f_{i,j}$ are pairwise orthogonal for $i = 1, \dots, d$ as functions with disjoint supports.

In this way, for a fixed vertex of length $n$, we have indicated exactly $n \cdot (d-1)$ linearly independent eigenfunctions associated with this vertex. In the entire subtree $\Lambda_x$ there are $d^{n-k}$ vertices of length $k$. Of course, the eigenfunctions corresponding to two such vertices of the same length $k$ are orthogonal as their supports are disjoint. Moreover, if two vertices are such that one is in the subtree associated with the other, the corresponding eigenfunctions are also orthogonal, because on each level of the smaller tree one function has a constant value while the values of the other sum to zero (cf. the proof of Fact 1.14). Therefore, the number of all eigenfunctions of $J$ thus defined with support in $\Lambda_x$ is equal to

$$(d-1) \cdot \sum_{k=1}^{n} k \cdot d^{n-k}$$

and they are all linearly independent.

Let $V_x \subseteq M_x$ denote the linear subspace spanned by the eigenvectors defined above and with support in $\Lambda_x$. Then

$$\dim V_x = (d-1) \cdot \sum_{k=1}^{n} k \cdot d^{n-k} = (1 + d + d^2 + \dots + d^n) - (n+1).$$

Since there are $n+1$ levels in $\Lambda_x$, there exist exactly $n+1$ linearly independent functions in $M_x$ which are constant on the levels of $\Lambda_x$. Therefore, the equality

$$\dim M_x = \dim V_x + (n+1),$$

obtained above, means that the orthogonal complement of $V_x$ in $M_x$ consists only of functions constant on levels of $\Lambda_x$.

To complete the proof it suffices to show that no square-summable and nonzero function is orthogonal to every $V_x$. Assume that $f \in \ell^2(\Lambda_d)$ satisfies

$$\forall x \in \Lambda_d \quad f \perp V_x.$$

Then $f$ is constant on levels of $\Lambda_x$ for each $x \in \Lambda_d$. Hence $f$ is constant on levels of $\Lambda_d$. But $f$ is square-summable. Therefore $f \equiv 0$. ∎

**Acknowledgments.** The authors are grateful to Christian Berg and Paweł Głowacki for many helpful remarks which improved the exposition.

This research project has been partially supported by European Commission Marie Curie Host Fellowship for the Transfer of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability" MTKD-CT-2004-013389. The authors were also supported by MNiSW Grants N201 3950 33 and N201 054 32/4285. This work is based on the PhD thesis of the first author defended in the Institute of Mathematics of the University of Wrocław in October 2008.

## REFERENCES

[1] N. I. Akhiezer, *The Classical Moment Problem*, Hafner, New York, 1965.

[2] C. Berg, *Moment problems and orthogonal polynomials*, lecture notes, 2007.

[3] T. Chihara, *An Introduction to Orthogonal Polynomials*, Math. Appl. 13, Gordon and Breach, New York, 1978.

[4] H. Hamburger, *Über eine Erweiterung der Stieltjesschen Momentenproblems*, Math. Ann. 81 (1920), 235–319; 82 (1920), 120–164; 82 (1921), 168–187.

[5] R Nevanlinna, *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem*, Ann. Acad. Sci. Fenn. (A) 18 (1922), no. 5, 52 pp.

[6] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I. Functional Analysis*, Academic Press, New York, 1972.

[7] —, —, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[8] M. Riesz, *Sur le problème des moments. Première note*, Arkiv för matematik, astronomi och fysik 16 (1921), no. 12, 23 pp.

[9] —, *Sur le problème des moments. Deuxième note*, ibid. 16 (1922), no. 19, 21 pp.

[10] —, *Sur le problème des moments. Troisième note*, ibid. 17 (1923), no. 16, 62 pp.

[11] B. Simon, *The classical moment problem as self-adjoint finite difference operator*, Adv. Math. 137 (1998), 82–203.

[12] T. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Univ. Toulouse 8 (1894), J1–J122; 9 (1894), A5–A47.

[13] M. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis*, Colloq. Publ. 15, Amer. Math. Soc., New York, 1932.

[14] R. Szwarc, *Orthogonal polynomials and moment problem*, lecture notes, Inst. Math. Univ. of Wrocław, 2004 (in Polish); http://www.math.uni.wroc.pl/~szwarc/pdf/momentw.pdf.

Agnieszka M. Kazun
Institute of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: agnieszka.kazun@gmail.com

Ryszard Szwarc
Institute of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
Institute of Mathematics and Computer Science
University of Opole
Oleska 48
45-052 Opole, Poland
E-mail: szwarc2@gmail.com

*Received 20.3.2009;*
*revised 20.1.2010* (5184)